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A curious implication of Spitzers identity

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Abstract. Spitzer’s identity can be read as follows: Let \( W_n \) denote the waiting time of the \( n \)-th customer in a \( G|G|1 \)-queue, and let \( N \) be geometrically distributed on \((0,1,...)\) and independent of \( W_n \). Then \( W_{N+1} \) is infinitely divisible.

0. Introduction and summary

It is well known that random sums of the form

\[ S_n = X_1 + ... + X_N \]

are infinitely divisible (inf div) if \( X_1, X_2, ... \) are i.i.d. and \( N \) is geometrically distributed on \( \{0,1,...\} \) and independent of \( (X_n)^\infty \).

In this note it will appear that from Spitzer’s identity it follows that

\[ W_{N+1} = W_1 + W_2 - W_1 + ... + W_{N+1} - W_N \]

is inf div, where \( W_n \) denotes the waiting time of the \( n \)-th customer in a \( G|G|1 \)-queue, and \( N \) is independent of \( (W_n) \); here, however the \( W_{n+1} - W_n \) are dependent, and do not have the same distribution.

In Section 1, Spitzer’s identity is given with some necessary context, Section 2 contains basic facts on inf div distributions, and in Section 3 the two ingredients are combined. Section 4 gives some additional remarks.

In what follows \( F \), with or without suffix, will denote a distribution function, and \( \hat{F} \) its Laplace-Stieltjes transform (LSt).

1. Spitzer’s identity

In the well-known \( G|G|1 \)-queueing system customers arrive at times \( 0, A_1, A_1 + A_2, ... \) and are served during periods \( B_1, B_2, ... \); all \( A \)'s and \( B \)'s are independent. We write \( S_0 = 0 \),

\[ S_n = \sum_{k=1}^{n} (B_k - A_k), \quad n = 1, 2, ... \]

and \( S_k^+ = \max (0, S_k), \quad k = 1, 2, ... \). If it is assumed that the first customer finds the server free, then the waiting time \( W_n \) of the \( n \)-th customer is given by \( W_1 = 0 \), and

\[ W_{n+1} \overset{d}{=} \max(S_0, S_1, ..., S_n), \quad n = 1, 2, ... \quad (1.1) \]

Now Spitzer’s identity (Loève (1977)) reads, for \(|z| < 1 \) and \( Re \ s \geq 0 \),
\[
\sum_{n=0}^{\infty} E e^{-sW_{n+1}} z^n = \exp \left\{ \sum_{k=1}^{\infty} \frac{1}{k} E e^{-sS_k^+} z^k \right\}.
\] (1.2)

2. Infinite divisibility

A random variable \( X \) is called inf div if for every \( n \in \mathbb{N} \) one has

\[
X \overset{d}{=} X_{1,n} + \ldots + X_{n,n},
\]

where the \( X_{j,n} \) are iid. We only need the following results (Feller (1971), Steutel (1970)).

**Lemma 1.** A nonnegative random variable is inf div if and only if it has a LST of the form

\[
\hat{F}(s) = E e^{-sx} = \exp \left\{ \int_{0}^{\infty} \frac{e^{-sx} - 1}{x} dK(x) \right\},
\] (2.1)

where \( K \) is a nondecreasing function, which, necessarily, has the property \( \int \frac{1}{x} dK(x) < \infty \).

**Lemma 2.** A nonnegative, integer-valued random variable \( M \) with \( P(M = n) = p_n, \) and \( p_0 > 0 \) is inf div if and only if its probability generating function has the form,

\[
P(z) := \sum_{n=0}^{\infty} p_n z^n = \exp \left\{ \sum_{n=0}^{\infty} \frac{r_n}{n+1} (z^{n+1} - 1) \right\},
\] (2.2)

with \( r_n \geq 0, n = 0, 1, 2,..., \) and, necessarily, \( \sum_{0}^{\infty} r_n /(n + 1) < \infty \).

3. \( W_{N+1} \) is infinitely divisible

We rewrite (1.2) as follows. Put \( z = p \in (0, 1) \) and multiply by \( (1 - p) \); this yields (use \(- \sum_{1}^{\infty} p^k / k = \log(1 - p))\),

\[
\sum_{n=0}^{\infty} (1 - p) p^n E e^{-sW_{n+1}} = \exp \left\{ \sum_{k=1}^{\infty} \frac{p_k}{k} (E e^{-sS_k^+} - 1) \right\}.
\] (3.1)

The left-hand side of the equation above is equal to the LST of \( W_{N+1} \), where \( N \) is independent of \((W_n)_{n=0}^{\infty} \), and

\[
P(N = n) = (1 - p)p^n \quad (n = 0, 1, \ldots, n).
\] (3.2)

The right-hand side can be rewritten as

\[
\exp \left\{ \int_{0}^{\infty} \frac{e^{-sx} - 1}{x} dK(x) \right\},
\]
with $K$ given by

$$K(x) = \sum_{k=1}^{\infty} \frac{p_k}{x} \int_{0}^{x} y \, dF_{S_k^+}(y)$$  \hspace{1cm} (3.3)$$

Combining the results above we obtain the main result of this note.

**Theorem.** If $W_n$ is the waiting time of the $n$-th customer in a $G|G|1$-queue, started empty, and $N$ is a rv independent of $(W_n)_{n=1}^{\infty}$ satisfying (3.2), then the rv $W_{N+1}$ is infinitely divisible.

It should be pointed out that for fixed $n$ the $W_n$ are in general not inf div, since they are bounded if the $B$'s are bounded; if $W_n \overset{q}{\to} W$ as $n \to \infty$, then $W$ is inf div (see end of Section 4).

4. **Further remarks**

Spitzer's identity (1.2) can also be related to Lemma 2. Apart from a multiplicative constant in the right-hand side, (1.2) is of the form (2.2) with

$$p_n = p_n(s) = E e^{-s W_{n+1}}; \quad r_n = E e^{-S^+_{n+1}}.$$  

So we see that, for every $s$, the sequence $(E e^{-s W_{n+1}})_{n=1}^{\infty}$ is an infinitely divisible sequence (not necessarily summing to 1).

Another result follows if we take logarithms in (1.2) and (2.2), and differentiate. Equation (2.2) then yields

$$n p_n = \sum_{k=0}^{n-1} p_k r_{n-k-1} \quad (n = 1, 2, ...).$$

Similarly, (1.2) leads to

$$n E e^{-s W_{n+1}} = \sum_{k=0}^{n-1} E e^{-s W_{k+1}} \cdot E e^{-s S^+_{n-k}},$$

or in terms of distribution functions

$$F_{W_{n+1}}(w) = \frac{1}{n} \sum_{k=1}^{n-1} (F_{W_{k+1}} \ast F_{S^+_{n-k}}) \quad (n = 1, 2, ...),$$ \hspace{1cm} (4.1)$$

which is not easily verified directly for $n > 1$. In fact, one can derive (4.1) from (1.1) and obtain a proof of Spitzer’s identity. This was done by Heinrich (1985), who proves (4.1) by induction; in Kotz et al. (1985), where I found the reference to Heinrich, the identity is misquoted: the factor $1/k$ in the right-hand side of (1.2) is missing. Finally, letting $p \nmid 1$ in (3.1) we obtain the Lst $\hat{F}_W$ of the limiting waiting time (assume $ES_1 < 0$):
\[ \hat{F}_W(s) = \exp \left\{ \int_0^\infty (e^{-ix} - 1) d \sum_{k=1}^\infty \frac{1}{k} F_{S_k^+}(x) \right\}, \]

which shows that \( W \) is inf div, as is well known.

References


Loève, M. (1977), Probability theory 1, Springer-Verlag, New York, etc.