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Citation for published version (APA):

Document status and date:
Published: 01/01/1992

Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
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Memorandum COSOR 91-38
(revised version)

On the asymptotically uniform distribution modulo 1 of extreme order statistics

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Eindhoven, December 1991
Revised June 1992
Revised January 1993
The Netherlands
ON THE ASYMPTOTICALLY UNIFORM DISTRIBUTION MODULO 1

OF EXTREME ORDER STATISTICS

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Abstract

Let \((X_m)\) be a sequence of independent and identically distributed random variables. We give sufficient conditions for the fractional part of \(\max(X_1,\ldots,X_n)\) to converge in distribution, as \(n \to \infty\), to a random variable with a uniform distribution on \([0,1)\).

Key Words & Phrases: distribution (modulo 1), Fourier-Stieltjes coefficients, fractional part.
1. Introduction and notation.

The concept of asymptotically uniform distribution (or equidistribution) modulo 1 (mod 1) of a sequence is well known in number theory, and is also important in random number generation (Ripley (1987)). The celebrated Weyl criterion states a necessary and sufficient condition for a sequence $(x_n)_{n=1}^{\infty}$ of real numbers to be asymptotically uniformly distributed (mod 1) (see Kuipers and Niederreiter (1974)). Holewijn (1969) generalizes this criterion to a criterion for a sequence $(X_n)_{n=1}^{\infty}$ of independent random variables to be uniformly distributed (mod 1) almost surely.

Let $\{X\}$ denote the fractional part of a random variable (r.v.) $X$, defined by $\{X\} := X - \lfloor X \rfloor$, where $\lfloor X \rfloor$ denotes the integer part of $X$, the largest integer not exceeding $X$. A sequence $(Z_n)_{n=1}^{\infty}$ of r.v.'s is said to be asymptotically uniformly distributed in distribution (mod 1) if

$$(1) \quad \{Z_n\} \overset{d}{\longrightarrow} U \quad (n \to \infty),$$

where $U$ is uniformly distributed on $[0,1)$. It is well known and easily proved (see e.g. Schatte (1983)) that (1) holds when $Z_n = \sum_{m=1}^{n} X_m$, where the $X_m$ are independent and identically distributed (i.i.d.) and non-lattice. For $Z_n = \sum_{m=1}^{n} X_m$ relation (1) does not generally hold; Jagers (1990), solving a problem by Steutel, shows that (1) does not hold for exponentially distributed $X_m$. Since, in this case $\sum_{m=1}^{n} X_m \overset{d}{\longrightarrow} \max(X_1, \ldots, X_n)$, it is of interest to consider $\max(X_1, \ldots, X_n)$ in general.

The object of this paper is to give sufficient conditions for $(Z_n) := (\max(X_1, \ldots, X_n))$ to be asymptotically uniform in distribution (mod 1). Clearly, if the right endpoint of distribution function
(d.f.) $F$, $\omega(F) := \sup\{x: F(x) < 1\}$, is finite, then \( \{Z_n\} \) converges almost surely to $\{\omega(F)\}$. Therefore, we shall assume that $\omega(F)$ is infinite.

Throughout this paper, $X_1, X_2, \ldots$ will be a sequence of i.i.d. r.v.'s with right-continuous d.f. $F = F_X$ on $\mathbb{R}$ with $\omega(F) = \infty$. In addition, we denote: $Z = \max(X_1, \ldots, X_n)$; $U$ is the r.v. with uniform distribution on $[0,1)$, and $V$ is the r.v. with standard exponentional distribution, i.e. $V \sim \text{Exp}(1)$. Further we use the following abbreviations:

\[ H = \{ h: \mathbb{R}_+ \rightarrow \mathbb{R}; h \text{ is piecewise continuous and non-decreasing} \}, \]
\[ G_t(x) = \exp(-e^{-tx}), \quad g_t(x) = G'_t(x) = e^{-tx} \exp(-e^{-tx}) \quad (x \in \mathbb{R}, t \in \mathbb{R}_+) \],
\[ \beta_k(x) = \exp(2\pi ikx) \quad (x \in \mathbb{R}, k \in \mathbb{Z}), \]
\[ x_+ = \max(x, 0) \quad (x \in \mathbb{R}). \]

We represent $F$ in terms of hazard functions: For any $X$ we can write

\[ X \overset{d}{=} h(V), \]

for some non-decreasing function $h: \mathbb{R}_+ \rightarrow \mathbb{R}$. The right-continuous (generalized) inverse function of $h$, $\tilde{h}: \mathbb{R} \rightarrow \mathbb{R}_+$, defined by $\tilde{h}(x) = -\log(1-F(x))$, is the cumulative hazard function of $F$; when $F$ has a derivative $F'$, then $\tilde{h}'(x) = F'(x)/(1-F(x))$ is called the hazard rate of $F$.

In section 2 we give some properties of Fourier-Stieltjes sequences (F.S.S.'s). In section 3 we prove the main result of this paper: the condition $h((Y+\log n)_+) \overset{d}{=} U$, i.e. the F.S.S. of $h((Y+\log n)_+)$ tends to zero as $n \rightarrow \infty$, is necessary and sufficient for $\{Z_n\} \overset{d}{=} U$. Here $Y$ is a r.v. with d.f. $G_0$, one of the three possible types of extreme value distributions (see Resnick
In addition, the requirement that \( h(V+t) \xrightarrow{d} U \) for all \( t \to \infty \) is sufficient for this condition. This leads to a rather simple sufficient condition: if \( F \) has a hazard rate that tends monotonically to zero, then \( \{ Z_n \} \xrightarrow{d} U \) for \( n \to \infty \). In section 4 we give some examples and briefly consider the connection with extreme value theory.

2. Properties of Fourier-Stieltjes Sequences.

We start by giving some notations and definitions. Let \( \mathcal{F}[0,1) \) denote the class of right-continuous d.f.'s on \( \mathbb{R} \) with support in \([0,1)\), and \( F(1-0)=1 \). We recall the definition of the F.S.S. of such d.f.'s.

**Definition 1.** Let \( F \) be a d.f. in \( \mathcal{F}[0,1) \). The F.S.S. \( c_F=(c_F(k))_{k=-\infty}^{\infty} \) of \( F \) is defined by

\[
    c_F(k) = \int_{[0,1)} e^{2\pi ikx} \, dF(x) \quad (k \in \mathbb{Z}).
\]

We write \( c_{F_X} \) instead of \( c_{F_X} \). Clearly \( c_F(0)=1 \), \( |c_F(k)| \leq 1 \), and \( c_F(-k)=\overline{c_F(k)} \) for \( k \in \mathbb{Z} \). Further we have \( c_{F_U}(k)=0 \) for \( k \neq 0 \). Since \( \exp(2\pi ikx) = \exp(2\pi ik) \) for \( x \in \mathbb{R} \), we have for any \( X \) the trivial but useful identity

\[
    E e^{2\pi ikX} = e^{2\pi ik(\mathbb{E}X)}.
\]

Next, we state the uniqueness and the continuity theorems for
F.S.S.'s of d.f.'s in $\mathcal{F}[0,1)$. For the proofs we refer to Zygmund (1968).

**Proposition 1.** Let $F, G \in \mathcal{F}[0,1)$. If $c_F = c_G$, then $F(x) = G(x)$ ($x \in \mathbb{R}$).

**Proposition 2.** Let $(F_n)_{n=1}^{\infty}$ be a sequence of d.f.'s in $\mathcal{F}[0,1)$ and let $(c_n)_{n=1}^{\infty}$ be the corresponding sequence of F.S.S.'s. The sequence $(F_n)_{n=1}^{\infty}$ converges weakly to a d.f. $F \in \mathcal{F}[0,1)$ iff $c_n(k) \to c(k)$ for $k \in \mathbb{Z}$ ($n \to \infty$). This sequence $c$ then is the F.S.S. of $F$.

The foregoing propositions justify the following definition of asymptotic uniformity (mod 1) in distribution.

**Definition 2.** A sequence $(Y_n)_{n=1}^{\infty}$ of r.v.'s is said to be asymptotically uniform in distribution (a.u.d.) (mod 1) if

$$
\lim_{n \to \infty} c_{(Y_n)}(k) = 0 \quad (k \neq 0).
$$

3. **Convergence of the sequence $(Z_n)$.**

In this section we give rather weak sufficient conditions on $F$ for $(Z_n)$ to be a.u.d. (mod 1). We now state the main result.

**Theorem 1.** Let $X \overset{d}{=} h(V)$ for some non-decreasing function $h: \mathbb{R}_+ \to \mathbb{R}$, and $V \sim \text{Exp}(1)$. Then
\[ (3) \quad \lim_{n \to \infty} \int_{-\log n}^{\infty} \beta_k(h(s+\log n))g_0(s)\,ds = 0 \quad (k \neq 0) \]

iff \( \{Z_n\} \overset{d}{\to} U(n \to \infty) \).

**Proof:** We first note that

\[ \{Z_n\} = \{\max(X_1, \ldots, X_n)\} = \{\max(h(V_1), \ldots, h(V_n))\} = \{h(\max(V_1, \ldots, V_n))\}, \]

where \( (V_n) \) is a sequence of i.i.d. r.v.'s with d.f. \( F_V \). By definition 2 it suffices to prove that for \( k \neq 0 \) \( c_{\{Z_n\}}(k) \to 0 \) \( (n \to \infty) \).

Using the notation introduced in section 1 we obtain

\[ c_{\{Z_n\}}(k) = \mathbb{E} e^{-S(1-\frac{1}{n}e^{-S})n-1}, \]

where

\[ R_k(n) = \int_{-\log n}^{\infty} \beta_k(h(s+\log n)) \left( e^{-S(1-\frac{1}{n}e^{-S})n-1} - g_0(s) \right)\,ds. \]

By dominated convergence we have for \( k \in \mathbb{Z} \)

\[ |R_k(n)| \leq \int_{0}^{n} \left| \left(1-\frac{x}{n}\right)^n e^{-x} \right|\,dx \to 0 \quad (n \to \infty). \]

Hence

\[ c_{\{Z_n\}}(k) = \int_{-\log n}^{\infty} \beta_k(h(s+\log n))g_0(s)\,ds + o(1) \quad (n \to \infty). \]

In the following lemmas we give sufficient conditions on the functions \( h \) for (3) to hold. These conditions are more easily

Lemma 1. Let $h \in \mathbb{H}$. If

\begin{equation}
\lim_{t \to \infty} \int_{0}^{\infty} \beta_{k}(h(s+t))e^{-s}ds = 0 \quad (k \neq 0),
\end{equation}

then

\begin{equation}
\lim_{t \to \infty} \int_{-t}^{\infty} \beta_{k}(h(s+t))g_{0}(s)ds = 0 \quad (k \neq 0)
\end{equation}

holds, and hence so does (3).

Proof: We define for $x \geq 0$

$$\Phi(x) = \int_{x}^{\infty} \beta_{k}(h(s))e^{-s}ds \quad \text{and} \quad \Psi(x) = \sup\{|e^{\psi}(y)| : y \geq x\}.$$

Substitution of $s = y - t$ in (4) yields

$$\int_{0}^{\infty} \beta_{k}(h(s+t))e^{-s}ds = e^{t} \int_{t}^{\infty} \beta_{k}(h(y))e^{-y}dy = e^{t} \Phi(t).$$

Let $x \in \mathbb{R}_{+}$. Then in (5) we have

$$\left| \int_{-t}^{x} \beta_{k}(h(s+t))g_{0}(s)ds \right| = \left| \int_{0}^{x} \beta_{k}(h(s))g_{t}(s)ds \right| \leq G_{t}(x).$$

Integrating by parts we find

$$\left| \int_{x-t}^{\infty} \beta_{k}(h(s+t))g_{0}(s)ds \right| = \left| \int_{x}^{\infty} \beta_{k}(h(s))g_{t}(s)ds \right|$$

$$= \left| e^{\Phi(x)}g_{t}(x) + \int_{x}^{\infty} e^{s} \Phi(s) \left[ g'_{t}(s) + g_{t}(s) \right] ds \right|$$

$$\leq \Psi(x) \left[ g_{t}(x) + \int_{x}^{\infty} \left| g'_{t}(s) + g_{t}(s) \right| ds \right] = \Psi(x) (1-G_{t}(x)),$$

and so it follows that
\[ \int_0^\infty \beta_k(h(s))g_t(s)ds \leq G_t(x) + \Psi(x)(1-G_t(x)). \]

Taking \( x = \frac{1}{2}t \), and writing \( e(t) = \exp(-e^{1/2t}) \) we get
\[ \int_{-t}^\infty \beta_k(h(s+t))g_0(s)ds \leq e(t) + \Psi\left(\frac{1}{2}t\right)(1-e(t)) \to 0 \quad (t\to\infty). \]

This lemma can be interpreted as follows: if \( h(V+t) \xrightarrow{d} U \), then \( h((Y+t)_+) \xrightarrow{d} U \ (t\to\infty) \), where \( Y \) has d.f. \( G_0 \).

**Corollary.** Let \( X \xrightarrow{d} h(V) \) for some \( h \in H \), and \( V \sim \text{Exp}(1) \). If condition (4) holds, then \( \{Z_n\} \xrightarrow{d} U \ (n\to\infty) \).

**Lemma 2.** Let \( h \in H \). If
\[ \lim_{k \to \infty} \int_0^{t^2} \beta_k(h(s))ds \quad (k \neq 0) \]
exists, then condition (4) holds, and hence so does (3).

**Proof:** Let \( \varepsilon > 0 \). Since \( \int_t^\infty \beta_k(h(s))ds \to 0 \ (t\to\infty) \), there is a constant \( A > 0 \) such that
\[ \left| \int_t^\infty \beta_k(h(s))ds \right| < \varepsilon \quad (t \geq A). \]

Integrating by parts we have
\[ \int_t^\infty \beta_k(h(s))e^{-s}ds = -\int_t^\infty e^{-s}d\left(\int_s^\infty \beta_k(h(y))dy\right) = e^{-t}\int_t^\infty \beta_k(h(s))ds - \int_t^\infty e^{-s}\left(\int_s^\infty \beta_k(h(y))dy\right)ds, \]
whence for \( t \geq A \)
\[ \left| \int_0^\infty \beta_k(h(s+t))e^{-s}ds \right| = \left| e^t \int_t^\infty \beta_k(h(s))e^{-s}ds \right|. \]
We note that for \( h(x) = \frac{-x^2 - 2\cos x^2}{x} \) (\( x \geq 8 \)) condition (6) fails, whereas (4) holds (cf. Brands (1991)).

**Lemma 3.** Let \( h: \mathbb{R}^+ \to \mathbb{R} \) be a continuous non-decreasing function with inverse function \( \tilde{h} \), and suppose \( \tilde{h}''(x) \) exists if \( x \geq A \), for some \( A \in \mathbb{R} \).

If

\[
\lim_{x \to \infty} \tilde{h}'(x) = 0,
\]

and

\[
\int_A^\infty |\tilde{h}''(x)| \, dx < \infty,
\]

then condition (3) holds.

**Proof:** By substituting \( s = \tilde{h}(x) \) and setting \( c = h(0) \) we have for \( k \neq 0 \)

\[
\int_0^\infty \beta_k(h(s)) \, ds = \int_c^\infty \tilde{h}'(x) \beta_k(x) \, dx.
\]

Integrating by parts we find

\[
\left| \int_A^\infty \tilde{h}'(x) \beta_k(x) \, dx \right| \leq \int_A^\infty \left( |\tilde{h}'(x)| \frac{1}{2\pi i k} \beta_k(x) \right) \, dx
\]

\[
\leq \frac{1}{2\pi k} \left( \int_A^{\tilde{h}'(A)} + \int_A^\infty |\tilde{h}''(x)| \, dx \right),
\]

which exists because of the conditions (7) and (8). Now lemma 2 yields the assertion.

Next, we state a corollary of theorem 1 and lemma 3, which says that \( (Z_n) \) is a.u.d. (mod 1) for distributions with hazard rates that decrease monotonically to zero (i.e. \( \tilde{h}'(x) \downarrow 0 \) as \( x \to \infty \)).
Theorem 2. Let $X \overset{d}{\sim} h(V)$ with positive derivative $F'$, and $V \sim \text{Exp}(1)$. Suppose $\tilde{h}''(x)$ exists if $x \geq A$, for some $A \in \mathbb{R}$. If conditions (7) and (8) hold, then $\{Z_n\} \overset{d}{\rightarrow} U (n \to \infty)$.

4. Examples and final remarks.

We give some explicit examples to illustrate the scope of the results of section 3.

1. For the Pareto d.f. $F(x) = 1 - x^{-\beta}$, $x > 1$, $\beta > 0$, (2) holds with $h(x) = e^{x/\beta}$, and $\tilde{h}(x) = \beta \log x$, $\tilde{h}'(x) = \beta / x$. Clearly $\tilde{h}(x)$ satisfies the assumptions of theorem 2, and thus $\{Z_n\} \overset{d}{\rightarrow} U (n \to \infty)$.

2. For $F(x) = 1 - \exp(-x^v)$, $x > 0$, $0 < v < 1$, we have $h(x) = x^v$, and theorem 2 yields $\{Z_n\} \overset{d}{\rightarrow} U (n \to \infty)$. If $v = 1$, then for $F = F_V$, it is shown in Jagers (1990) that $\{Z_n\}$ diverges. If $v > 1$, it is also possible to show that $\{Z_n\}$ diverges. Also, if $v = 1$, we see that $\tilde{h}'(x) = v x^{v-1}$ does not converge to zero as $x \to \infty$, and that condition (8) does not hold. So, it would seem that conditions (7), (8), and hence condition (4) are not too far away from being necessary for $\{Z_n\}$ to be a.u.d. (mod 1).

3. For $F(x) = 1 - \frac{1}{\alpha \log x}$, $x > e^{1/\alpha}$, $\alpha > 0$, the conditions of theorem 2 are satisfied. Thus $\{Z_n\}$ is a.u.d. (mod 1) even though $F$ does not belong to the domain of attraction of an extreme value distribution.
Obviously, example 2 suggests that there is a relation with extreme value theory. In fact, if $F$ is absolutely continuous and belongs to the domain of attraction of $G_0$, and if $1/h'(x)$ tends to some $c \geq 0$ monotonically as $x \to \infty$, then $(Z_n)$ diverges in distribution. These results, which need rather long proofs, will be published elsewhere.

Finally, for any r.v. $W$ with $P(0 \leq W < 1) = 1$ it is possible to construct a sequence of i.i.d. r.v.'s $(X_n)_{1}^{\infty}$ such that $(Z_n) \xrightarrow{d} W (n \to \infty)$ (see Brands, Steutel and Wilms (1992)).

Acknowledgement.

The authors are indebted to F.W. Steutel and J.G.F. Thiemann for suggesting the problem treated here, and for reading an earlier version of this paper.
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