An Efficient Solution of a Class of Integrals Arising in Antenna Theory

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Abstract

A novel analytical solution for a class of power-radiation integrals, arising in antenna theory, is presented. These integrals are then applied to the analysis of circular-loop and circular-microstrip antennas, and the results are compared to published results obtained using numerical integration. The analytical solution is shown to be sufficiently accurate and efficient for the calculation of the radiation characteristics of these types of antennas. Possible applications to other types of antennas are discussed.

Keywords: Analytical methods; antenna theory; power radiation integrals; directivity; radiation resistance; circular loop antenna; circular microstrip antenna; loop antennas; cylindrical antennas; microstrip antennas; Bessel functions

1. Introduction

A novel analytical method for the calculation of a class of power radiation integrals that arise in antenna theory is considered in this paper. These integrals are over a finite range, and involve a product of Bessel functions. Such integrals often appear when cylindrical coordinates are used in the analysis of the radiated power (particularly, in the case of a circular loop and a circular microstrip antenna).

In this paper, the general case is explored first, and an analytical solution, represented by a single power series, is obtained. Then, this analysis is applied to the special case of a loop antenna, for two different excitations: first, a constant current distribution, and second, a co-sinusoidal current distribution.
(Because of duality, the last result is also valid for the microstrip antenna.) After the solution of the power radiation integrals is found, it is applied to obtain the radiation resistance and the directivity of the antenna. The results thus obtained are compared with available numerical results. The new analytical method is simple and accurate, and provides a stable and efficient solution for the power radiation integral. The numerical evaluation of this is a difficult task in the case of large antenna dimensions, due to the rapid oscillation of the integrand.

2. Introduction of New Auxiliary Functions $Q$

A class of new auxiliary functions, which arise in the expressions for the radiated power of some antennas, are introduced as follows:

$$Q_{mn}^{(1)}(\xi) = \frac{n}{2} \int_{0}^{\pi/2} J_{m}(\xi \sin \theta)J_{n}(\xi \sin \theta) \sin^{n} \theta d\theta,$$

(1)

where $J_{m}(\cdot)$ is the Bessel function of the first kind, $m$th order. A closed-form solution of this integral is not available in standard handbooks on integrals and special functions [1-2]. The numerical solution of these integrals becomes a difficult numerical task with increasing value of the parameter $\xi$. The crucial point in the analytical method, proposed here, is the fact that the product of the Bessel functions can be represented by a single power series [3]:

$$J_{m}(z)J_{n}(z) = \sum_{l=0}^{\infty}(-1)^{l}\left(\frac{z}{2}\right)^{m+n+2l} B_{mn,l},$$

(2)

($z = \xi \sin \theta$), where the coefficients are given by the following expression:

$$B_{mn,l} = \frac{\Gamma(m+n+2l+1)}{\Gamma(l+1)\Gamma(m+l+1)\Gamma(n+l+1)\Gamma(m+n+l+1)},$$

(3)

and $\Gamma(\cdot)$ is the gamma function [1]. After a substitution of Equation (2) into Equation (1), the following power-series expression is obtained:

$$Q_{mn}^{(1)}(\xi) = \sum_{l=0}^{\infty}(-1)^{l} B_{mn,l} W_{m+n+2l+1} \left(\frac{\xi}{2}\right)^{m+n+2l},$$

(4)

where Wallis’s integrals are introduced [2]:

$$W_{p} = \frac{\pi^{2}}{\sin^{2} \theta} \theta d\theta = \frac{\pi^{2}(p+1)}{2^{p+1} \Gamma\left(\frac{p+1}{2}\right)^{2}}.$$

(5)

Now, the solution of these new auxiliary functions, $Q_{mn}^{(1)}(\xi)$, defined by Equation (1), is given by the series expression in Equation (4), where the coefficients $\{B\}$ are obtained by Equation (3), and the coefficients $\{W\}$ are obtained by Equation (5). Below, we will show a few important applications of these new functions.

3. Application of the New Functions to the Circular Loop and to Microstrip Antennas

The thin circular-loop antenna is a simple antenna, used for many years because of the convenience that it offers: it is small, light, and easy to fabricate [4]. The problem of the current excitation of a loop antenna is very well treated in the general case in [5]. However, the problem of finding a closed-form expression for the radiated power and related characteristics (such as the radiation resistance and directivity) is treated only approximately or numerically [6]. Interest in this classical type of antenna was demonstrated recently in [7], where a new analytical approach, applied to the electromagnetic field (but not to the radiated power), was proposed.

In the next sections, a rigorous analytical method for the calculation of the radiated-power characteristics of the circular loop is proposed for the case of two different excitations: a constant current distribution, and a co-sinusoidal current distribution. A comparison of the results obtained by this efficient analytical method with available numerical results will be presented, to verify the accuracy of the method.

3.1 Constant Current

It is known that the radiation resistance, $R_{0}$, of a circular loop with a radius $a$, excited by a uniform axial current $I_{0} = I_{0}$, can be expressed by the following equation ($\xi = ka$, $k = 2\pi/\lambda$, $\eta = 377 \Omega$) [6]:

$$R_{0}(\xi) = \frac{\pi}{2} \xi^{2} Q_{11}^{(1)}(\xi).$$

(6)

We have used the new auxiliary integral of Equation (1) (in the special case when $m = n = s = 1$), which can be also expressed in the form [6]

$$Q_{11}^{(1)}(\xi) = \frac{\pi}{2} \int_{0}^{\infty} J_{2}(\xi \sin \theta) \sin \theta d\theta = \frac{\xi}{2} \int_{0}^{\infty} J_{2}(x) dx.$$

(7)

Only for this special case can another series solution be found, different from the power series of Equation (4). In [6, p. 221], the author states: “Even though (5-59) [our Equation (7)] still cannot be integrated, approximations can be made.” The last integration, however, can be performed analytically in terms of a series of Bessel functions [2]:

$$Q_{11}^{(1)}(\xi) = \frac{2}{\xi} \sum_{m=0}^{\infty} J_{2m+3}(2\xi).$$

(8)

In Figure 1, the results of simulations of the radiation resistance, $R_{0}$, versus the normalized radius $\xi = ka$ are shown. Because of the fast convergence of the series, the method is very efficient. At the same time, the results are very close to those in Figure 5.8.a in [6], obtained by numerical integration. Now, an analytical expression for the directivity, $D_{0}$, is also available [6]:
3.2 Co-Sinusoidal Current

Let us assume now that the thin loop has a co-sinusoidal axial current distribution, \( I_\phi (\varphi) = I_0 \cos \varphi \). The far field of the loop can then be expressed in terms of Bessel functions \([7]\):

\[
E_\phi = -j \frac{I_0}{4} e^{-jkr} \frac{\xi \cos \varphi}{r} \left[ J_0 (\xi \sin \theta) - J_2 (\xi \sin \theta) \right],
\]

\[
E_\theta = -j \frac{I_0}{2} e^{-jkr} \sin \varphi \cos \theta \frac{J_1 (\xi \sin \theta)}{\sin \theta},
\]

where \( r \) is the distance \((kr \gg 1)\). The radiated-power integral is

\[
P_r (\xi) = \frac{2 \pi x_0^2}{\eta} \int_0^\infty E_E^2 r^2 \sin \theta d\theta d\varphi,
\]

which, after simple integration over \( \varphi \), can be expressed in terms of five \( \vartheta \) integrals:

\[
P_r (\xi) = C \left[ \frac{\xi^2}{2} \left[ Q^{(1)}_{00} (\xi) - 2Q^{(1)}_{02} (\xi) + Q^{(1)}_{22} (\xi) \right] + Q^{(1)}_{11} (\xi) \right]
\]

where \( C = \pi \eta | I_0|^2 / 4 \). Then, the radiated power can be written in the following series expansion:

\[
P_r (\xi) = 4C \left( \frac{\xi^2}{2} \right)^2 T(\xi).
\]

After a substitution of the five auxiliary integrals from Equation (4) into Equation (13), and performing analytical manipulations, we arrive at the following final power-series expression for the new auxiliary function:

\[
T(\xi) = \frac{1}{4} \sum_{j=0}^{\infty} (-1)^j \left( \frac{\xi}{2} \right)^{2j} \left[ B_{00j} W_{2j+1} + B_{11j} (W_{2j+1} - W_{2j+3}) \right]

-2 \left( \frac{\xi}{2} \right)^4 B_{02j} W_{2j+3} + \left( \frac{\xi}{2} \right)^4 B_{22j} W_{2j+5} \right].
\]

Following the definition of the radiation resistance, we have \([6]\)

\[
R_{r1} (\xi) = \eta \frac{\pi x_0^2}{2} T(\xi),
\]

which is reminiscent of Equation (6) with \( Q^{(1)}_{11} (\xi) \) replaced by \( T(\xi) \).

The dashed line in Figure 1 shows the results of the simulations for the radiation resistance, \( R_{r1} \), of a loop antenna with a co-sinusoidal current distribution. The series expression exhibits rapid convergence, confirming the efficiency of the proposed method. To find the solution with good accuracy for even the worst-case scenario \((\xi = 20)\), only \( N = 30 \) terms in the series representation of Equation (4) are necessary.
It is easily seen that this new current distribution decreases the radiation resistance (this is not true only for values of $\xi < 1$). The present author found that the predictions for the radiation resistance of a circular-loop antenna, based on triple numerical integration, made in [8], are incorrect, probably because of accumulated numerical error. Application of the definition of the directivity, for a reference direction $\theta = 0$, yields the following expression [6]:

$$D_1(\xi) = \frac{1}{T(\xi)},$$

again reminiscent of Equation (9) when $F_m(\xi) = 1$ is assumed.

In Figure 2, the results for the directivity, $D_1$, are shown. One can easily observe that the new current distribution leads to higher values for the directivity. This plot corresponds very well to Figure 14.25 in [6]. There, the directivity of a circular microstrip antenna is obtained by numerical integration. This is because a thin magnetic-current distribution in the slot is used, which is a dual case of the previous situation with an electric-current distribution.

### 4. Conclusion

In this paper, a novel analytical solution for a class of power-radiation integrals, arising in antenna theory, has been obtained. These integrals were then applied to the analysis of circular-loop and circular-microstrip antennas, for which results obtained using numerical integration were available [6]. The analytical result, obtained in this paper, has been shown to be sufficiently accurate and efficient for the calculation of the radiation characteristics of these antennas. The auxiliary power-radiation integrals, $\{Q_\alpha(\xi)\}$, might be also useful in the analysis of other types of antennas, when the far field is expressed in terms of cylindrical harmonics.

### 5. References