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Some Extremal Problems about Differential Equations with Perturbations

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I. INTRODUCTION

In Research Problem 26, R. Bellman [1] proposed a number of questions, some of which will, at least partly, be answered in this note.

We consider the differential equation

$$\frac{d^2u}{dt^2} + (1 + f(t)) u = 0$$

with the initial conditions

$$u(0) = 1, \quad u'(0) = 0. \quad (1.2)$$

The variable $t$ will be referred to as the time. The function $f$ is integrable over any finite interval $0 < t < T$. We consider the functional

$$J_T(f) = \max_{0 \leq t \leq T} |u(t)|,$$

and we ask for the maximum of $J(f)$ over all $f$ subject to certain constraints. In Sections II, III, and IV the constraint will be

$$|f(t)| \leq b, \quad (0 \leq t \leq T), \quad (1.3)$$

where $b$ is a fixed number, $0 < b < 1$. In Section V we take the constraint

$$\int_0^T |f(t)| dt \leq b,$$

and in Section VI we take

$$\int_0^T |f(t)|^p dt \leq b,$$

where $1 < p < \infty$.

Throughout the paper we describe the solutions by polar coordinates $r$ and $\varphi$ in the phase plane:

$$u = r \cos \varphi, \quad -\frac{du}{dt} = r \sin \varphi; \quad (1.4)$$

the definition of $\varphi$ is completed by the condition that $\varphi = 0$ at $t = 0$. 

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We obtain the differential equations
\[ \frac{d\varphi}{dt} = 1 + f(t) \cos^2 \varphi, \quad (1.5) \]
\[ \frac{d}{dt} \log r = f(t) \cos \varphi \sin \varphi, \quad (1.6) \]
with initial conditions \( r = 1, \varphi = 0 \) at \( t = 0 \).

In the remaining part of this introduction we restrict ourselves to a discussion of the constraint (1.3). Then \( |f(t)| < 1 \), so by (1.5) the relation between \( t \) and \( \varphi \) is monotonic and one-to-one. The successive extrema of \( u \) are attained for \( \varphi = 0, \pi, 2\pi, \ldots \), positive maxima and negative minima alternating. The time interval elapsing between two consecutive extrema will be called a semicycle.

The problem to maximize \( J(f) \) finds its physical interpretation in the question how to raise the amplitude of the oscillations when standing on a swing. By lowering and raising one's barycenter, the length of the pendulum can be altered. The obvious way to have the maximal effect seems to be to keep it low when descending and high when ascending. Indeed, this gives the maximal effect per semicycle, but on the other hand it increases the duration of a semicycle. If we ask for the maximal effect in a given time interval the best policy turns out to be what might be called "advanced ignition": always lower the barycenter a moment before the highest point is reached, but, remarkably, still raise it again exactly at the lowest point.

II. Constraint \(|f(t)| \leq b\). Logarithmic Gain per Unit of Time

The number \( Q \) is defined as the length of the first semicycle, i.e., the lowest positive value of \( Q \) for which \( u'(Q) = 0 \). And \( P \) is defined by the "logarithmic gain"

\[ P = \log u(Q). \]

\( P \) and \( Q \) obviously depend on \( f \).

The problem we shall solve in this section is to choose \( f \) (satisfying (1.3)) such that the quotient \( P/Q \) (which can be called the logarithmic gain per unit of time, though it is evaluated for a full semicycle) is maximal.

Since the relation between \( t \) and \( \varphi \) is one-to-one, we can write \( f(t) = g(\varphi) \), whence \( g \) is an arbitrary integrable function of \( \varphi \), restricted by \( |g(\varphi)| \leq b \). The values \( t = 0, t = Q \) correspond to \( \varphi = 0, \varphi = \pi \), respectively. It is easy to verify that

\[ P = \int_0^\pi \frac{g(\varphi) \cos \varphi \sin \varphi}{1 + g(\varphi) \cos^2 \varphi} \, d\varphi, \quad (2.1) \]

\[ Q = \int_0^\pi \frac{1}{1 + g(\varphi) \cos^2 \varphi} \, d\varphi. \quad (2.2) \]
In order to emphasize that $P$ and $Q$ depend on the function $g$, we write $P[g], Q[g]$. Now consider two functions of $\varphi$, viz., $g$, $h$, such that
\[ |g(\varphi)| \leq b, \quad |g(\varphi) + h(\varphi)| \leq b. \]

We have
\[ P[g + h] - P[g] = \int_0^1 d\mu \int_0^\pi \frac{h(\varphi) \cos \varphi \sin \varphi d\varphi}{1 + (g(\varphi) + \mu h(\varphi)) \cos^2 \varphi}; \]
\[ Q[g + h] - Q[g] = -\int_0^1 d\mu \int_0^\pi \frac{h(\varphi) \cos^2 \varphi d\varphi}{1 + (g(\varphi) + \mu h(\varphi)) \cos^2 \varphi}. \]

It follows that
\[ P[g + h] Q[g] - P[g] Q[g + h] \]
\[ = \int_0^1 d\mu \int_0^\pi \frac{Q[g] \sin \varphi + P[g] \cos \varphi}{1 + (g(\varphi) + \mu h(\varphi)) \cos^2 \varphi} h(\varphi) \cos \varphi d\varphi. \]

Now assume that $g$ has the property that for all $\varphi$
\[ g(\varphi) = b \text{ sgn } [(Q[g] \sin \varphi + P[g] \cos \varphi) \cos \varphi] \quad (2.3) \]
(where $\text{sgn } x$ denotes $1, 0, -1$ if $x > 0$, $x = 0$, $x < 0$, respectively). If $h$ is admissible, i.e., if it satisfies $|g(\varphi) + h(\varphi)| \leq b$ for all $\varphi$, then we have $g(\varphi) h(\varphi) \leq 0$ for all $\varphi$. Therefore
\[ (Q[g] \sin \varphi + P[g] \cos \varphi) h(\varphi) \cos \varphi \leq 0, \]
and hence
\[ P[g + h] Q[g] - P[g] Q[g + h] \leq 0 \quad (2.4) \]
for every admissible function $h$. There is strict inequality in (2.4) unless $h$ vanishes almost everywhere.

Thus we have proved that if $g$ satisfies (2.3), then $g$ maximizes the expression $P[g]/Q[g]$. In order to show the existence of a function satisfying (2.3) we first take any number $\omega$ ($0 \leq \omega < \frac{1}{2} \pi$) and we consider the function $g_\omega$, with period $\pi$, described by
\[ g_\omega(\varphi) = b \quad \text{if} \quad -\omega < \varphi < \frac{1}{2} \pi, \]
\[ g_\omega(\varphi) = -b \quad \text{if} \quad \frac{1}{2} \pi < \varphi < \pi - \omega, \]
\[ g_\omega(-\omega) = g_\omega\left(\frac{1}{2} \pi\right) = 0. \]
Furthermore we put
\[ \tan \omega = \theta. \]

We easily obtain
\[ P[\omega] = \frac{1}{2} \log \frac{1 + b + \theta^2}{1 - b + \theta^2}, \]
\[ Q[\omega] = (1 - b)^{-1/2} \left( \frac{1}{2} \pi - \arctan \{\theta(1 - b)^{-1/2}\} \right) \]
\[ + (1 + b)^{-1/2} \left( \frac{1}{2} \pi + \arctan \{\theta(1 + b)^{-1/2}\} \right). \]

If \( \omega \) runs from 0 to \( \frac{1}{2} \pi \), then \( P[\omega] \) runs from \( \frac{1}{2} \log \{(1 + b)/(1 - b)\} \) to 0 and \( Q[\omega] \) runs from \( \frac{1}{2} \pi\{(1 - b)^{-1/2} + (1 + b)^{-1/2}\} \) to \( \pi(1 + b)^{-1/2} \). Thus \( P[\omega]/Q[\omega] \) moves from a positive number to zero. Since \( \theta = \tan \omega \) moves from 0 to \( \infty \), it follows that there is at least one value of \( \omega \) such that
\[ P[\omega] = \theta Q[\omega]. \] (2.5)

If \( \omega \) has this property, then we have
\[ b \text{ sgn } [(Q[\omega] \sin \varphi + P[\omega] \cos \varphi) \cos \varphi] = b \text{ sgn } [\sin (\varphi + \omega) \cos \varphi] = g_\omega(\varphi). \]

Thus \( g_\omega \) satisfies (2.3), whence it solves our maximum problem. The maximum of \( P/Q \) turns out to be equal to \( \theta = \tan \omega \), according to (2.5).

It is now obvious that \( \omega \) is determined uniquely by (2.5), since different solutions \( \omega_1, \omega_2 \) would lead to different values of the maximum, which is absurd. Thus we have

**Theorem 1.** The maximum of \( P/Q \) under the constraint (1.3) (where \( 0 \leq b < 1 \)) is equal to the unique positive root \( \theta \) of the equation
\[ \frac{1}{2} \log \frac{1 + b + \theta^2}{1 - b + \theta^2} - \theta(1 - b)^{-1/2} \left( \frac{1}{2} \pi - \arctan \{\theta(1 - b)^{-1/2}\} \right) \]
\[ - \theta(1 + b)^{-1/2} \left( \frac{1}{2} \pi + \arctan \{\theta(1 + b)^{-1/2}\} \right) = 0, \] (2.6)

and the maximum is attained by taking \( f(t) = g_\omega(\varphi) \), with \( \omega = \arctan \theta \). The maximum is strict: if \( g \) is not equal to \( g_\omega \) almost everywhere on \( 0 \leq \varphi \leq \pi \), then \( g \) produces a strictly smaller value of \( P/Q \).
III. Discussion of the Result

For small values of $b$ and $\theta$, Eq. (2.6) can be expanded into powers of $\theta$ and $b$, and then it takes the form

$$bf_1(b, \theta) + \theta f_2(b^2) + bf_3(b, \theta) = 0,$$

where $f_1, f_2, f_3$ denote power series in two, one, two variables, respectively. It follows that for small values of $b$ the solution $\theta$ has the form

$$\theta = c_1b + c_2b^3 + c_3b^5 + \cdots.$$

We easily evaluate the first two coefficients:

$$0 = n^{-9}b + (n^{-3} - n^{-1}/24)b^3 + \cdots. \quad (3.1)$$

It is not difficult to argue that $\theta$ depends monotonically on $b$ throughout the interval $0 < b < 1$. For, a smaller value of $b$ means a stronger constraint on $f$ and therefore a smaller value of $\max P/Q$.

If $b \to 1$ we prefer to write $\arctan \{\theta^{-1}(1 - b)^{1/2}\}$ instead of

$$\frac{1}{2} \pi - \arctan \{\theta(1 - b)^{-1/2}\}.$$

We then observe that the left-hand side of (2.6) is a continuous function of $b$ and $\theta$ for $\theta > 0$, $0 < b \leq 1$. Since $0 < \theta < \frac{1}{2} \pi$, and since $\theta$ increases if $b$ increases, $\theta$ tends to a positive limit $\theta_1$. It is not difficult to show that $\theta_1$ satisfies the equation arising from (2.6) by making $b \to 1$. With $\theta_1 = 2^{1/2} \pi$, this equation becomes

$$\frac{1}{2} \log (1 + \eta^{-2}) - 1 - \frac{1}{2} \pi \eta - \eta \arctan \eta = 0.$$

Its unique positive root is $\eta = 0.244\ldots$, leading to $\theta = 0.345\ldots$.

If, instead of maximizing the logarithmic gain per unit of time, we maximize the logarithmic gain per cycle, we simply have to maximize $P$ itself. From (2.1) it is obvious that the maximum (under the constraint $|g| \leq b$) is obtained by taking $g = b \text{ sgn } (\cos \varphi \sin \varphi)$. This is our function $g_\omega$ specialized by taking $\omega = 0$. Thus

$$P = P[g_0] = \frac{1}{2} \log \{(1 + b)/(1 - b)\},$$

$$Q = Q[g_0] = \frac{1}{2} \pi (1 - b)^{-1/2} + \frac{1}{2} \pi (1 + b)^{-1/2}.$$ 

If $b$ is small this gives

$$P[g_0]/Q[g_0] = n^{-1}b - n^{-1}b^3/24 + \cdots,$$

and the difference with (3.1) turns out to be only very slight. If $b \to 1$, however, we have $P[g_0]/Q[g_0] \to 0$. So if $b$ is close to 1 the function producing the maximal logarithmic gain per semicycle has only a very poor logarithmic gain per time unit. In this case $g_\omega$ is really superior to $g_0$. 

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IV. MAXIMAL GAIN OVER A LONG TIME INTERVAL

Theorem 2. Assume $0 < b < 1$, and let $u$ satisfy (1.1) and (1.2). Then there exist positive numbers $C_1$ and $C_2$ (depending on $b$ only), with the following property. For all $f$ satisfying (1.3) and for all $T > 0$ we have

$$\max_{0 \leq t \leq T} |u(t)| \leq C_1 e^{\theta T}; \quad (4.1)$$

on the other hand it is possible to choose $f$ such that

$$\max_{0 \leq t \leq T} |u(t)| \geq C_2 e^{\theta T}. \quad (4.2)$$

The number $\theta$ is the number introduced in Theorem 1.

Proof. By (2.2) the length of any semicycle is at least $\pi(1 + b)^{-1/2}$ and at most $\pi(1 - b)^{-1/2}$. Thus the semicycle to which the point $t = T$ belongs, ends at a point $t = T_1$, $T \leq T_1 \leq T + \pi(1 - b)^{-1/2}$.

The semicycles between 0 and $T_1$ may all have different length, but for each one of them the increment of $\log u$ is at most $\theta$ times the length of that semicycle (by Theorem 1). Thus the increment from 0 to $T_1$ is at most $\theta$ times the sum of the lengths, i.e., $\theta T_1$. This proves (4.1), with

$$C_1 = \exp (\pi \theta (1 - b)^{-1/2}).$$

On the other hand we can consider the special $g$ for which the maximum in theorem 1 is attained. Let $Q_0$ denote the length of the semicycle in that case. There is an integral multiple $T_2$ of $Q$, with $T - \pi(1 + b)^{-1/2} < T_2 \leq T$. The value of $u$ at that point $T_2$ equals $\exp (\theta T_2)$. This proves (4.2), with

$$C_2 = \exp [- \pi \theta (1 + b)^{-1/2}].$$

V. CONSTRAINT $\int_0^T |f(t)| \, dt \leq b$

Theorem 3. If $b > 0$, $T > 0$, if $f$ is integrable on $0 \leq t < T$, with $\int_0^T |f(t)| \, dt \leq b$, then the solution of (1.1) and (1.2) satisfies

$$|u(t)| \leq e^{b/2} \quad (0 \leq t \leq T).$$

The bound is best-possible if $T \geq \pi + \frac{1}{b} b$.

Proof. With $u = r \cos \varphi$, $-du/dt = r \sin \varphi$ we have

$$|d (\log r)/dt| = |f(t) \cos \varphi \sin \varphi| \leq \frac{1}{b} |f(t)|,$$

whence $|u| \leq r \leq e^{b/2}$ for $0 \leq t \leq T$. 

The possibility of finding an $f$ for which this bound is attained, lies in the fact that $\varphi$ can be kept constant over an arbitrary large time interval. We have $d\varphi/dt = 1 + f(t) \cos^2 \varphi$, so if $\varphi = \varphi_0$ at a point $t = t_0$, and $f(t) = -\cos^{-2} \varphi_0$ if $t_0 \leq t \leq t_1$, then $\varphi$ keeps the value $\varphi_0$ throughout that interval $t_0 \leq t \leq t_1$.

In order to maximize the increment of $\log r$, we require that

$$f(t) \cos \varphi_0 \sin \varphi_0 = \frac{1}{2} |f(t)|,$$

whence $\varphi_0 = \frac{3}{4} \pi$, $f(t) = -2$. Thus we obtain the following example:

For $0 \leq t \leq \frac{3}{4} \pi$ we take $f(t) = 0$, whence $\varphi = t$, $r = 1$. If $\frac{3}{4} \pi \leq t \leq \frac{3}{4} \pi + \frac{1}{2} b$

we take $f(t) = -2$, whence $\varphi = \frac{3}{4} \pi$, $r = e^{t-3\pi/4}$. If $\frac{3}{4} \pi + \frac{1}{2} b \leq t \leq T$

we again take $f(t) = 0$, whence $\varphi = t - \frac{1}{2} b$, $r = e^{b/2}$. At the point $t = \pi + \frac{1}{2} b$

we have $\varphi = \pi$, $r = e^{b/2}$, whence $u = -e^{b/2}$.

VI. CONSTRAINT $\int_0^T |f|^p \, dt \leq b$

Let $T$, $b$, $p$ be positive numbers, $p > 1$. We consider the solution of (1.1) and (1.2) and we ask for the maximum of

$$J_T(f) = \max_{0 \leq t \leq T} |u(t)|$$

over all $f$ subject to the constraint $\int_0^T |f|^p \, dt \leq b$.

For the time being, we discuss

$$K_T(f) = \max_{0 \leq t \leq T} r(t)$$

instead of $J_T(f)$. By (1.6) we have

$$\log K_T(f) \leq \frac{1}{2} \int_0^T |f(t) \sin 2\varphi| \, dt,$$

(6.1)

and it follows by Hölder's inequality (with $q = (1 - p^{-1})^{-1}$) that

$$\log K_T(f) \leq \frac{1}{2} b^{1/p} \left( \int_0^T |\sin 2\varphi|^q \, dt \right)^{1/q}.$$

(6.2)

Assuming $p$ to be fixed ($p > 1$), we shall show that (6.1) is best possible provided that $T > 1$ and that $T/b$ is sufficiently large. It follows from the discussion of the equality case in Hölder's inequality that the equality sign
in (6.2) holds if and only if the following three conditions are satisfied simultaneously:

(i) \( f(t) \sin 2\varphi \geq 0 \) for almost all \( t \),

(ii) \( |f(t)|^p \) is almost everywhere equal to a constant multiple of \( |\sin 2\varphi|^q \),

(iii) \( \int_0^T |f(t)|^p \, dt = b \).

In combination with (1.5), the set of conditions (i), (ii), (iii) is equivalent to the following set:

\[
\begin{align*}
 f(t) &= \alpha \text{sngn} (\sin 2\varphi) \sin 2\varphi |^{q/p} \\
 &= a^q (\sin 2\varphi) |^{q/p} \\
\end{align*}
\]

for almost all \( t \), where \( \alpha \) is a positive constant, \( \varphi = \varphi(t) \) is a solution of the differential equation

\[
\frac{d\varphi}{dt} = 1 + \alpha \text{sngn} (\sin 2\varphi) \sin 2\varphi |^{q/p} \cos^q \varphi,
\]

with \( \varphi(0) = 0 \), and where finally

\[
\alpha^q \int_0^T |\sin 2\varphi(t)|^q \, dt = b.
\]

In order to show the existence of \( f, \alpha, \varphi \), we first remark that if \( \alpha \) is chosen in the interval \( 0 < \alpha < \frac{1}{2} \), say, then (6.4) has a unique solution with \( \varphi(0) = 0 \), and the question remains whether we can satisfy (6.5). First we try \( \alpha = \frac{1}{2} \). Then \( \frac{1}{2} \leq \frac{d\varphi}{dt} \leq \frac{3}{2} \), whence

\[
\frac{1}{2} \leq \int_0^T |\sin 2\varphi(t)|^q \, dt \leq \int_0^{T/2} |\sin 2\varphi(t)|^q \cdot \frac{2}{3} \, dt.
\]

If \( T > 1 \) the latter integral is at least \( C_q T \), where \( C_q \) is a positive number depending on \( q \) only. Next we consider the interval \( 0 < \alpha < \frac{1}{2} \). In that interval, the left-hand side of (6.5) depends continuously on \( \alpha \) (note that also \( \varphi \) depends on \( \alpha \)); it tends to 0 if \( \alpha \to 0 \), and it is \( > b \) at \( \alpha = \frac{1}{2} \), provided that \( T < 1 \), \( (\frac{1}{2})^q C_q T > b \). So if \( T > 1 \), \( T/b > 2^q C_q^{-1} \), there exists a \( \alpha \) with \( 0 < \alpha \leq \frac{1}{2} \) such that (6.5) holds, and this means that in (6.2) the equality sign can be attained.

Next we return to the general case (i.e. \( f \) is an arbitrary function satisfying \( \int_0^T |f|^p \, dt \leq b \)) and we shall estimate the integral occurring on the right-hand side of (6.2). We shall again assume that \( T > 1 \) and that \( T/b \) is large (it now suffices that \( T/b > 1 \)). We consider an interval \( \pi N \leq t \leq \pi (N + 1) \), where \( N \) is an integer. Let \( \varphi_0 \) be the value of \( \varphi \) at the point \( t_0 = \pi N \). By (1.5) we have, for all \( t \) in that interval,

\[
|\varphi - t - \varphi_0 + t_0| \leq \int_{\pi N}^{\pi (N+1)} |f(s)| \, ds.
\]
and it follows that upon replacing \( \varphi \) by \( t + \varphi_0 - t_0 \) in \( \int_{\pi N}^{\pi (N+1)} | \sin 2\varphi |^q \, dt \), we make an error less than \( C \int_{\pi N}^{\pi (N+1)} | f(s) | \, ds \), where \( C \) depends on \( q \) only. Since

\[
\int_{\pi N}^{\pi (N+1)} | \sin 2(t + \varphi_0 - t_0) |^q \, dt = A,
\]

we have

\[
A = 2^q \Gamma\left(\frac{q+1}{2}\right)^{2/q} \Gamma(q+1),
\]

and it follows that upon replacing \( \varphi \) by \( t + \varphi_0 - t_0 \) in \( \int_{\pi N}^{\pi (N+1)} | \sin 2\varphi |^q \, dt \), we make an error less than \( C \int_{\pi N}^{\pi (N+1)} | f(t) | \, dt \).

Using this result for \( N = 0, 1, \ldots, [T/\pi] - 1 \), we obtain

\[
\left| \int_0^T | \sin 2\varphi |^q \, dt - AT/\pi \right| < \pi + A + C \int_0^T | f(t) | \, dt
\]

(this is still true if \( 0 < T < \pi \)). Again by Hölder’s inequality, we have

\[
\int_0^T | f(t) | \, dt \leq b^{1/p} T^{1/q}.
\]

It follows that the right-hand side of (6.2) is, if \( T > 1 \) and \( T/b > 1 \),

\[
\frac{1}{2} (A/\pi)^{1/q} (b/T)^{1/p} T\{1 + O(T^{-1}) + O((b/T)^{1/p})\}.
\]

Since the equality sign in (6.2) can be attained for a special \( f \), it follows that

\[
\sup_r \log K_T(f)
\]

can be expressed as (6.6), provided that \( T > 1 \) and \( T/b \) is sufficiently large.

We finally show that we have exactly the same result for \( J_T \) instead of \( K_T \). If \( T > 1 \), \( T/b > 2^p C_\alpha \), there has been shown to exist an \( \alpha \) such that \( 0 < \alpha < \frac{1}{2} \) and (6.5) hold, where \( \varphi \) is defined by (6.4) with \( \varphi(0) = 0 \). With \( f \) defined by (6.3), and \( r \) by (1.6), \( u \) by \( u = r \cos \varphi \), we get a special set of functions \( f, u, r, \varphi \). We shall denote these by \( f^*, u^*, r^*, \varphi^* \), in order to distinguish them from the general case.

Since \( dq^*/dt > \frac{1}{2} \) we have at least one point \( t_1 \) in the interval \( T - 2\pi \leq t_1 \leq T \) such that \( | \cos \varphi^*(t_1) | = -1 \). Hence \( | u^*(t_1) | = r^*(t_1) \).

By (6.3) and (1.6), \( r^* \) is an increasing function of \( t \). It follows that, if \( T > 4\pi \), and if \( (T - 4\pi)/b \) is sufficiently large,

\[
\sup_r J_T(f) \geq J_T(f^*) \geq | u^*(t_1) | = r^*(t_1) \geq r^*(T - 4\pi) = K_{T-4\pi}(f^*) = \sup_r K_T=f^*(f).
\]

In the opposite direction we have obviously \( K_T(f) \geq J_T(f) \) for all \( f \). The loss caused by passing from \( K_{T-4\pi} \) to \( K_T \) is unimportant, for (6.7) is described by
(6.6), and (6.6) does not change if $T$ is replaced by $T - 4\pi$ (provided that $T$ is large enough). We formulate the final result as a theorem:

**Theorem 4.** Let $p$ be a fixed number, $p > 1$. Then for $b > 0$, $T > 0$ we have

$$\sup_{0 \leq t \leq T} \log \max_{f} |u(t)| = \frac{1}{2} \left( \frac{A}{\pi} \right)^{1-1/p} (b/T)^{1/p} T \left( 1 + O(T^{-1}) + O((b/T)^{1/p}) \right),$$

provided that $T$ and $T/b$ are sufficiently large. The supremum is taken over all $f$ satisfying $\int_0^T |f(t)|^p \, dt \leq b$.

We did not prove any uniformity with respect to $p$ in the above theorem, so Theorem 3 does not follow from Theorem 4. Nevertheless there is some continuity if $p \to 1$. Then we have $q \to \infty$ whence

$$\max_{t} |\sin t| = 1.$$ 

Therefore, if $b$ and $T$ are fixed, we have

$$\lim_{p \to 1} \frac{1}{2} \left( \frac{A}{\pi} \right)^{1-1/p} (b/T)^{1/p} T = \frac{1}{2} b,$$

and indeed, Theorem 3 states that

$$\sup_{f} \log \max_{0 \leq t \leq T} |u(t)| = \frac{1}{2} b$$

if $p = 1$, $T \geq \pi + \frac{1}{2} b$.

**Reference**