Performance analysis of tandem queues with small buffers

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PERFORMANCE ANALYSIS OF TANDEM QUEUES WITH SMALL BUFFERS

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Abstract: In this paper we present an approximation for the performance analysis of single-server tandem queues with small buffers and generally distributed service times. The approximation is based on decomposition of the tandem queue in subsystems, the parameters of which are determined by an iterative algorithm. By employing a detailed description of the service process of each subsystem, we obtain an accurate approximation of performance characteristics such as throughput and mean sojourn time, which substantially outperforms former methods.

Keywords: tandem queues, approximation, decomposition, small finite buffers, blocking, production lines.

1 Introduction

Queueing networks with finite buffers have been studied extensively in the literature; see, e.g., [2, 7, 9]. These models have many applications in manufacturing, communication and computer systems. In most cases queueing networks with finite buffers are intractable; therefore the majority of the literature is devoted to approximations of performance characteristics such as throughput and mean sojourn times. To the best of our knowledge, the average errors in throughput and mean sojourn time approximations reported in the literature are usually around 5%. Typically, the errors are large (up to 30%) for systems with small buffers; see e.g. [12]. But, in manufacturing systems, it is common to have small buffers. Hence, good approximations for such systems are definitely needed. In this paper we propose a method for the approximative analysis of single-server tandem queues with general service times, *small* finite buffers and Blocking After Service (BAS). We are interested in the queue-length distribution of each buffer; these distributions may be used to determine the throughput and mean sojourn time.

In this paper we consider a tandem queue (L) with N servers in tandem and $N - 1$ buffers B_i , $i = 1, \dots, N - 1$, of size b_i in between. The servers are labeled M_i , $i = 0, \dots, N - 1$. The random variable S_i denotes the generic service time of server M_i ; S_i is generally distributed with rate $\mu_{p,i}$ and coefficient of variation $c_{p,i}$. Each server can serve one customer at a time. Server M_0 is never starved and we consider the BAS blocking protocol. Figure 1 shows a tandem queue with four servers in tandem.

Figure 1: A tandem queue with four servers.

The method to approximate the queue-length distribution of the buffers is based on decomposition of the tandem queue in single-buffer subsystems. Each buffer B_i is considered in isolation, where, to take into account the relation with the upstream line and downstream line, the service times of server M_{i-1} in front of buffer B_i and server M_i after buffer B_i are adapted by including possible starvation of M_{i-1} before service and possible blocking of M_i after service. It is important to note that dependencies between service time and blocking after service are carefully taken into account; this seems to be the most important source of performance improvement of the proposed approximation. By fitting the first two

moments, the distributions of the service times including starvation and blocking are approximated by a simple phase-type distributions, the parameters of which are tuned by means of an iterative algorithm.

Decomposition techniques for single-server queueing networks have also been used by a.o. Perros [8], and Kerbache and MacGregor Smith [4]. Van Vuuren et al. [12] extend the decomposition technique to multi-server tandem queues with finite buffers, and show that the quality of the approximation substantially degrades when the buffers become small, i.e., errors can get as large as 30% in the throughput and mean sojourn time. The results of the current method will be compared with the ones of [12] and also with the ones of the approximation developed by Helber [3], which is the most recent and best method available; the comparison shows that the current method performs substantially better than former methods, with average errors around 1%.

This paper is organized as follows. In Section 2 we explain the decomposition of the tandem queue in subsystems. In the section thereafter we take a closer look at the subsystems. Section 4 describes the iterative algorithm. Numerical results are presented in Section 5 and they are compared with simulation and other approximation methods. Finally, Section 6 contains some concluding remarks.

2 Decomposition of the tandem queue

We decompose the original tandem queue L into $N - 1$ subsystems $L_1, L_2, \dots, L_{N-1}$. Subsystem L_i consists of buffer B_i of size b_i , an arrival-server in front of the buffer, and a departure-server after the buffer. In Figure 2 we show the decomposition of line L of Figure 1.

Figure 2: Decomposition of the tandem queue of Figure 1.

The arrival-server of subsystem L_i is, of course, server M_{i-1} , but, to take into account the connection with the upstream line, its service time is different from S_{i-1} . The random variable A_i denotes the service time of the arrival-server in subsystem L_i , $i = 1, \dots, M - 1$. This random variable represents the original service time S_{i-1} of server M_{i-1} *including possible starvation of this server before the service start*. Similarly, the random variable D_i denotes the service time of the departure-server in subsystem L_i ; it represents the service time S_i of server M_i *including possible blocking of this server after service completion*.

In the next section we elaborate further on the arrivals at and the departures from the subsystems.

3 The subsystems

In this section we describe how the service times of the arrival and departure server in subsystem L_i are modeled. Also, we describe the method to fit the first two moments, the computation of the first two moments of the maximum of two independent phase-type random variables, and the detailed analysis of each subsystem.

3.1 The arrivals at and the departures from the subsystems

In the description of the service time A_i of arrival-server M_{i-1} of subsystem L_i we try to use all information available. Note that an arrival in buffer B_i , i.e., a customer being served by M_{i-1} moves to buffer B_i when space becomes available, corresponds to a departure from the upstream subsystem L_{i-1} . Just after this departure, two situations may occur: subsystem L_{i-1} is empty with probability q_{i-1}^e , or it is not empty with probability $1 - q_{i-1}^e$ (where, of course, we do not count the customer at arrival-server M_{i-2} as being in L_{i-1}). In the former situation, server M_{i-1} has to wait for a residual service time of arrival-server M_{i-2} of subsystem L_{i-1} , denoted as RA_{i-1} , before the actual service can start. In the latter situation, the actual service S_{i-1} can start immediately. Hence, the service time A_i is equal to $RA_{i-1} + S_{i-1}$ with probability q_{i-1}^e and equal to S_{i-1} with probability $1 - q_{i-1}^e$. Figure 3 shows a schematic representation of the service time A_i .

Figure 3: The service time A_i of the arrival-server of subsystem L_i .

We now describe the service time D_i of the departure-server M_i of subsystem L_i . Here we distinguish between the service time of a customer for which the previous customer left behind an empty subsystem L_i or not; the reason for doing so is that, when time elapses between the departure of the previous customer and the service start of the current customer, it will be less likely that the current one will be blocked. Let the random variable D_i^e denote the service time of departure-server M_i if just after the previous service completion subsystem L_i is empty; D_i^{ne} is the service time if L_i is not empty.

If subsystem L_i is empty just after a departure, the next service time starts when a new customer enters subsystem L_i . Upon arrival of this customer, we have two possibilities: buffer B_{i+1} of the downstream subsystem L_{i+1} is full with probability p_{i+1}^f , or not. However, this event does not correspond to an arrival or departure in L_{i+1} . So p_{i+1}^f cannot be calculated as an arrival or departure probability in L_{i+1} , but instead, it will be approximated by the equilibrium probability that buffer B_{i+1} is full, given that arrival-server M_i in subsystem L_{i+1} is not blocked. If buffer B_{i+1} is full, the service time D_i^e is the maximum of S_i and the residual service time of departure-server M_{i+1} in subsystem L_{i+1} ; the residual service time of M_{i+1} is approximated by the equilibrium residual service time, denoted as ED_{i+1} . If B_{i+1} is not full, the current customer can immediately move to B_{i+1} after completion of S_i , so then D_i^e is equal to S_i . Figure 4 shows a schematic representation of the service time D_i^e .

If subsystem L_i is not empty just after a departure, the next service starts immediately. Note that a departure from L_i is an arrival at L_{i+1} , and *on arrival* three situations may occur. First, it is possible that the arrival is triggered by a service completion of departure-server M_{i+1} of L_{i+1} , i.e., server M_i was blocked and waiting for M_{i+1} to complete service. In this situation we say that the status of downstream

Figure 4: The service time of the departure-server of subsystem L_i when the previous customer leaves behind an empty subsystem.

subsystem L_{i+1} changes from blocked to full; the probability that this occurs is denoted by r_{i+1}^{bf} . Then the next service of M_i and M_{i+1} start simultaneously, so D_i^{ne} is equal to the maximum of S_i and D_{i+1} . Second, just before the arrival there is only one position left in B_{i+1} , so right after the arrival B_{i+1} is full; the probability that this happens is denoted by r_{i+1}^{ef} . Then, on arrival, departure-server M_{i+1} is already busy with a residual service time RD_{i+1} , and thus D_i^{ne} is equal to the maximum of S_i and RD_{i+1} . Finally, when neither of these two situations occurs, server M_i can start servicing and after service completion, the customer can immediately move to buffer B_{i+1} . Hence, in this case, D_i^{ne} is equal to S_i . Figure 5 shows a schematic representation of the service time D_i^{ne} .

Figure 5: The service time of the departure-server of subsystem L_i when the previous customer does not leave behind an empty subsystem.

Remark: There is one exception to the representation of D_i^{ne} , namely when $b_{i+1} = 0$. In the second situation, just before the arrival, server M_{i+1} is available and thus, right after the arrival, both servers M_i and M_{i+1} start the next service simultaneously. Hence, D_i^{ne} is equal to the maximum of S_i and D_{i+1} (instead of RD_{i+1}).

Summarizing, the service time D_i of the current customer is equal to D_i^e if the previously served customer left behind an empty subsystem and otherwise, it is equal to D_i^{ne} .

We will approximate the distribution of A_i , D_i^e and D_i^{ne} by fitting simple phase-type distributions to the first two moments. In particular, we develop an efficient method to compute the first two moments of the maximum of two phase-type random variables; this will be explained in the following sections.

3.2 Two moment fit

We will model the distribution of a random variable with rate λ and coefficient of variation c as a mixed Erlang distribution if $c^2 \leq 1$, and otherwise, as a Hyperexponential distribution (see, e.g., [11]).

More specifically, if $1/k \leq c^2 \leq 1/(k-1)$ for some $k = 2, 3, \dots$, then the rate and squared coefficient

of variation of the Erlang $_{k-1,k}$ distribution with density

$$f(t) = p\mu^{k-1} \frac{t^{k-2}}{(k-2)!} e^{-\mu t} + (1-p)\mu^k \frac{t^{k-1}}{(k-1)!} e^{-\mu t}, \quad t \geq 0,$$

matches with λ and c_a^2 , provided the parameters p and μ are chosen as

$$p = \frac{1}{1+c^2} [kc^2 - \{k(1+c^2) - k^2c^2\}^{1/2}], \quad \mu = (k-p)\lambda. \quad (1)$$

If $c^2 > 1$, the rate and squared coefficient of variation of the Hyper-exponential distribution with density

$$f(t) = p\mu_1 e^{-\mu_1 t} + (1-p)\mu_2 e^{-\mu_2 t}, \quad t \geq 0$$

matches with λ and c^2 , provided the parameters p , μ_1 and μ_2 are chosen as

$$p = \frac{1}{2} \left(1 + \sqrt{\frac{c^2 - 1}{c^2 + 1}} \right), \quad \mu_1 = 2p\lambda, \quad \mu_2 = 2(1-p)\lambda. \quad (2)$$

There exist also other parameter choices for fitting these distributions, and other distributions for fitting. In our experience, use of other distributions or parameters does not essentially affect the quality of the approximation.

Both Erlang and Hyper-exponential distribution belong to the family of Coxian distributions. A random variable is said to have a Coxian $_k$ distribution if it has to go through at most k exponential phases, where phase i has rate ν_i , $i = 1, \dots, k$. It starts in phase 1 and after phase i , $i = 1, \dots, k-1$, it enters phase $i+1$ with probability p_i , whereas it ends with probability $1-p_i$; phase k is the last phase, so $p_k = 0$. Clearly, the Erlang $_{k-1,k}$ distribution is a Coxian $_k$ distribution with $\nu_i = \mu$ for all i and $p_i = 1$ for $i = 1, \dots, k-2$ and $p_{k-1} = 1-p$; the Hyper-exponential distribution is a Coxian $_2$ distribution with

$$\nu_1 = \mu_1, \quad \nu_2 = \mu_2, \quad p_1 = (1-p) \frac{\mu_1 - \mu_2}{\mu_1},$$

where, without loss of generality, $\mu_1 \geq \mu_2$. The representation of Erlang and Hyper-exponential distributions in terms of Coxians will be convenient for the description of subsystem L_i as a finite-state Markov process in Section 3.4.

3.3 The maximum of two phase-type random variables

In this section we calculate the first two moments of the maximum of two independent Erlang distributed random variables. Let E_i denote an Erlang $_{k_i}$ distributed random variable with scale parameter μ_i , $i = 1, 2$, and assume that E_1 and E_2 are independent. The maximum of E_1 and E_2 is phase-type distributed, the first (random) number of exponential phase have rate $\mu_1 + \mu_2$. These phases are followed by a (random) number of exponential phases with rate μ_1 or rate μ_2 , depending on which of the random variables E_1 and E_2 finishes first. Let $q_{1,j}$ with $0 \leq j \leq k_2 - 1$ be the probability that E_2 has completed j phases when E_1 completes its final phase, and similarly, let $q_{2,i}$ with $0 \leq i \leq k_1 - 1$ be the probability that E_1 has completed i phases when E_2 completes its final phase. It is easily verified that $q_{1,j}$ and $q_{2,i}$ both follow a Negative Binomial distribution, i.e.,

$$\begin{aligned} q_{1,j} &= \binom{k_1 - 1 + j}{k_1 - 1} \left(\frac{\mu_2}{\mu_1 + \mu_2} \right)^j \left(\frac{\mu_1}{\mu_1 + \mu_2} \right)^{k_1}, \quad 0 \leq j \leq k_2 - 1, \\ q_{2,i} &= \binom{k_2 - 1 + i}{k_2 - 1} \left(\frac{\mu_1}{\mu_1 + \mu_2} \right)^i \left(\frac{\mu_2}{\mu_1 + \mu_2} \right)^{k_2}, \quad 0 \leq i \leq k_1 - 1. \end{aligned}$$

Conditioned on the event that E_1 finishes first and E_2 has then completed j phases, the maximum of E_1 and E_2 is Erlang distributed with $k_1 + k_2$ phases, the first $k_1 + j$ of which have rate $\mu_1 + \mu_2$ and the last $k_2 - j$ have rate μ_2 . Let $M_{1,j}$ denote this conditional maximum, then

$$\begin{aligned}\mathbb{E}M_{1,j} &= \frac{k_1 + j}{\mu_1 + \mu_2} + \frac{k_2 - j}{\mu_2}, \\ \mathbb{E}M_{1,j}^2 &= \frac{(k_1 + j)(k_1 + j + 1)}{(\mu_1 + \mu_2)^2} + \frac{(k_1 + j)(k_2 - j)}{(\mu_1 + \mu_2)\mu_2} + \frac{(k_2 - j)(k_2 - j + 1)}{\mu_2^2}.\end{aligned}$$

Similarly, let $M_{2,i}$ denote the maximum of E_1 and E_2 , conditioned on the event that E_2 finishes first and E_1 has then completed i phases. For the first two moments of $M_{2,i}$ we have

$$\begin{aligned}\mathbb{E}M_{2,i} &= \frac{k_2 + i}{\mu_1 + \mu_2} + \frac{k_1 - i}{\mu_1}, \\ \mathbb{E}M_{2,i}^2 &= \frac{(k_1 + j)(k_1 + j + 1)}{(\mu_1 + \mu_2)^2} + \frac{(k_1 + j)(k_2 - j)}{(\mu_1 + \mu_2)\mu_2} + \frac{(k_2 - j)(k_2 - j + 1)}{\mu_2^2}.\end{aligned}$$

The first two moments of the maximum of E_1 and E_2 can now easily be computed by conditioning on the above events, yielding

$$\begin{aligned}\mathbb{E}(\max\{E_1, E_2\}) &= \sum_{j=0}^{k_2-1} q_{1,j} \mathbb{E}M_{1,j} + \sum_{i=0}^{k_1-1} q_{2,i} \mathbb{E}M_{2,i}, \\ \mathbb{E}(\max\{E_1, E_2\}^2) &= \sum_{j=0}^{k_2-1} q_{1,j} \mathbb{E}M_{1,j}^2 + \sum_{i=0}^{k_1-1} q_{2,i} \mathbb{E}M_{2,i}^2.\end{aligned}$$

Note that, if E_1 and E_2 are both probabilistic mixtures of Erlang random variables, then the first two moments of the maximum of E_1 and E_2 can be computed from the above equations by conditioning on the composition of the maximum.

3.4 Analyzing a subsystem

In this section we describe the steady-state analysis of subsystem L_i ; in the remainder of this section we drop the subscript i in the notations. By fitting phase-type distributions on the service times A , D^e , D^{ne} , the subsystem can be described by a finite state Quasi-Birth-and-Death process (QBD), with states (i, j, k) . The state variable i denotes the total number of customers in the subsystem. Clearly, $0 \leq i \leq b + 2$ where $i = b + 2$ indicates that the arrival-server is blocked. The state variable j (k) indicates the phase of the service time of the arrival(departure)-server. The possible values of j are the phases of A ; the possible values of k are the phases of D^e plus the phases of D^{ne} .

To define the generator of the QBD we use the Kronecker product: If A is an $n_1 \times n_2$ matrix and B is an $n_3 \times n_4$ matrix, the Kronecker product $A \otimes B$ is defined by

$$A \otimes B = \begin{pmatrix} A(1,1)B & \cdots & A(1,n_2)B \\ \vdots & & \vdots \\ A(n_1,1)B & \cdots & A(n_1,n_2)B \end{pmatrix}.$$

To specify the generator $\mathbf{Q}$ of the QBD, we order the states lexicographically and partition the state space into levels, where level $= 0, 1, \dots, b + 2$ is the set of all states with i customers in the system. Then $\mathbf{Q}$

has the form:

$$\mathbf{Q} = \begin{pmatrix} B_{00} & B_{01} & & & & \\ B_{10} & A_1 & A_0 & & & \\ & A_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & A_0 & \\ & & & A_2 & A_1 & C_{10} \\ & & & & C_{01} & C_{00} \end{pmatrix}$$

Below we specify the submatrices in $\mathbf{Q}$. To describe the service process of the arrival and departure server we use the concept of a Markovian Arrival Process (MAP); see [1]. In general, a MAP is defined in terms of a continuous-time Markov process with finite state space $\{0, \dots, m-1\}$ and generator $G_0 + G_1$. The element $G_1(i, j)$ denotes the intensity of transitions from i to j accompanied by an arrival, whereas for $i \neq j$ element $G_0(i, j)$ denotes the intensity of the remaining transitions from i to j and the diagonal elements $G_0(i, i)$ are negative and chosen such that the row sums of $G_0 + G_1$ are zero.

The MAP for the service process of the departure-server can be described as follows. Assume that the distribution of D^{ne} can be represented by a Coxian distribution with n_{ne} phases, numbered $1, \dots, n_{ne}$; the starting phase is 1, the rate of phase i is μ_i and q_i is the probability to proceed to the next phase $i+1$, and $1 - q_i$ is the probability that D^{ne} is finished. Of course, since n_{ne} is the last phase, $q_{n_{ne}} = 0$. Similarly, the distribution of D^e can be represented by a Coxian distribution with n_e phases; the starting phase is 1, the rates are ν_i and the transition probabilities p_i . Then the states of the MAP are numbered $1, \dots, n_{ne} + n_e$; the first n_{ne} states correspond to the phases of D^{ne} and the last n_e states to D^e . The generator can be expressed as $DE_0 + DE_1$, where the transitions in DE_1 correspond to departures. So, in $\mathbf{Q}$, the transitions in DE_1 lead to a transition from level i to $i-1$, whereas the ones in DE_0 correspond to transitions within level i . The non-zero elements of DE_0 and DE_1 are specified below.

$$\begin{aligned} DE_0(i, i) &= -\mu_i, & i &= 1, \dots, n_{ne}, \\ DE_0(i, i+1) &= q_i \mu_i, & i &= 1, \dots, n_{ne} - 1, \\ DE_1(i, 1) &= (1 - q_i) \mu_i, & i &= 1, \dots, n_{ne}, \\ DE_0(i, i) &= -\nu_i, & i &= n_{ne} + 1, \dots, n_{ne} + n_e, \\ DE_0(i, i+1) &= p_i \nu_i, & i &= n_{ne} + 1, \dots, n_{ne} + n_e - 1, \\ DE_1(i, 1) &= (1 - p_i) \nu_i, & i &= n_{ne} + 1, \dots, n_{ne} + n_e. \end{aligned}$$

The MAP for the service process of the arrival-server can be described similarly. Its generator can be expressed as $AR_0 + AR_1$, where the transition rates in AR_1 are the ones corresponding to a service completion, i.e., an arrival in the buffer. The size of the square matrices RA_0 and RA_1 is n_a (which is equal to the number of phases of A). Hence, in $\mathbf{Q}$, the transitions in AR_1 lead to a transition from level i to $i+1$, whereas the ones in AR_0 correspond to transitions within level i .

Now we can describe the submatrices in $\mathbf{Q}$. The transition rates from levels $1 \leq i \leq b$ are given by

$$\begin{aligned} A_0 &= AR_1 \otimes I_{n_{ne}+n_e}, \\ A_1 &= AR_0 \otimes I_{n_{ne}+n_e} + I_{n_a} \otimes DE_0, \\ A_2 &= I_{n_a} \otimes DE_1, \end{aligned}$$

where I_n is the identity matrix of size n .

In level 0 and level $b+2$ the transition rates are different, because at level 0 the departure-server is idle and at level $b+2$ the arrival-server is blocked. Also, jumps from level 0 to 1 should be directed to the starting phase of D^e (instead of the starting phase of D^{ne}). Hence,

$$\begin{aligned}
B_{01} &= AR_1 \otimes I_{n_{ne}+n_{se}}(n_{ne}+1, :), \\
B_{00} &= AR_0, \\
B_{10} &= I_{n_a} \otimes DE_1(:, 1), \\
C_{10} &= AR_1(:, 1) \otimes I_{n_{ne}+n_{se}}, \\
C_{00} &= DE_0, \\
C_{01} &= I_{n_a}(1, :) \otimes DE_1,
\end{aligned}$$

where $X(:, y)$ is the y -th column of matrix X and $X(z, :)$ is the z -th row of X .

This completes the description of the QBD. The steady-state distribution can be determined by the matrix-geometric method. More specifically, we use the efficient techniques developed by Latouche and Ramaswami [5], and Naoumov et al. [10]. If we denote the equilibrium probability vector of level i by π_i , then π_i has a matrix-geometric form

$$\pi_i = x_1 R^{i-1} + x_{b+1} \hat{R}^{b+1-i}, \quad i = 1, \dots, b+1, \quad (3)$$

where R is the minimal nonnegative solution of the matrix-quadratic equation

$$A_0 + RA_1 + R^2 A_2 = 0,$$

and $\hat{R}$ is the minimal nonnegative solution of

$$A_2 + \hat{R}A_1 + \hat{R}^2 A_0 = 0.$$

The matrices R and $\hat{R}$ can be determined by using an iterative algorithm developed by Naoumov et al. [10]. The algorithm for R is listed in Figure 6.

```

N := A1
L := A0
M := A2
W := A1
dif := 1

while dif > ε
{
    X := -N-1L
    Y := -N-1M
    Z := LY
    dif := ||Z||
    W := W + Z
    N := N + Z + MX
    Z := LX
    L := Z
    Z := MY
    M := Z
}
R := -A0W-1

```

Figure 6: Algorithm of Naoumov et al. [10] to determine the rate matrix R , where $\|\cdot\|$ denotes a matrix-norm and ϵ some small positive number.

The final step is to determine x_1 and x_{b+1} . The balance equations at the boundary levels 0, 1, $b+1$ and $b+2$ are given by

$$\begin{aligned} 0 &= \pi_0 B_{00} + \pi_1 B_{10}, \\ 0 &= \pi_0 B_{01} + \pi_1 A_1 + \pi_2 A_2, \\ 0 &= \pi_b A_0 + \pi_{b+1} A_1 + \pi_{b+2} C_{01}, \\ 0 &= \pi_b C_{10} + \pi_{b+1} C_{00}. \end{aligned}$$

Eliminating π_0 and π_{b+2} from the equations above, and then substituting (3) for π_1 and π_{b+1} yields

$$\begin{aligned} 0 &= x_1(A_1 + RA_2 - B_{10}B_{00}^{-1}B_{01}) + x_{b+1}(\hat{R}^b A_1 + \hat{R}^{b-1} A_2 - \hat{R}^b B_{10}B_{00}^{-1}B_{01}), \\ 0 &= x_1(R^{b-1}A_0 + R^b A_1 - R^b C_{10}C_{00}^{-1}C_{01}) + x_{b+1}(\hat{R}A_0 + A_1 - C_{10})C_{00}^{-1}C_{01}. \end{aligned}$$

These equations have, together with the normalization equation, a unique solution x_1 and x_{b+1} . Once the queue-length probabilities are known, we can easily derive performance measures and quantities required to describe the service times of the arrival- and departures-server. This is described below.

The Throughput T follows from

$$T = \pi_1 B_{10}e + \sum_{i=2}^{b+1} \pi_i A_2 e + \pi_{b+2} C_{01} e, \quad (4)$$

or equivalently,

$$T = \pi_0 B_{01}e + \sum_{i=1}^b \pi_i A_0 e + \pi_{b+1} C_{10} e,$$

where e is the all-one vector.

The probability q^e that the subsystem is empty just after a departure, is equal to the mean number of departures per time unit leaving behind an empty system divided by the mean total number of departures per time unit; so

$$q^e = \pi_1 B_{10}e / T.$$

The moments of RA can be computed straight-forwardly, once the phase of the service time of the arrival-server, just after a departure leaving behind an empty system, is known. To determine the distribution of the service phase, note that component (j, k) of the vector $\pi_1 B_{10}$ gives the mean number of departures per time unit from state (j, k) at level 1. By adding all components with $j = i$ and dividing by $\pi_1 B_{10}e$, we obtain the probability that the arrival-server is in phase i just after a departure leaving behind an empty system.

The probability p^f that the buffer is full, given that the arrival-server is not blocked, satisfies

$$p^f = \frac{\pi_{b+1}e}{\sum_{i=0}^{b+1} \pi_i e}.$$

The moments of ED can be calculated once the phase of the service time of the departure-server is known. To obtain the distribution of the service phase, note that component (j, k) of vector π_{b+1} is the

probability that the arrival-server is in phase j , the departure-server is in phase k and the buffer is full. Hence, addition of all components of π_{b+1} with $k = i$ and division by $\pi_{b+1}e$ yields the probability that the departure-server is in phase i , given that the buffer is full and arrival-server is not blocked.

The probabilities r^{bf} and r^{ef} follow from

$$r^{bf} = \pi_{b+2}C_{01}e/T, \quad r^{ef} = \pi_b A_0 e/T.$$

The moments of RD , finally, require the phase of the service time of the departure-server just after an arrival saturating the buffer. To find the distribution of the service phase, note that component (j, k) of $\pi_b A_0$ is equal to the mean number of arrivals per time unit from state (j, k) at level b . So, by adding over all components with $k = i$ and dividing by $\pi_b A_0 e$, yields the probability that the departure-server is in phase i , just after an arrival saturating the buffer.

4 The iterative algorithm

Now we describe the iterative algorithm to approximate the performance of tandem queue L . The algorithm is based on decomposition of L in $N - 1$ subsystems $L_1, L_2, \dots, L_{N-1}$. Before going into detail in Section 4.2, we present the outline of the algorithm in Section 4.1.

4.1 Outline of the algorithm

- Step 0: Choose initial characteristics of the service times of the departure-servers for all subsystems $L_1, \dots, L_{N-1}$.
- Step 1: For subsystem $L_i = L_1, \dots, L_{N-1}$:
 1. Determine the first two moments of the service time A_i of the arrival-server, given the queue-length distribution and throughput of subsystem L_{i-1} .
 2. Determine the queue-length distribution of subsystem L_i .
 3. Determine the throughput T_i of subsystem L_i .
- Step 2: Determine the new characteristics of the service times of the departure-servers for all subsystems $L_{N-1}, \dots, L_1$.
- Repeat Step 1 and 2 until the characteristics of the service times of the departure-servers have converged.

4.2 Details of the algorithm

Step 0: Initialization

The first step of the algorithm is to initially assume that there is no blocking. This means that the random variables D_i^e and D_i^{ne} are initially the same as the service times S_i .

Step 1: Evaluation of subsystems

We know the first two moments of D_i^e and D_i^{ne} , but we also need to know the first two moments of A_i , before we are able to determine the queue-length distribution of L_i .

(a) The arrival process

For the first subsystem L_1 , the service time A_1 is equal to S_0 , because server M_0 cannot be starved.

For the other subsystems we proceed as follows in order to determine the first two moments of A_i . By Little's law we have for the throughput T_i of subsystem L_i ,

$$T_i = (1 - p_{i,b_i+2})\mu_{a,i},$$

where p_{i,b_i+2} denotes the fraction of time the arrival-server of subsystem L_i is blocked and $\mu_{a,i}$ is the mean arrival rate at subsystem L_i , i.e., $\mu_{a,i} = 1/\mathbb{E}(A_i)$. By substituting the estimate $T_{i-1}^{(k)}$ for T_i , which is the principle of conservation of flow, and $p_{i,b_i+2}^{(k-1)}$ for p_{i,b_i+2} we get as new estimate for $\mu_{a,i}$,

$$\mu_{a,i}^{(k)} = \frac{T_{i-1}^{(k)}}{1 - p_{i,b_i+2}^{(k-1)}},$$

where the superscripts indicate in which iteration the quantities have been calculated.

The second moment of A_i cannot be determined by using Little's law; instead we calculate q_{i-1}^e and the first two moments of RA_{i-1} from the queue length distribution of L_{i-1} and then determine the second moment of A_i according to the representation in Figure 3.

(b) Analysis of subsystem L_i

Based on the (new) estimates for the first two moments of A_i , D_i^e and D_i^{ne} we can determine the steady-state queue length distribution of subsystem L_i . To do so we first fit mixed Erlang or Hyper-exponential distributions on the first two moments of A_i , D_i^e and D_i^{ne} , as described in Section 3.2. Then we calculate the equilibrium probability vectors π_i as described in Section 3.4.

(c) Determining the throughput of L_i

Once the steady-state distribution is known, we can determine the new throughput $T_i^{(k)}$ according to (4) or equivalently,

$$T_i^{(k)} = (1 - p_{i,0}^{(k)})\mu_{d,i}^{(k-1)},$$

where $p_{i,0}$ denotes the fraction of time subsystem L_i is empty, and $\mu_{d,i}$ is the mean departure rate of subsystem L_i , i.e., $\mu_{d,i} = 1/(q_i^e \mathbb{E}(D_i^e) + (1 - q_i^e) \mathbb{E}(D_i^{ne}))$.

Further we determine new estimates for the probabilities p_i^f , r_i^{bf} and r_i^{ef} and the first two moments of ED_i and RD_i , as explained in Section 3.4.

We perform Step 1 for every subsystem from L_1 up to L_{N-1} .

Step 2: The departure process

Now we have new estimates for the probabilities p_i^f , r_i^{bf} and r_i^{ef} and the first two moments of ED_i and RD_i , and hence, we can recalculate the first two moments of D_i^e and D_i^{ne} , starting from subsystem L_{N-2} down to L_1 . Note that D_{N-1}^e and D_{N-1}^{ne} are always the same as S_{M-1} , because server M_{N-1} can never be blocked.

The calculation of the first two moments of D_i^e and D_i^{ne} is based on the representation shown in Figures 4 and 5.

Convergence

After Step 1 and 2 we can check whether the iterative algorithm has converged or not. We check this by comparing the mean departure rates in the $(k-1)$ -th and k -th iteration. When the sum of the absolute values of the differences between these rates is less than ε we stop; otherwise we repeat Step 1 and 2.

Of course, we may use other stop-criteria as well; for example, we may consider the throughput instead

of the departure rates. Bottom line is that we go on until 'nothing' changes anymore.

5 Numerical Results

In this section we investigate the quality of the proposed approximation by comparing it with discrete event simulation. We also compare the results with an approximation of Van Vuuren et al. [12] and an approximation of Helber [3]. The approximation of Van Vuuren et al. [12] was originally developed for multi-server systems, and it appeared to be less accurate for small-buffered single-server tandem queues. The approximation of Helber [3] is the most recent and best available method in the literature; therefore it is worthwhile to compare the present method with this one. Both methods are based on decomposition, but differ in the description of the subsystems and the iterative method.

In order to investigate the quality of the current method we will evaluate a large set of examples; In each example we assume that only mean and the squared coefficient of variation of the service times at each server are known, and we fit mixed Erlang or Hyper-exponential distributions on the first two moments of the service times, depending on whether the coefficient of variation is less or greater than 1; see Section 3.2. Then we will compare the throughput and the mean sojourn time (i.e., the mean time that elapses from the service start at server M_1 until service completion at server M_{N-1}) produced by the approximation with the ones produced by discrete event simulation. We are especially interested in investigating for which set of input parameters the current method gives satisfying results. Each simulation run is sufficiently long such that the widths of the 95% confidence intervals of the mean sojourn time and the delay probability are smaller than 1%.

We use a broad set of parameters for the tests. The mean service times of the servers are all 1. We vary the number of servers in the tandem queue between 4, 8, 12 and 16. The squared coefficient of variation (SCV) of the service times of each server is the same and is varied between 0.2, 0.4, 1 and 2. The buffer sizes between the servers are the same and varied between 0, 1, 3, and 5. We can also test three kinds of *imbalance* in the tandem queue. We test imbalance in the mean service times by increasing the average service time of the 'even' servers from 1 to 1.2. Imbalance in the SCV is tested by increasing the SCV of the service times of the 'even' servers by 0.5. Finally, imbalance in the buffer sizes is tested by increasing the buffers size of the 'even' buffers by 2. This leads to a total of $4^3 2^3 = 512$ test cases. The results for each category are summarized in Tables 1 up to 4. Each table lists the average error in the throughput and the mean sojourn time compared with simulation results. Each table also gives for 3 error-ranges the percentage of the cases which fall in that range, the average error of the approximation from Van Vuuren et al.[12], which is denoted by VAR and the average error of the approximation from Helber [3], which is denoted by Hel. Note that the approximation from Helber [3] can only handle cases with squared coefficients of variation of the service times larger than 0.5.

Buffer sizes	Error (%) in the throughput						Error (%) in mean sojourn time					
	Avg.	0-2	2-4	> 4	VAR	Hel	Avg.	0-2	2-4	> 4	VAR	Hel
0, 0, ...	1.15	81.3	14.1	4.7	11.17	5.02	2.06	59.4	28.1	12.5	12.22	7.14
1, 1, ...	1.21	79.7	20.3	0.0	5.64	7.40	1.24	75.0	25.0	0.0	7.03	10.54
3, 3, ...	1.49	70.3	29.7	0.0	2.89	6.90	1.13	79.7	20.3	0.0	3.95	11.35
5, 5, ...	1.65	62.5	34.4	3.1	2.12	5.84	1.29	75.0	23.4	1.6	2.81	10.87
0, 2, ...	1.18	81.3	7.8	10.9	5.54	5.77	1.19	87.5	12.5	0.0	6.92	8.71
1, 3, ...	1.02	87.5	12.5	0.0	3.55	6.64	0.84	93.8	6.3	0.0	5.02	10.48
3, 5, ...	1.45	70.3	29.7	0.0	2.29	6.10	1.33	76.6	20.3	3.1	3.51	10.88
5, 7, ...	1.60	64.1	32.8	3.1	1.84	5.28	1.76	64.1	28.1	7.8	2.87	10.56

Table 1: Overall results for tandem queues with different buffer sizes.

Overall we can conclude from the above results that the approximation method works very well. The average error in the throughput is around 1.4% and the average error in the mean sojourn time is around 1.3%. In most cases the errors are within 1%-width confidence interval of the simulation results. When

SCVs	Error (%) in the throughput						Error (%) in mean sojourn time					
	Avg.	0-2	2-4	> 4	VAR	Hel	Avg.	0-2	2-4	> 4	VAR	Hel
0.2, 0.2, ...	0.84	90.6	9.4	0.0	1.57	-	1.71	75.0	15.6	9.4	1.58	-
0.4, 0.4, ...	0.70	92.2	7.8	0.0	0.85	-	0.97	84.4	14.1	1.6	2.72	-
1, 1, ...	0.99	87.5	12.5	0.0	3.54	3.32	0.78	92.2	7.8	0.0	5.79	8.50
2, 2, ...	2.12	48.4	46.9	4.7	9.34	7.49	1.75	59.4	34.4	6.3	9.91	11.20
0.2, 0.7, ...	1.22	76.6	20.3	3.1	1.54	-	1.48	71.9	26.6	1.6	2.97	-
0.4, 0.9, ...	1.12	81.3	18.8	0.0	2.22	-	1.14	84.4	15.6	0.0	3.86	-
1, 1.5, ...	1.47	70.3	29.7	0.0	5.24	4.91	1.05	84.4	15.6	0.0	6.75	8.91
2, 2.5, ...	2.27	50.0	35.9	14.1	10.74	8.75	1.96	59.4	34.4	6.3	10.76	11.66

Table 2: Overall results for tandem queues with different SCVs of the service times.

Mean service times	Error (%) in the throughput						Error (%) in mean sojourn time					
	Avg.	0-2	2-4	> 4	VAR	Hel	Avg.	0-2	2-4	> 4	VAR	Hel
1, 1, ...	1.07	84.0	14.1	2.0	4.06	5.67	1.12	81.3	17.2	1.6	5.10	9.74
1, 1.2, ...	1.62	65.2	31.3	3.5	4.70	6.57	1.59	71.5	23.8	4.7	5.98	10.40

Table 3: Overall results for tandem queues with different service rates.

the results of the approximation are compared with those of the VAR and Hel approximations, we see that the new approximation performs substantially better than both other approximations.

Let us look at the results in more detail. If we look at Table 1, we see that the quality of the results for the throughput is nearly insensitive to the buffer sizes. The errors in the mean sojourn time are slightly larger for cases with zero buffers than for cases with non-zero buffers. But the results are still highly acceptable. Another remark is that the approximation does not seem to be very sensitive to imbalance in the buffer sizes.

In Table 2 we see that the error in the throughput is slightly increasing in the SCVs of the service times. For the mean sojourn the times the approximation performs best when the SCVs are around 1. Here, also, the quality of the approximation is not significantly sensitive to imbalance in the SCVs of the service times. On the other hand, Table 3 shows that quality of the approximation slightly depends on imbalance in the mean service times.

Finally in Table 4, it is shown that, as expected, the errors in both the throughput and the mean sojourn time increase when the tandem queue increases in length. This is the case, because an increasingly larger part of the line is described by a single server in the subsystems. But, as the length of the line increases, the results remain highly acceptable.

6 Concluding remarks

In this paper we described an algorithm for approximating single-server tandem queues with small buffers. We used a decomposition approach and developed an iterative algorithm to approximate the performance characteristics of the tandem queue. To improve the algorithm over existing ones, we modeled the service processes at the subsystems in more detail, in particular taking into account dependencies between service times and blocking after service. The queue-length distributions of the subsystems are determined by using the matrix-analytic method.

We tested the algorithm by comparing it with a discrete-event simulation and with existing methods and the results are very good. The average errors in both the throughput and the mean sojourn time are around 1.3%, where the average errors in existing methods are between 5 and 10%. Hence, it now seems possible to get reliable approximations for single-server tandem queues with small or zero buffers.

Servers in line	Error (%) in the throughput						Error (%) in mean sojourn time					
	Avg.	0-2	2-4	> 4	VAR	Hel	Avg.	0-2	2-4	> 4	VAR	Hel
4	0.52	96.9	3.1	0.0	1.48	1.55	0.73	94.5	5.5	0.0	1.78	1.44
8	1.04	88.3	11.7	0.0	3.91	4.91	0.98	92.2	7.0	0.8	4.71	6.47
12	1.67	64.1	32.8	3.1	5.42	7.93	1.48	74.2	21.1	4.7	6.69	13.11
16	2.15	49.2	43.0	7.8	6.71	10.08	2.23	44.5	48.4	7.0	8.98	19.24

Table 4: Overall results for tandem queues of different length.

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