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Abstract

We give a simple rigourous treatment of the classical results of the abelian sandpile model. Although we treat results which are well-known in the physics literature, in many cases we did not find complete proofs in the literature. The paper tries to fill the gap between the mathematics and the physics literature on this subject, and also presents some new proofs. It can also serve as an introduction to the model.

keywords: abelian sandpile, recurrent configurations, burning algorithm.

Mathematics Subject Classification: 60K35

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1 Introduction

Since its introduction in [BTW (1988)], the abelian sandpile model has been one of the archetype models of self organised criticality. In words, the model can loosely be described as follows. Each vertex in some finite subset $V$ of the $d$-dimensional integer lattice contains a certain number of sand grains. At discrete times, we add a sand grain to a randomly chosen vertex in $V$. Each vertex has a maximal capacity of sand grains, and when we add a grain to a vertex which has already reached this maximal capacity, grains of this site move to the neighbouring vertices, starting an avalanche. This moving of grains to neighbours is called a toppling and it can in turn cause neighbouring vertices to exceed their capacity. In this case, these neighbouring vertices send their grains to their neighbours, etcetera. At the boundary, grains are lost. The avalanche continues as long as there is at least one vertex which exceeds its capacity. A configuration in which no vertex exceeds its capacity is called stable.

Physicists are very interested in the statistics associated with the avalanches, see [Dhar (1999b)]. They study the size and duration of these avalanches, and try to describe these in terms of power laws (see e.g. [Priezzhev (1994)]). The spatial correlations in the stationary state are also believed to decay as a power law. The presence of this power law decay of correlations - typical for models at the critical point - without “fine tuning” of parameters (such as temperature or magnetic field) has led to the term “self-organised criticality”.

The abelian sandpile model allows, to some extent at least, for rigorous mathematical analysis. It can be described in terms of an abelian group of addition operators. The abelianness is an essential simplifying property, which allows for many exact results. We noted, however, that many results in the physics literature that are claimed as being exact, are not always rigorous and/or complete. Sometimes, it turns out that the ideas can be turned
into a rigorous proof simply by being a bit more precise. But sometimes, it seems that more is needed to do that. Since we think it is important that mathematicians take up the subject of self-organised criticality, we want to make sure that at least in the basic model of self-organised criticality, there is a reference containing a mathematically rigorous analysis of the model. We hope and expect that this note increases the interest of mathematicians for self-organised criticality. We treat the following aspects.

First, we consider the abelianness of the model. It will be clear from the precise definition of the model below, that if two vertices $x$ and $y$ exceed their capacity, and we only topple these two vertices (so we do not topple vertices which exceed their capacity as a result of the toppling of $x$ and/or $y$), then it doesn't matter in which order we do this: the resulting configuration after toppling $x$ and $y$, and only these, is always the same. This elementary fact does not imply that if we have multiple vertices exceeding their capacity, then the final stable configuration, obtained by toppling until no vertex exceeds its capacity anymore, is independent of the order in which we topple. Indeed, by toppling $x$ first, say, we have to take into account the possibility that a certain vertex needs to be toppled, which would never have been toppled, if $y$ had been toppled first. The essential point is to prove that irrespective of the order in which we perform the topplings, the same sites are toppled the same number of times.

After having proved the abelian property, we define the Markov chain associated with the sandpile model. In Section 4, we investigate the recurrent configurations of this Markov chain, and show that Dhar's definition of recurrence (see [DR (1989)]) is in this case the same as classical recurrence in the language of Markov chains. The number of recurrent configurations is proved to equal the number of group elements of the "group of addition operators". Our proof is in the spirit of [DR (1989)].
rent configurations. We shall call a configuration allowed if it passes a certain test via the well known *burning algorithm*. The equivalence between allowed and recurrent was open in [DR (1989)], and has been settled via a correspondence between allowed configurations and spanning trees in [IP (1998)]. We give an alternative proof of the equivalence allowed/recurrent, not using spanning trees.

2 The model

Let $V$ be a finite subset of $\mathbb{Z}^d$. An integer valued matrix $\Delta_{x,y}^V$ indexed by the sites of $x, y \in V$ is a *toppling matrix* if it satisfies the following conditions:

1. For all $x, y \in V$, $x \neq y$, $\Delta_{x,y}^V = \Delta_{y,x}^V \leq 0$,
2. For all $x \in V$, $\Delta_{x,x}^V \geq 1$,
3. For all $x \in V$, $\sum_{y \in V} \Delta_{x,y}^V \geq 0$,
4. $\sum_{x,y \in V} \Delta_{x,y}^V > 0$.

The fourth condition ensures that there are sites (so-called *dissipative sites*) for which the inequality in the third condition is strict. This is fundamental for having a well defined toppling rule later on. In the rest of the paper we will choose $\Delta^V$ to be the lattice Laplacian with open boundary conditions. More explicitly:

$$\Delta_{x,x}^V = 2d \text{ if } x \in V,$$

$$\Delta_{x,y}^V = -1 \text{ if } x \text{ and } y \text{ are nearest neighbors},$$

$$\Delta_{x,y}^V = 0 \text{ otherwise.} \tag{2.1}$$

The dissipative sites then correspond to the boundary sites of $V$. This restriction is for convenience only: the essential features on which proofs are based are symmetry and existence of dissipative sites.
2.1 Configurations

A height configuration \( \eta \) is a mapping from \( V \) to \( \mathbb{N} = \{1, 2, ... \} \) assigning to each site a natural number \( \eta(x) \geq 1 \) ("the number of sand grains" at site \( x \)). A configuration \( \eta \in \mathbb{N}^V \) is called stable if, for all \( x \in V \), \( \eta(x) \leq \Delta^V_{x,x} \). Otherwise \( \eta \) is unstable. We denote by \( \Omega_V \) the set of all stable height configurations. The maximal element of \( \Omega_V \) is denoted by \( \eta^{\max} \) (i.e., \( \eta^{\max}(x) = \Delta^V_{x,x} \) for all \( x \in V \)). For \( \eta \in \mathbb{N}^V \) and \( V' \subset V \), \( \eta|_{V'} \) denotes the restriction of \( \eta \) to \( V' \).

2.2 The toppling rule

The toppling rules corresponding to the toppling matrix \( \Delta^V \) are the mappings \( T_x \)

\[
T_x : \mathbb{N}^V \to \mathbb{N}^V,
\]

indexed by \( V \), and defined by

\[
T_x(\eta)(y) = \begin{cases} 
\eta(y) - \Delta^V_{x,y} & \text{if } \eta(x) > \Delta^V_{x,x}, \\
\eta(y) & \text{otherwise.}
\end{cases}
\]

In words, site \( x \) topples if and only if its height is strictly larger than \( \Delta^V_{x,x} \), by transferring \(-\Delta^V_{x,y}\) grains to site \( y \neq x \) and losing itself \( \Delta^V_{x,x} \) grains. Toppling rules commute on unstable configurations. This means for \( x, z \in V \) and \( \eta \) such that \( \eta(x) > \Delta^V_{x,x} \) and \( \eta(z) > \Delta^V_{z,z} \),

\[
T_x \circ T_z(\eta) = T_z \circ T_x(\eta).
\]

Choose some enumeration \( \{x_1, \ldots, x_n\} \) of the set \( V \). The toppling transformation is the mapping

\[
T_{\Delta^V} : \mathbb{N}^V \to \Omega_V
\]

defined by

\[
T_{\Delta^V}(\eta) = \lim_{N \to \infty} \left( \prod_{i=1}^{n} T_{x_i}(\cdot) \right)^N(\eta).
\]

Remarks:
1. The limit in (2.4) exists, i.e. there are no cycles, this is an easy consequence of the presence of dissipative sites.

2. The stable configuration $T_{\Delta V}(\eta)$ is independent of the chosen enumeration of $V$. This is the famous abelian property which will be proved first.

2.3 The abelian property

In this section, we shall prove that equation (2.4) properly defines a transformation from unstable to stable configurations.

Theorem 2.1 The operator $T_{\Delta V}$ is well defined.

Proof: Suppose that a certain configuration $\eta$ has more than one unstable site. In that situation, the order of the topplings is not fixed. Clearly, if we only topple site $x$ and site $y$, the order of these two topplings doesn't matter and both orders yield the same result. In the physics literature, this is often presented as a proof that $T_{\Delta V}$ is well defined. But clearly, more is needed to guarantee this. The problem is that toppling $x$ first, say, could possibly lead to a new unstable site $z$, which would never have become unstable if $y$ had been toppled first. This is the key problem we have to address. More precisely, we have to prove the following statement: no matter in which order we perform topplings, we always topple the same sites the same number of times, and thus obtain the same final configuration. Our proof is inductive, and runs as follows.

Let $\eta$ be an unstable configuration, and suppose that

$$T_{x_N} \circ \cdots \circ T_{x_2} \circ T_{x_1}(\eta)$$

and

$$T_{y_M} \circ \cdots \circ T_{y_2} \circ T_{y_1}(\eta)$$
are both stable, and both sequences are minimal in the sense that $T_{x_1} \circ \cdots \circ T_{x_2} \circ T_{x_1}(\eta)$ and $T_{y_j} \circ \cdots \circ T_{y_2} \circ T_{y_1}(\eta)$ are not stable, for all $i < N$ and $j < M$. We need to show that $M = N$, and that the sequences $x_1, x_2, \ldots, x_N$ and $y_1, y_2, \ldots, y_N$ are permutations of each other. To do this, we choose $N$ minimal with the property that there exists a sequence $x_1, \ldots, x_N$ with the property that $T_{x_N} \circ \cdots \circ T_{x_2} \circ T_{x_1}(\eta)$ is stable. We now perform induction with respect to $N$. For $N = 1$, there is nothing to prove. Suppose now that $N > 1$ and that the result has been shown for minimal length $N - 1$. Let $y_1, y_2, \ldots, y_M$ be a sequence so that $T_{y_M} \circ \cdots \circ T_{y_2} \circ T_{y_1}(\eta)$ is stable. Since $\eta(x_1) > \Delta_{x_1, x_1}$, $x_1$ must appear at least once in the sequence $y_1, y_2, \ldots, y_M$. Choose $k$ minimal so that $y_k = x_1$. Now we claim that

$$T_{y_M} \circ \cdots \circ T_{y_{k+1}} \circ T_{x_1} \circ T_{y_{k-1}} \circ \cdots \circ T_{y_2} \circ T_{y_1}(\eta)$$

and

$$T_{y_M} \circ \cdots \circ T_{y_{k+1}} \circ T_{y_{k-1}} \circ T_{x_1} \circ \cdots \circ T_{y_2} \circ T_{y_1}(\eta)$$

are the same. To see this, define $\eta' = T_{y_{k-2}} \circ \cdots \circ T_{y_2} \circ T_{y_1}(\eta)$. $x_1$ has not been toppled at this point, hence $\eta'(x_1) > \Delta_{x_1, x_1}$. We also have $\eta'(y_{k-1}) > \Delta_{y_{k-1}, y_{k-1}}$, and therefore we are allowed to interchange $T_{x_1}$ and $T_{y_{k-1}}$. Repeating this argument, we can transfer $T_{x_1}$ to the right completely, and this leads to the conclusion that

$$T_{y_M} \circ \cdots \circ T_{y_{k+1}} \circ T_{y_k} \circ T_{y_{k-1}} \circ \cdots \circ T_{y_2} \circ T_{y_1}(\eta)$$

and

$$T_{y_M} \circ \cdots \circ T_{y_{k+1}} \circ T_{y_{k-1}} \circ \cdots \circ T_{y_2} \circ T_{x_1}(\eta)$$

are the same stable configuration. Now apply the induction hypothesis to $T_{x_1}(\eta)$ and the proof is complete.
2.4 Addition operators

For $\eta \in \mathbb{N}^V$ and $x \in V$, let $\eta^x$ denote the configuration obtained from $\eta$ by adding one grain to site $x$, i.e. $\eta^x(y) = \eta(y) + \delta_{x,y}$. The addition operator defined by

$$a_{x,V}: \Omega_V \to \Omega_V; \eta \mapsto a_{x,V}\eta = T_{\Delta V}(\eta^x)$$

(2.5)

represents the effect of adding a grain to the stable configuration $\eta$ and letting the system topple until a new stable configuration is obtained. By abelianness, the composition of addition operators is commutative: for all $\eta \in \Omega_V$, $x, y \in V$,

$$a_{x,V}(a_{y,V}\eta) = a_{y,V}(a_{x,V}\eta).$$

2.5 The Markov chain

Let $p$ denote a probability measure on $V$ with support $V$, i.e. numbers $p_x$, $0 < p_x < 1$ with $\sum_{x \in V} p_x = 1$. We define a discrete time Markov chain $\{\eta_n : n \geq 0\}$ on $\Omega_V$ by picking a point $x \in V$ according to $p$ at each discrete time step and applying the addition operator $a_{x,V}$ to the configuration. This Markov chain has the transition operator

$$P_V f(\eta) = \sum_{x \in V} p_x f(a_{x,V}\eta).$$

(2.6)

We will denote by $\mathbb{P}_\eta$ the Markov measure of the chain with transition operator $P_V$ starting from $\eta$.

A configuration $\eta \in \Omega_V$ is called recurrent for the (discrete) Markov chain if

$$\mathbb{P}_\eta(\eta_n = \eta \text{ for infinitely many } n) = 1.$$  

(2.7)

A configuration which is not recurrent is called transient. Let us denote by $\mathcal{R}_V$ the set of all recurrent configurations of the Markov chain with transition operator (2.6). As we will show later on, this set is independent of the chosen $p_x$, as long as $p_x > 0$ for all $x$. 

Let $\eta, \zeta \in \Omega_V$. We say that $\zeta$ can be reached from $\eta$ in the Markov chain (notation $\eta \rightarrow^* \zeta$) if there exists $n \in \mathbb{N}$ such that $P_\eta(\eta_n = \zeta) > 0$. Two configurations $\eta, \zeta \in \Omega_V$ are said to communicate in the Markov chain (notation $\eta \sim \zeta$) if $\eta \rightarrow^* \zeta$ and $\zeta \rightarrow^* \eta$. The relation $\sim$ defines an equivalence relation on configurations, which satisfies the following property: if $\eta \in \mathcal{R}_V$ and $\eta \sim \zeta$, then $\zeta \in \mathcal{R}_V$. In fact, every configuration that can be reached from a recurrent configuration is recurrent, and hence on $\mathcal{R}_V$ the relations $\rightarrow^*$ and $\sim$ coincide. The set $\mathcal{R}_V$ can be partitioned into equivalence classes $\mathcal{C}_i$, $i = 1, \ldots, n$ which do not communicate.

If $p_x > 0$ for all $x \in V$, then from any $\eta \in \mathcal{R}_V$ we can reach the maximal configuration $\eta_{\max}$, therefore $\eta_{\max}$ is recurrent and hence the Markov chain defined by (2.6) has only one recurrent class containing the maximal configuration.

A subset $A$ of $\Omega_V$ is called closed under the Markov chain if for any $\eta \in A$ and $n \in \mathbb{N}$, $P_\eta(\eta_n \in A) = 1$. A recurrent class is closed under the Markov chain, and any set closed under the Markov chain contains at least one recurrent class. A probability measure $\mu$ on $\Omega_V$ is called invariant for the Markov chain if for any $f : \Omega_V \to \mathbb{R}$ one has

$$\int (P_V f) d\mu = \int f d\mu.$$  

(2.8)

If the Markov chain has a unique recurrent class, then it also has a unique invariant measure concentrating on that class and any initial probability measure converges exponentially fast to this unique invariant measure. In the next section we show that the invariant measure of the Markov chain (2.6) is the uniform probability measure on $\mathcal{R}_V$.

3 The group of toppling operators

In this section we show the group property of the addition operators working on the set of recurrent configuration, and some related results on subsets of
addition operators. For notational convenience we will skip the indices \( V \) referring to the finite volume in what follows.

By the abelian property, the set

\[
S = \left\{ \prod_{x \in V} a_x^{n_x} : n_x \in \mathbb{N} \right\}
\]

(3.9)

working on the set of all stable configurations is an abelian semigroup. We first show that \( S \) working on the set of recurrent configurations is a group.

**Proposition 3.1**

1. \( S \) restricted to \( \mathcal{R} \) is an abelian group (denoted by \( G \)).

2. For all \( x \in V \), there exist \( n_x \geq 1 \) such that for all \( \eta \in \mathcal{R} \):

\[
a_x^{n_x} \eta = \eta
\]

(3.10)

3. The cardinality of \( G \) equals the cardinality of \( \mathcal{R} \).

4. We have the following closure relation: for all \( x \in V \)

\[
\prod_y a_y^{\Delta x,y} = e,
\]

(3.11)

where \( e \) denotes the neutral element in \( G \).

**Proof:** First of all notice that \( \eta \in \mathcal{R} \) and \( g \in S \) implies (by positivity of the addition probabilities \( p_x \)) that \( \eta \rightarrow g\eta \), and hence \( g\eta \) is recurrent. Therefore \( \mathcal{R} \) is closed under the action of \( S \). Let \( \eta \in \mathcal{R} \). Since in the Markov chain (2.6) we add on any site with positive probability, there exist \( n_x \geq 1 \) such that

\[
\prod_{x \in V} a_x^{n_x(\eta)} \eta = \eta
\]

(3.12)

Consider the set

\[
A = \{ \zeta \in \mathcal{R} : \prod_{x \in V} a_x^{n_x(\eta)} \zeta = \zeta \}
\]

(3.13)
This set is non-empty and by the abelian property, it is closed under the action of the semigroup $S$ and hence under the Markov chain. Therefore it contains $\mathcal{R}$ and thus, by definition, equals $\mathcal{R}$. Hence the product

$$\prod_{x \in V} a_x^{n_x(\eta)}$$

acts on $\mathcal{R}$ as the neutral element, and inverses of $a_x$ acting on $\mathcal{R}$ are defined by

$$a_x^{-1} = a_x^{n_x(\eta)-1} \prod_{y \in V, y \neq x} a_y^{n_y(\eta)}$$

(3.15)

This proves the group property. To prove statement (2) of the proposition, note that $G$ is a finite group, so every element is of finite order. To prove point (3), suppose that $g\eta = g'\eta$ for some $\eta \in \mathcal{R}$, $g, g' \in G$. Then by abelianness:

$$g(h\eta) = g'(h\eta),$$

(3.16)

for any $h \in G$. The set $\{h\eta : h \in G\}$ is closed under the working of $S$, and contains $\eta$. Therefore it coincides with $\mathcal{R}$. We conclude that $g\zeta = g'\zeta$ for any $\zeta \in \mathcal{R}$, and hence by definition of $G$ this implies $g = g'$. Therefore the mapping

$$\Psi_\eta : G \to \mathcal{R} : g \mapsto g\eta$$

(3.17)

is bijective. Finally (as explained already in [Dhar (1990a)]) the closure relation is the consequence of the observation that adding $\Delta_{x,x}$ grains to a site $x$ makes the site topple, which results in a transfer of $-\Delta_{x,y}$ particles to any neighboring site $y$. This gives

$$a_x^{\Delta_{x,x}} = \prod_{y \neq x} a_y^{-\Delta_{x,y}},$$

(3.18)

which yields (3.11).

**Corollary 3.2** *The unique invariant measure of the Markov chain (2.6) is the uniform measure on $\mathcal{R}_V$.***
Proof: The invariant measure is unique since there is only one recurrent class. The uniform measure is invariant under the working of any individual addition operator \( a_x \) because

\[
\sum_{\eta \in \mathcal{R}_V} f(\eta)g(a_x \eta) = \sum_{\eta \in \mathcal{R}_V} f(a^{-1}_x \eta)g(\eta), \tag{3.19}
\]

and we can choose \( f = 1 \). Hence the uniform measure on \( \mathcal{R} \) is invariant under the working of the Markov transition operator \( P_v \) of (2.6), independently of the chosen \( p \).

Remark: From the implication \( \eta \in \mathcal{R}, g \in \mathcal{S}, \) then \( g\eta \in \mathcal{R}, \) it follows that \( \eta \in \mathcal{R} \) and \( \zeta \geq \eta \) implies \( \zeta \in \mathcal{R} \).

Definition 3.3 Let \( A \subset \Omega \) and \( \mathcal{S}' \subset \mathcal{S} \). We say that \( A \) has the \( \mathcal{S}' \)-group property if \( \mathcal{S}' \) restricted to \( A \) forms a group.

Definition 3.4 Let \( \mathcal{S}' \subset \mathcal{S} \), and \( A, B \subset \Omega \). We say that \( A \) is \( \mathcal{S}' \)-connected to \( B \) if for any \( \eta \in A \) there exists \( g \in \mathcal{S}' \) such that \( g\eta \in B \).

Proposition 3.5 Let \( \mathcal{S}' \subset \mathcal{S} \), and \( A \subset \Omega \). Suppose \( A \) has the \( \mathcal{S}' \)-group property and is \( \mathcal{S}' \)-connected to \( \mathcal{R} \). Then \( A \) is a subset of \( \mathcal{R} \). If, in addition, \( A \) is closed under the action of \( \mathcal{S} \), then \( A \) equals \( \mathcal{R} \).

Proof: Let \( \eta \in A \). Then there exists \( g \in \mathcal{S}' \) such that \( \zeta = g\eta \in \mathcal{R} \). Since \( g \) acting on \( A \) can be inverted, \( \eta = g^{-1}\zeta \). Therefore, \( \zeta \) and \( \eta \) communicate in the Markov chain. Since \( \zeta \in \mathcal{R} \), it follows \( \eta \in \mathcal{R} \). Therefore, \( A \) is a subset of \( \mathcal{R} \). If \( A \) is closed under the action of \( \mathcal{S} \), then it is closed under the Markov chain, and hence contains \( \mathcal{R} \).

4 Recurrent configurations

We first show that Dhar's definition of recurrence in [DR (1989)] is the same as the classical definition in terms of the Markov chain.
The abelian sandpile

Theorem 4.1 We have the following identity:

\[ \mathcal{R} = \{ \eta \in \Omega : \forall x \in V \exists n_x \geq 1 : a_x^{n_x} \eta = \eta \} \] (4.20)

Proof: Denote the set in the right hand site of (4.20) by \( A \). Remark that
the \( n_x \) can be chosen independent of \( \eta \). Indeed, if \( a_x^{n_x} \eta = \eta \) for all \( \eta \in A \),
then by abelianness, for all \( \zeta \in A \) we obtain

\[ a_x^{\prod_{\eta \in A} n_x(\eta)} \zeta = \zeta. \] (4.21)

By Proposition 3.1, \( \mathcal{R} \subset A \). Moreover, restricted to \( A \), inverses on \( S \) can
be defined by \( a_x^{-1} = a_x^{n_x-1} \). Therefore, \( S \) restricted to \( A \) is a group, and
\( A \) is clearly \( S \)-connected to the maximal configuration which belongs to \( \mathcal{R} \). ■

The previous result showed that the recurrent configurations are precisely
those, for which repeated adding of grains at any vertex eventually leads to
the original configuration. The following lemma is related. It shows that if
we start with a configuration outside \( \mathcal{R} \), then by repeated addition at any
particular vertex, we eventually obtain a recurrent configuration. We shall
use this result later.

Lemma 4.2 Define

\[ \Omega' = \{ \eta \in \Omega : \forall x \in V \exists n_x : a_x^{n_x} \eta \in \mathcal{R} \} \] (4.22)

then \( \Omega' = \Omega \).

Proof: Certainly, \( \Omega' \) is not empty, since it contains \( \mathcal{R} \). Define, for \( x \in V \),
the "diminishing-operator" \( \dim_x(\eta) \) as follows:

\[ \dim_x(\eta)(y) = \max\{ (\eta(y) - \delta_{y,x}), 1 \}. \] (4.23)

In words, we substract one from \( \eta \) at site \( x \), if this is possible. We want
to prove now that for \( \eta \in \Omega' \), \( \dim_x(\eta) \) is still in \( \Omega' \). Since the maximal
configuration \( \eta^\text{max} \) is in \( \mathcal{R} \), this clearly implies the statement of the lemma.
Let $\eta \in \Omega'$. Clearly $a_x^{n_x+1}\dim_x(\eta) = a_x^{n_x}\eta \in \mathcal{R}$. Now let $y \in V$. By adding at $y$ we can create as many topplings as we want at any site $z \in V$, i.e., we can write

$$a_y^k = \prod_{z \in V} a_x^{r_x(k)},$$

(4.24)

where $r_z(k) \to \infty$ for any $z \in V$ as $k \to \infty$. Since $\eta \in \Omega'$, there exists $n_y$ such that $a_y^{n_y}\eta \in \mathcal{R}$. Now choose $k > n_y$ big enough such that $r_x(k) \geq 1$, and $r_y(k) \geq n_y$. Then we can write,

$$a_y^k \dim_x(\eta) = a_y^{n_y} \left( a_y^{r_y(k)-n_y} a_x^{r_x(k)-1} \prod_{z \in V, z \neq x, y} a_z^{r_z(k)}(a_x \dim_x(\eta)) \right)$$

$$= a_y^{n_y} \left( a_y^{r_y(k)-n_y} a_x^{r_x(k)-1} \prod_{z \in V, z \neq x, y} a_z^{r_z(k)}(\eta) \right)$$

$$= \left( a_y^{r_y(k)-n_y} a_x^{r_x(k)-1} \prod_{z \in V, z \neq x, y} a_z^{r_z(k)}(\eta) \right) a_y^{n_y}(\eta) \in \mathcal{R}.$$

Hence we conclude that $\Omega'$ is closed under the $\dim_x$-operation, for any $x \in V$.

Next, we prove Dhar's formula for the number of recurrent configurations ([DR (1989)]).

**Theorem 4.3** $|\mathcal{R}| = \det(\Delta)$.

**Proof:** Consider the following mapping:

$$\Psi : \mathbb{Z}^V \to G : n \mapsto \prod_x a_x^{n_x}.$$  

(4.25)

Clearly, $\Psi$ is a homomorphism, i.e., for $n, m \in \mathbb{Z}^V$,

$$\Psi(n + m) = \Psi(n)\Psi(m).$$

Since $\psi$ is also surjective, $G$ is isomorphic to the quotient $\mathbb{Z}^V/K$, where $K$ is the set of those vectors $n \in \mathbb{Z}^V$ for which $\Psi(n) = e$. By identity (3.11),
we conclude that
\[ K \supset \Delta Z^V, \]  
(4.26)
where
\[ \Delta Z^V = \{ \Delta n : n \in Z^V \} \]  
(4.27)
Suppose now that \( \Psi(n) = e \) for some \( n \in Z^V \). Then, writing \( n = n^+ - n^- \), where \( n^+(x) \geq 0, n^-(x) \geq 0 \) for all \( x \in V \), we have
\[ \prod_x a^+_{x} = \prod_x a^-_{x}. \]  
(4.28)
Let \( \eta \in R \). By (4.28), adding \( n^+ \) to \( \eta \) gives the same result as adding \( n^- \). Therefore we can write
\[ \eta + n^+ = \zeta + \Delta k^+, \]
\[ \eta + n^- = \zeta + \Delta k^-, \]  
(4.29)
where \( k^+(x) \), resp. \( k^-(x) \) represents the number of topplings at site \( x \) after addition of \( n^+ \), resp. \( n^- \). Subtracting the second from the first equation in (4.29) leads to the conclusion
\[ n = n^+ - n^- = \Delta(k^+ - k^-), \]  
(4.30)
i.e., \( K \subset \Delta Z^V \). We thus conclude that \( G \) is isomorphic to \( Z^V / \Delta Z^V \). The latter group has cardinality \( \det(\Delta) \), as is well known.

Remark:

From the fact that each equivalence class of \( Z^V / \Delta Z^V \) can be identified with a unique recurrent configuration, we deduce the following useful fact. If \( \eta \in R \) is and we add to \( \eta \) a configuration \( \zeta \in N^V \) (point-wise addition) and \( \xi \in R, \alpha \in N^V \) are such that
\[ \eta + \zeta - \Delta \alpha = \xi, \]  
(4.31)
then this means the following: if we add to \( \eta \) according to \( \zeta \), then we topple to \( \xi \), and the number of topplings at each site is given by \( \alpha \).
5 Allowed configurations

Let $\eta : V \to \mathbb{N}$ be a height configuration. For a subset $W \subset V$ we say that the restriction $\eta|_W$ is a forbidden subconfiguration if for all $x$ in $W$ we have the inequality

$$\eta(x) \leq \text{deg}_W(x), \quad (5.32)$$

where $\text{deg}_W(x)$ denotes the number of neighbours of $x$ in $W$. A configuration without forbidden subconfigurations is called allowed. The burning algorithm determines whether a configuration $\eta \in \Omega$ is allowed or not. It is described as follows: Pick $\eta \in \Omega$ and erase all sites $x \in V$ satisfying the inequality

$$\eta(x) > \sum_{y \in V : y \neq x} (-\Delta_{x,y}).$$

This means "erase the set $E_1$ of all sites $x \in V$ with a height strictly larger than the number of neighbors of that site in $V".\) Iterate this procedure for the new volume $V \setminus E_1$, and the new matrix $\Delta^{V \setminus E_1}$ defined by

$$\Delta^{V \setminus E_1} = \Delta^V \ 	ext{if } x, y \in V \setminus E_1$$

and so on. If $\eta$ contains a forbidden subconfiguration, then the algorithm will never remove vertices in this subconfiguration, and the limiting set is nonempty. On the other hand, if there is no such forbidden subconfiguration in $\eta$, then the algorithm will eventually remove all vertices. Hence in this case, the limiting set will be empty. So a configuration is allowed if and only of the burning algorithm erases (burns) all vertices. Let us denote by $A$ the set of all allowed configurations.

Lemma 5.1 1. The set of allowed configurations is closed under the action of $S$.

2. $A \supseteq \mathcal{R}$. 
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Proof: Let \( \eta \in \mathcal{A} \). Addition on a site \( x \in V \) for which \( \eta(x) < \Delta_{x,x} \) increases the height and thus cannot create a forbidden subconfiguration if the original \( \eta \) does not contain a forbidden subconfiguration. Suppose that by toppling the site \( x \), we create a forbidden subconfiguration in the subvolume \( V_f \subset V \). After toppling at site \( x \), the new height at site \( y \) satisfies

\[
T_x \eta(y) = \eta(y) - \Delta_{x,y}.
\]

If \( T_x \eta|_{V_f} \) is a forbidden subconfiguration, then for all \( y \in V_f \setminus \{x\} \) we have

\[
\eta(y) \leq \deg_{V_f}(y) + \Delta_{xy}
\]

i.e.,

\[
\eta(y) \leq \deg_{V_f \setminus \{x\}}(y)
\]

and we conclude that \( \eta|_{V_f \setminus \{x\}} \) is a forbidden subconfiguration for \( \eta \), which is not possible since \( \eta \) was supposed to be allowed. Since the operators \( a_x \) are products of additions and topplings, we conclude \( \eta \in \mathcal{A} \) implies \( a_x \eta \in \mathcal{A} \). Clearly, the maximal configuration \( \eta^{\text{max}} \in \mathcal{A} \). Therefore, \( g \eta^{\text{max}} \in \mathcal{A} \) for all \( g \in S \), and thus point (2) of the lemma follows. 

The following lemma is called “the multiplication by identity test” (see e.g., [Dhar (1999b)])

Lemma 5.2 For \( x \in V \), let \( \alpha_x \) denote the number of neighbors of \( x \) in \( V \).

The following two assertions are equivalent

1. \( \eta \in \mathcal{A} \)

2. \( \prod_{x \in V} a_x^{\Delta_{x,x} - \alpha_x} \eta = \eta \).

Proof: Let \( \eta \in \Omega \). Upon addition of \( \sum_x (\Delta_{x,x} - \alpha_x) \delta_x \) to \( \eta \), we have to topple those boundary sites \( x \in V \) that satisfy the inequality

\[
\Delta_{x,x} - \alpha_x + \eta(x) > \Delta_{x,x}.
\]

\[
\tag{5.34}
\]
These are precisely the sites that can be burned in the "first step" of the burning algorithm. Let us call $B_1$ the set of those sites. After toppling all sites in $B_1$, we will have a toppling at those sites $x$ in $\partial V \setminus B_1$ that satisfy the inequality

$$\Delta_{x,x} - \alpha_x + \eta(x) + \alpha^{B_1}_x > \Delta_{x,x}$$ (5.35)

where $\alpha^{B_1}_x$ denotes the number of neighbors of $x$ in $B_1$. (5.35) is equivalent to

$$\eta(x) > \alpha^{V \setminus B_1}_x.$$ (5.36)

Those sites that topple after the toppling of sites in $B_1$ thus coincide with the sites that can be burned after burning of $B_1$. Continuing this reasoning, we arrive at the conclusion that $\eta$ does not contain a forbidden subconfiguration if and only if upon addition of $\sum_x (\Delta_{x,x} - \alpha_x)\delta_x$ every site topples at least once. We now show that for any configuration, any site topples also at most once upon addition of $\sum_x (\Delta_{x,x} - \alpha_x)\delta_x$. By the abelian property, it suffices to show this for the maximal configuration. Since the maximal configuration is recurrent, it is sufficient to prove the following equality (see (4.31)):

$$\eta^{\text{max}}(x) + \left( \sum_y (\Delta_{y,y} - \alpha_y)\delta_y \right)(x) - \sum_y \Delta_{y,x} = \eta^{\text{max}}(x),$$ (5.37)

or

$$\sum_y \Delta_{y,x} = \Delta_{x,x} - \alpha_x,$$ (5.38)

which is obvious. Therefore we conclude $\eta \in A$ to be equivalent with the fact that upon addition of $\sum_x (\Delta_{x,x} - \alpha_x)\delta_x$, every site topples precisely one time, and hence the resulting configuration is $\eta$.

**Corollary 5.3** Consider the following subset of $S$:

$$S_\theta = \{ \prod_{x \in \partial V} a^n_x : n_x \in \mathbb{N} \}.$$ (5.39)

Restricted to $A$, $S_\theta$ defines an abelian group.
Proof: By Lemma 5.2, restricted to \( A \), \( S_\theta \) has the neutral element

\[
\prod_{x \in \partial V} a_x^\Delta x - a_x = e. \quad (5.40)
\]

Because in the product (5.40) every operator appears with a power at least one, inverses of the boundary operators are defined by (5.40) and abelianness.

Finally, we can now prove the fact that "allowed" is the same as "recurrent"

**Theorem 5.4** \( A = R \)

**Proof:** By Corollary 5.3, \( S_\theta \) restricted to \( A \) is a group. By Lemma 4.2, \( A \) is \( S_\theta \)-connected to \( R \). Therefore, the theorem follows as an application of Proposition 3.5.

**Remark:** From combination of Proposition 3.5 and Lemma 4.2, we obtain the following generalization of the previous theorem. If \( A \) is any set closed under the action of \( S \), and has the \( S' \)-group property for some \( S' \subset S \), then \( A = R \).

References


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