Solution to Problem 72-10*: Conjectured monotonicity of a matrix

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Thus for

$a < \pi$ there is a unique minimum, $y(x) = 0$;

$a > \pi$ there is no minimum, i.e. $\inf J(y) = -\infty$;

while for

$a = \pi$ the functions $y(x) = b \sin(\pi x/a)$ satisfy $J(y) = 0$ (for all b , $-\infty < b < \infty$) and $0 = \inf J(y)$.

Also solved in essentially the same way by K. PARK (NASA, Langley Research Center), O. RUEHR (Michigan Technological University) and the proposer. However, only the proposer showed that J has no maxima, by observing that if $\Phi_n(x) = \sin(n\pi x/a)$, then $J(\Phi_n) = (a/2)((n\pi/a)^2 - 1) \rightarrow \infty$ as $n \rightarrow \infty$, for all $a > 0$.

II. Solution by R. T. SHIELD (University of Illinois).

The substitution $y = u \sin(\pi x/a)$ (cf. Jacobi's method of multiplicative variation [1]) is permissible in view of the properties of $y(x)$. The integral becomes

$$J = \int_0^a \left[u'^2 + \left(\frac{\pi^2}{a^2} - 1 \right) u^2 \right] \sin^2 \frac{\pi}{a} x \, dx,$$

and the extrema for $a < \pi$, $a = \pi$ and $a > \pi$ can be determined by inspection.

REFERENCE

- [1] R. COURANT AND D. HILBERT, *Methods of Mathematical Physics*, vol. I, Interscience, New York, 1953, p. 458.

*Problem 72-10**, *Conjectured Monotonicity of a Matrix*, by R. A. USMANI (University of Manitoba, Manitoba, Winnipeg, Canada).

Consider $A = (a_{ij})$, a five-band matrix of order N such that

$$a_{ij} = \begin{cases} 5 + \varepsilon_i, & i = j = 1, N, \\ 6 + \varepsilon_i, & i = j = 2, 3, \dots, N-1, \\ -4, & |i - j| = 1, \\ 1, & |i - j| = 2, \\ 0, & \text{otherwise,} \end{cases}$$

where $\varepsilon_i \geq 0$. It is conjectured that the matrix A is monotone, i.e., if $A^{-1} = (a_{ji}^*)$, then $a_{ji}^* \geq 0$ for all i and j .

Remark. It is known that the matrix A_0 obtained from A by setting $\varepsilon_i \equiv 0$ is monotone. For $A_0 = J^2$, where $J = (j_{mn})$ such that

$$j_{mn} = \begin{cases} 2, & m = n, \\ -1, & |m - n| = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and if $J^{-1} = (j_{mn}^*)$, then

$$j_{mn}^* = \begin{cases} \frac{n(N+1-m)}{(N+1)} > 0, & m \geq n, \\ \frac{m(N+1-n)}{(N+1)} > 0, & m \leq n. \end{cases}$$

Thus J and consequently A_0 are monotone matrices. The matrix A arises in computation of the norm of the inverse of a fourth order differential operator.

Solution by O. P. LOSSERS (Technological University, Eindhoven, the Netherlands).

The conjecture is false. Let $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_N = \lambda$, $\lambda > 0$, and let $A^0 = A - \lambda I$; then $A^{-1} = \lambda^{-1}(I + \lambda^{-1}A^0)^{-1} = \lambda^{-1}(I - \lambda^{-1}A^0 + R)$, where $R_{ij} = O(\lambda^{-2})$, for all i and j ($\lambda \rightarrow \infty$). Then $a_{i,i+2}^* = \lambda^{-1}(\delta_{i,i+2} - \lambda^{-1} + O(\lambda^{-2})) = -\lambda^{-2} + O(\lambda^{-3})$, which is less than 0 for sufficiently large λ .

Solution by A. A. JAGERS (Twente University of Technology, Enschede, the Netherlands).

A straightforward calculation of A^{-1} for $N \leq 4$ shows (i) that A is always monotone if $N \leq 2$, (ii) that, for $N = 3$, A is monotone if and only if $\varepsilon_2 \leq 10$ and (iii) that, for $N = 4$, A is monotone if and only if $\varepsilon_2 + \varepsilon_3 \leq 5$. For larger N the conditions become more complicated. Anyhow we shall see later that, if $N \geq 3$, then $a_{13}^* < 0$, as soon as $\varepsilon_i = 0$ for all $i \neq 2$ and ε_2 is large enough. So, if $N \geq 3$, the conjecture does not even hold if we restrict ourselves to the case where only one of the ε_i is positive. It is worthwhile to consider this phenomenon in a more general setting:

Let $B = (b_{ij})$ be a matrix of order N with $\det(B) \neq 0$. Choose a fixed k , $1 \leq k \leq N$. Let $B(\varepsilon_k) = (b_{ij}(\varepsilon_k))$ be the matrix given by $b_{kk}(\varepsilon_k) = b_{kk} + \varepsilon_k$ and $b_{ij}(\varepsilon_k) = b_{ij}$ if $i \neq k$ or $j \neq k$. Denote the ij th entry of the inverse matrix of $B(\varepsilon_k)$ by $b_{ij}^*(\varepsilon_k)$ and $b_{ij}^*(0)$ by b_{ij}^* . Suppose that $b_{kk}^* > 0$. Then we have the following two propositions. An unconventional but easy way of proving the second part of the first proposition is by successively solving the N^2 differential equations

$$\frac{db_{ij}^*(\varepsilon_k)}{d\varepsilon_k} = -b_{ik}^*(\varepsilon_k)b_{kj}^*(\varepsilon_k).$$

The second proposition is a consequence of the first.

PROPOSITION 1.

$$(i) \quad \det(B(\varepsilon_k)) = (1 + b_{kk}^* \cdot \varepsilon_k) \det(B) \neq 0.$$

$$(ii) \quad b_{ij}^*(\varepsilon_k) = \frac{b_{ij}^* + \Delta_{ij}^k \cdot \varepsilon_k}{1 + b_{kk}^* \cdot \varepsilon_k}$$

$$\text{with } \Delta_{ij}^k = b_{kk}^* b_{ij}^* - b_{ik}^* b_{kj}^*.$$

PROPOSITION 2. If B is monotone, then $B(\varepsilon_k)$ is monotone for all $\varepsilon_k \geq 0$ if and only if $\Delta_{ij}^k \geq 0$ for all i and j .

Note that $\Delta_{ik}^k = \Delta_{kj}^k = 0$.

Now a direct calculation using the given j_{mn}^* shows that in the original problem the quantity corresponding to Δ_{13}^2 is equal to

$$-\frac{1}{(N+1)^2} \sum_{k=3}^N (N+1-k)^2$$

which is less than 0. This proves the announced statement concerning a_{13}^* .

On the other hand, it follows at once from the corollary to the proposition below, that at least all diagonal entries a_{ii}^* are positive if $\varepsilon_i \geq 0$ for all i .

PROPOSITION 3. *If B is symmetric and positive definite, then $b_{ii}^*(\varepsilon_k) > 0$ for all i and $\varepsilon_k \geq 0$.*

Proof. Since B is symmetric and positive definite, a symmetric matrix M exists such that $B = M^2$. Then the condition $\Delta_{ii}^k \geq 0$ is equivalent to Schwarz's inequality applied to the i th and k th column vector of M^{-1} .

Now $B(\varepsilon_k)$ is again symmetric and positive definite. (Note that $\det(B(\varepsilon_k)) \neq 0$ for all $\varepsilon_k \geq 0$.) So we may apply Proposition 3 another time, this time changing another diagonal entry, etc. We finally obtain the following corollary.

COROLLARY. *If B is symmetric and positive definite, then the diagonal entries of the inverse C^{-1} of the matrix C obtained from B by adding $\varepsilon_i (\geq 0)$ to the i th diagonal entry ($i = 1, 2, \dots, N$) are all positive.*

Solution by A. N. WILLSON, JR. (University of California, Los Angeles).

The stated conjecture is false. This is easily verified by considering the case in which $N = 3$, with $\varepsilon_i = 11$, $i = 1, 2, 3$. In this case the elements $(A^{-1})_{13} = (A^{-1})_{31} < 0$. Also, it is clear that the conjecture cannot be true for *any* matrix of the type specified having order $N > 2$, as it would contradict the following theorem (which is proved in [1]).

THEOREM. *The matrix A is such that $(A + D)^{-1} \geq 0$ for every diagonal matrix D having positive elements on the main diagonal only if the off-diagonal elements of A are all nonpositive.*

REFERENCE

- [1] A. N. WILLSON, JR., *Pairs of matrices and matrices of monotone kind*, SIAM J. Numer. Anal., 10 (1973).

Also solved by L. ELSNER (Institut für Angewandte Mathematik I, Erlangen-Nürnberg), I. FARKAS (University of Toronto, Canada), M. J. MARSDEN (University of Pittsburgh), C. C. ROUSSEAU (Memphis State University), O. G. RUEHR (Michigan Technological University), J. N. SHOOSMITH (NASA, Langley Research Center), R. A. SWEET (National Center for Atmospheric Research), J. J. VANDERGRAFT (University of Maryland), L. RUBENFELD (Rensselaer Polytechnic Institute) submitted a solution co-authored by six NSF undergraduate research participants: M. FRAME, D. HALE, C. KELLER, D. RUTLEDGE, S. SENTOFF and D. WAGNER.

By [1], it is sufficient to construct a $2^{m+1} \times 2^{m+1}(-1, 1)$ matrix W for which $\det W = 2^{(m+1)2^m}$. Let A denote the 2×2 matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and define $W = \otimes^{m+1} A$, the $(m+1)$ th Kronecker power of A . W satisfies $\det W = (\det A)^{(m+1)2^m} = 2^{(m+1)2^m}$ which provides a desired matrix.

Problem 72-11, Limit of an Integral, by N. MULLINEUX and J. R. REED (University of Aston, Birmingham, England).

Determine

$$I = \lim_{\varepsilon \rightarrow 0} \int_0^\varepsilon \log \left\{ \frac{|\sin t - \varepsilon/2|}{\sin \varepsilon/2} \right\} \frac{dt}{\sin t}.$$

The problem arose in the numerical inversion of singular integral equations such as those which occur in aerodynamics and problems associated with plane transmission lines.

Editorial note. Most solvers (see later discussion) noted that the integral as given above does not exist since for fixed $\varepsilon \neq 0$, the integrand behaves like $t^{-1} \log \{(\varepsilon/2) \sin \varepsilon/2\}$ as $t \rightarrow 0$. The error is a typographical one since in the original proposal the term $|\sin t - t/2|$ appears as $|\sin(t - \varepsilon/2)|$. In the following solution of O. P. LOSSERS (Technological University of Eindhoven, Eindhoven, the Netherlands) the latter expression is used.

Replacing ε by 2ε we get

$$\begin{aligned} \int_0^{2\varepsilon} \log \left\{ \frac{|\sin(t - \varepsilon)|}{\sin \varepsilon} \right\} \frac{dt}{\sin t} &= \int_0^{2\varepsilon} \log |\cos t - \cot \varepsilon \cdot \sin t| \frac{dt}{\sin t} \\ &= \int_0^{2\varepsilon} \log \cos t \frac{dt}{\sin t} + \int_0^{2\varepsilon} \log |1 - \cot \varepsilon \cdot \tan t| \frac{dt}{\sin t}. \end{aligned}$$

Since $(\log \cos t)/\sin t = -t/2 + O(t^3)$ ($t \rightarrow 0$), we have that the first term is $O(\varepsilon^2)$, ($\varepsilon \rightarrow 0$). By substitution of $v = \cot \varepsilon \cdot \tan t$ we get

$$\begin{aligned} I &= \lim_{\varepsilon \rightarrow 0} \left\{ O(\varepsilon^2) + \int_0^{\cot \varepsilon \cdot \tan 2\varepsilon} \frac{\log |1 - v|}{v} (1 + v^2 \tan^2 \varepsilon)^{-1/2} dv \right\} \\ &= \int_0^2 \frac{\log |1 - v|}{v} dv = -\frac{\pi^2}{4}. \end{aligned}$$

In the solutions by N. M. BLACHMAN (Sylvania Electronics Systems), L. KHATCHATOORIAN (California State College, Los Angeles), S. L. PAVERIFONTANA (University of California, Berkeley), Z. C. MOTTELER and O. G. RUEHR (Michigan Technological University), J. W. RIESE (Kimberly-Clark Corporation), J. D. TALMAN (University of Western Ontario, London, Canada) and P. TH. L. M. VAN WOERKOM (National Aerospace Laboratories, Amsterdam, the Netherlands), it was remarked that the integral as originally stated does not exist. Various alternative statements of the problem were made. We do not list them all. Most solvers chose to replace $\sin \varepsilon/2$ by $\varepsilon/2$ (one could as well replace $\varepsilon/2$ by $\sin \varepsilon/2$) in