On the differential equation \( yy'' + 2xy' = 0 \)

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by

J. J. M. Brands

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ON THE DIFFERENTIAL EQUATION \( yy'' + 2xy' = 0 \)

by

J.J.A.M. Brands

Department of Mathematics, Eindhoven University of Technology,

The Netherlands

Abstract

The asymptotic behaviour for \( x \to \infty \) of solutions \( y(x) \) of the initial value problem \( yy'' + 2xy' = 0, \ y(0) = \alpha > 0, \ y'(0) = \beta \), and also the asymptotic behaviour for \( \beta \to +\infty \) and for \( \beta \to -\infty \) of \( L := \lim \limits_{x \to \infty} y(x) \) is investigated.

1. Introduction

In this paper we consider the following initial value problem

(1) \( yy'' + 2xy' = 0 \quad (x > 0) \),

(2) \( y(0) = \alpha > 0, \ y'(0) = \beta \).

Equation (1) occurs in boundary layer theory (see [1], p. 22–23) and arises from the diffusion equation

\[
\frac{\partial C}{\partial t} = \frac{\partial}{\partial z} \left( D \frac{\partial C}{\partial z} \right).
\]

If \( D = e^{-C} \), and if \( C \) only depends on \( x := \frac{1}{2}t^{-\frac{1}{2}} \), then \( y := e^{-C} \) satisfies (1).
In boundary layer theory one is also interested in the boundary value problem (1), (3) with

(3) \[ y(0) = \alpha > 0 \quad , \quad y(\infty) = L > 0 \, . \]

A case of special interest is

(2') \[ y(0) = 1 \quad , \quad y'(0) = \gamma \, , \]

since all solutions of (1), (2) are expressible in solutions of (1), (2') by means of the transformation (11) (see Section 2). We shall study the asymptotic behaviour of solutions \( y(x, \gamma) \) of (1), (2') for \( x \to \infty \). Moreover, we shall pay attention to the asymptotic behaviour of the limit

\[ L(\gamma) := \lim_{x \to \infty} y(x, \gamma) \quad \text{for} \quad \gamma \to \infty \quad \text{and} \quad \gamma \to -\infty \, , \]

since physicists seem interested in large values of \(|\gamma|\).
2. Results

In Section 3 the following fundamental results are proved. There exists a unique solution $y(x,a,\beta)$ of (1), (2), defined on $[0,\infty)$. This solution $y(x,a,\beta)$ tends monotonically to a positive limit $L(a,\beta)$ if $x \to \infty$. The transformation rule

$$y(x,a,\beta) = a y(a^{-\frac{1}{2}} x, \gamma) , \quad L(a,\beta) = a L(\gamma) , \quad \gamma = a^{-\frac{1}{2}} \beta ,$$

enables us to consider only the solution $y(\cdot,\gamma)$ of (1), (2') and its limit $L(\gamma)$. This limit is a continuous increasing bijection $L : \mathbb{R} \to (0,\infty)$. Hence the boundary value problem (1), (3) has a unique solution.

In Section 4 the asymptotic behaviour of $y(x,\gamma)$ for $x \to \infty$ is determined.

$$y(x,\gamma) = L - A \text{erfc} \left( L^{-\frac{1}{2}} x \right) + \sigma(x^{-2} \exp(-2 L^{-1} x^2)) \quad (x \to \infty) ,$$

where

$$A = \frac{1}{2} \pi^{\frac{1}{2}} L^{\frac{1}{2}} \gamma \exp \left[ 2 \int_{0}^{\infty} s(L^{-1} - (y(s,\gamma))^{-1}) ds \right] .$$

In Sections 5 and 6 bounds for the limit $L(\gamma)$ are determined.

$$2 L_0 \log(1 + L_0 \gamma^2) < L(\gamma) - L_0 \gamma^2 < 47 L_0 \log(1 + 3(16)^{-1} \gamma^2) + 47 L_0 \quad (\gamma > 1.8)$$

$$1 - \gamma^{-2} < L(\gamma) \exp \left[ \frac{1}{2} \gamma^2 + \frac{1}{2} \right] < 1 + \gamma^{-2} \quad (\gamma < -7) ,$$

where $L_0 = 0.3574\ldots$ denotes the limit of the special solution $y$ of (1) with initial values $y(0) = 0$, $y'(0) = 1$. 
3. Preliminaries

We shall investigate existence and uniqueness properties, and prove the continuous dependence of the limit as functions on the initial values.

(4) Theorem. There exists a unique solution \( y \) of the initial value problem (1), (2) which is defined on \([0, \infty)\). This solution \( y \) is positive and monotone, and \( y(x) \) tends to a positive limit \( L \) if \( x \to \infty \).

Proof. According to well-known existence and uniqueness theorems (see for example Sections 1.1 and 1.2 of [2]), there is a positive number, say \( \delta \), such that there is a unique solution, say \( y \), of (1), (2), defined on \([0, \delta)\).

Let \( I \) be the maximal interval of the form \([0, a)\), (possibly \( a \) may be \( \infty \)) on which \( y \) is positive and satisfies (1), (2). Dividing both sides of (1) by \( y y' \), integrating over \([0, x]\) with \( x \in I \) we get

\[
y'(x) = \beta \exp \left[ -2 \int_0^x s(y(s))^{-1} ds \right] \quad (x \in I),
\]

a result also correct if \( y' = 0 \). It follows that

\[
0 \leq y'(x) \text{ sgn } \beta \leq |\beta| \quad (x \in I).
\]

Let \( a > 0, a \in I \). Dividing both sides of (1) by \( xy \) integrating from \( a \) to \( x \), \( x \in I, x > a \), taking exponentials, we find

\[
y(x) = y(a) \exp \left[ \frac{1}{2} a^{-1} y'(a) - \frac{1}{2} x^{-1} y'(x) - \frac{1}{2} \int_a^x s^{-2} y'(s) ds \right].
\]
From (6) and (7) we infer that

\[(8) \quad y(x) \leq y(a) \exp[\int_a^1 y'(a)] \quad (a < x, x \in I),\]

where \(\leq\) holds if \(\beta \geq 0\) and \(>\) if \(\beta < 0\). It follows from (5) and (8) that \(y\) is bounded, and increasing if \(\beta > 0\), and that \(y\) is bounded away from zero and decreasing if \(\beta < 0\), and that \(I = [0, \infty)\). Clearly \(\lim_{x \to \infty} y(x)\) exists and is positive.

We shall denote the solution of (1), (2) by \(y(x, \alpha, \beta)\) and its limit by \(L(\alpha, \beta)\). In the case of (1), (2') we write \(y(x, \gamma)\) and \(L(\gamma)\). Simply by inspection we can prove

(9) Theorem. For every \(\lambda > 0\)

\[(10) \quad y(x, \lambda^2 \alpha, \lambda \beta) = \lambda^2 y(\lambda^{-1} x, \alpha, \beta), \quad L(\lambda^2 \alpha, \lambda \beta) = \lambda^2 L(\alpha, \beta).\]

Theorem (9) enables us to consider, without loss of generality, only the initial value problem (1), (2'). We have

\[(11) \quad y(x, \alpha, \beta) = \alpha y(a^{-1} x, \gamma), \quad L(\alpha, \beta) = \alpha L(\gamma), \quad \gamma = a^{-1} \beta.\]

(12) Theorem. \(L(\cdot): \mathbb{R} \to (0, \infty)\) is an increasing continuous bijection.

Proof. Let \(\gamma_2 > \gamma_1\). Then there is a \(\delta > 0\) such that \(y(x, \gamma_2) > y(x, \gamma_1)\) on \([0, \delta]\). Suppose that there exists a positive \(x_0\) such that \(y(x_0, \gamma_2) = y(0, \gamma_1)\) and \(y(x, \gamma_2) > y(x, \gamma_1)\) for \(0 < x < x_0\). Clearly \(y'(x_0, \gamma_2) < y'(x_0, \gamma_1)\).

Dividing (1) by \(y\) and integrating from 0 to \(x\) we find

\[(13) \quad y'(x, \gamma) - \gamma + 2x \log y(x, \gamma) = 2 \int_0^x s \log y(s, \gamma)ds.\]
Applying (13) twice with \( Y = Y_2 \) and \( Y = Y_1 \) and subtracting both results we get for \( x = x_0 \)

\[
y'(x_0, Y_2) - y'(x_0, Y_1) - Y_2 + Y_1 = 2 \int_0^x s \log\left(\frac{y(s, Y_2)}{y(s, Y_1)}\right) ds ,
\]

a contradiction since both sides have different signs. It follows that \( y(x, Y_2) > y(x, Y_1) \) for all \( x > 0 \).

In the sequel of the proof we need the formula

\[
(14) \quad \log L = \frac{1}{2} \int_0^\infty (\gamma - y'(s)) s^{-2} ds ,
\]

where \( y = y(\cdot, \gamma) \), \( L = L(\gamma) \), and which is obtained by integration of

\[
y'y^{-1} = -\frac{1}{2} x^{-1} y''. \quad \text{The integral in (14) exists since } y''(s) = -2Ys + \sigma(s) \quad (s \rightarrow 0) \quad \text{and } \gamma - y'(s) = \gamma s^2 + \sigma(s^2) \quad (s \rightarrow 0). \quad \text{Applying (14) to } Y_2 := y(\cdot, Y_2) \quad \text{and } Y_1 := y(\cdot, Y_1) \quad \text{respectively, and subtracting the results we get}
\]

\[
(15) \quad \log L_2^{-1} = \frac{1}{2} \int_0^\infty \left[ Y_2 - Y_1 + y_1'(s) - y_2'(s) \right] s^{-2} ds ,
\]

where \( L_2 := L(Y_2) \) and \( L_1 := L(Y_1) \).

If \( Y_1 > 0 \) then by (5), \( y_2' > Y_2^{-1} y_1' \) which implies directly \( L_2 > 1 + Y_2^{-1}(L_1 - 1) \), and, by (15) and (14)

\[
\log L_2^{-1} < (Y_2 - Y_1)Y_1^{-1} \log L_1 < (Y_2 - Y_1)Y_1^{-1} \log L_2 .
\]
If $\gamma_1 = 0$ then by (5), $y_2' < \gamma_2 \exp[-x^2 L_2]$ which upon integration gives $L_2 < (1 + \frac{1}{2} \pi \gamma_2)^2$.

If $\gamma_2 = 0$ then by (5) $y_1' > \gamma_1 \exp[-x^2]$ which leads to $L_1 > 1 + \frac{1}{2} \pi \gamma_1$.

If $\gamma_2 < 0$ then by (5) $y_2' < \gamma_2 \gamma_1^{-1} y_1'$ which implies

$$L_2 - L_1 < (1 - \gamma_2 \gamma_1^{-1})(1 - L_1) < (1 - \gamma_2 \gamma_1^{-1})$$,

and by (15) and (14) $\log L_2 \gamma_1^{-1} > (1 - \gamma_1 \gamma_2^{-1}) \log L_2$.

It follows that $L(\cdot)$ is continuous and strictly increasing on $\mathbb{R}$, and, moreover $L(\gamma) \to \infty$ if $\gamma \to \infty$, $L(\gamma) \to 0$ if $\gamma \to -\infty$. 
4. Asymptotic behaviour

First we investigate the asymptotic behaviour of solutions \( y := y(\cdot, \gamma) \) with \( \gamma \geq 0 \). Using the inequality \( y(x) \leq L := L(\gamma) \) in (5) we get

\[
y'(x) \leq \gamma \exp\left[-\frac{x^2}{L}\right],
\]

from which it follows by integration over \([x, \infty)\) that

\[
y(x) \geq L - \frac{1}{2} \pi^{1/2} L^{1/2} \text{erfc}(x L^{-1/2}).
\]

Hence

\[
y(x) = L + \sigma(x^{-1} \exp[-x^2/L]) \quad (x \to \infty).
\]

It follows that

\[
Q := 2 \int_{0}^{\infty} s((y(s))^{-1} - L^{-1}) \, ds
\]

exists and that

\[
2 \int_{x}^{\infty} s((y(s))^{-1} - L^{-1}) \, ds = \sigma(x^{-1} \exp[-x^2/L]) \quad (x \to \infty).
\]

Using (17) and (18) in (5) we get

\[
y'(x) = \gamma \exp[-Q - x^2/L] (1 + \sigma(x^{-1} \exp[-x^2/L])) \quad (x \to \infty),
\]

which upon integration from \(x\) to \(\infty\) gives

\[
y(x) = L - \frac{1}{2} \pi^{1/2} L^{1/2} \gamma e^{-Q} \text{erfc}(x L^{-1/2}) + \sigma(x^{-2} \exp[-2x^2/L]) \quad (x \to \infty).
\]
Secondly we consider the case $\gamma < 0$. Then using $L < y(x) < 1$ in (5) we get $y'(x) \geq \gamma \exp[-x^2]$. By integration over $[x, \infty)$ it follows that

$$y(x) = L + \sigma(x^{-1} \exp[-x^2]) \quad (x \to \infty).$$

Repeating the same kind of arguments (after (16)) as in the case $\gamma \geq 0$ we arrive at the same Formula (20) except for the order term which is now $\sigma(x^{-2} \exp[-x^2(L^{-1} + 1)])$. Hence Formula (16) holds. Again by repeating the arguments after (16) we get (20).
5. Bounds for the limit if \( \gamma > 0 \)

By (10) we have

\[(21) \quad \L(1,\gamma) = \gamma^2 \L(\gamma^{-2},1). \]

We shall prove that there are positive constants \( C_1, C_2 \) such that for \( \alpha > 0 \) sufficiently small

\[(22) \quad C_1 \alpha | \log \alpha | < \L(\alpha,1) - \L(0,1) < C_2 \alpha | \log \alpha |. \]

For simplicity we write \( \gamma, \L, \gamma_0, \L_0 \) instead of \( \gamma(\cdot;\alpha,1), \L(\alpha,1), y(\cdot;0,1), \L(0,1) \) respectively. It is easily seen that

\[(23) \quad \gamma_0(x) = x - x^2 + \sigma(x^3) \quad (x > 0). \]

Clearly, Formula (5) with \( \beta = 1 \) holds for \( \gamma_0 \). It follows by (5) that initially \( \gamma'(x) > \gamma_0'(x) \), since initially \( \gamma(x) > \gamma_0(x) \). By standard reasoning we can conclude that

\[(24) \quad \gamma'(x) > \gamma_0'(x) \quad , \quad \gamma(x) > \gamma_0(x) + \alpha \quad (x > 0). \]

For later use we shall prove the following facts:

\[(25) \quad \gamma_0(x) > x - x^2 + \frac{1}{3}x^3 > x - x^2 \quad (0 < x \leq 1) \]

\[(26) \quad \gamma_0(x) < \frac{1}{4}(1 - e^{-2x}) \quad (x > 0) \]

\[(27) \quad \L_0 = 0.35742210059... \]
Proof of (25), (26) and (27). From \( y''_0 > 0 \) and \( y_0(0) = 0 \) it follows that 
\[ y_0(x) > xy'_0(x) \quad (x > 0). \] 
From (1) we deduce \( y''_0 > -2 \) which leads to 
\[ y_0 > x - x^2. \] 
Using this last result in (5) we find \( y'_0 > (1 - x)^2 \) 
\( (0 < x \leq 1) \) whence \( y_0 > x - x^2 + \frac{1}{3}x^3 \) \( (0 < x \leq 1). \) Clearly \( L_0 > y_0(1) > \frac{1}{3}. \) 
Using \( y_0(x) < x \) \( (x > 0) \) in (5) we get \( y'_0 < e^{-2x} \) and \( y_0 < \frac{1}{4}(1 - e^{-2x}). \) 
Hence \( L_0 < \frac{1}{4}. \) A numerical computation gives (27). We also need

\[ L < (1 + 2\sqrt{a})^2 L_0. \]

Let \( 0 < a \leq \frac{1}{2}. \) The line \( y = a - a^2 + (1 - 2a)(x - a) \) is tangent to the curve 
\( y = x - x^2 \) in the point \( (a, a - a^2). \) It follows by (25) that 
\( y_2(x) := y(x; a^2, 1 - 2a) \) equals \( y_0(x) \) for some \( x = b < a \) since \( y_2(a) < y_0(a). \) 
At \( x = b \) we have \( y_0(b) = y_2(b) \) and \( y'_0(b) > y'_2(b). \) We show that \( y_0(x) > y_2(x) \) 
for all \( x > b. \) Immediately to the right of \( x = b \) we have \( y_0(x) > y_2(x). \) Then 
in some interval \( (b, b + \delta) \)

\[ y'_0(x) = y'_0(b) \exp \left[ -2 \int_b^x s(y_0(s))^{-1} ds \right] > y'_0(b) \exp \left[ -2 \int_0^x s(y_2(s))^{-1} ds \right] \]

\[ = y'_0(b) (y'_2(b))^{-1} y'_2(x) > y'_2(x). \]

By standard arguments it follows that \( y_0(x) > y_2(x) \) \( (x > b). \)

Letting \( x \to \infty \) we find \( L(a^2, 1 - 2a) < L_0. \) By (10) we have \( L(a^2, 1 - 2a) = \)
\( = (1 - 2a)^2 L(a^2(1 - 2a)^{-2}, 1). \) Substituting \( a = \frac{1}{4}(1 + 2a^2)^{-1} \) we get (28).

Now we proceed with the proof of (22).
The function $K$ defined by (29) satisfies (30).

(29) \[ K(x) := \frac{y'(x)}{y_0'(x)} \, . \]

(30) \[ K'(x)/K(x) = 2x((y_0(x))^{-1} - (y(x))^{-1}) \, , \quad K(0) = 1 \, . \]

Using (24) and the inequalities $y_0(x) < x$, $y_0'(x) < 1$ for $x > 0$, we see that

\[ K'/K > 2a (y_0 + a)^{-1} y_0' \, , \]

from which it follows by integration

\[ K > (1 + y_0/a)^{2a} \, . \]

Using (29) and integrating we get

\[ y > \alpha + \alpha(2\alpha + 1)^{-1} [(1 + y_0/a)^{2\alpha+1} - 1] \, . \]

Letting $x \to \infty$ we find for all $\alpha > 0$

\[ L > \alpha + \alpha(2\alpha + 1)^{-1} [(1 + L_0/a)^{2\alpha+1} - 1] \, . \]

Using that $L_0 < \frac{1}{2}$ we derive

\[ L > L_0 + 2L_0 \alpha \log(1 + L_0/a) \, . \]

Now we turn to the proof of the right hand inequality of (22).

Let $u := y - y_0$. Then $u$ satisfies

(32) \[ y u'' = -2x u' + y_0'' u \quad ; \quad u(0) = \alpha \, , \quad u'(0) = 0 \, . \]
We want an upper bound for \( u'(\frac{1}{4}) \).

From (24) we know that \( u' > 0 \), and also \( y > \alpha + y_0 \). Moreover, we have 
\[-y_0'' \leq 2 \text{, and, using (30), } y_0(x) > x(1 - x + \frac{1}{3}x^2) > \frac{3}{4}x \text{ on } [0, \frac{1}{4}] \text{. Hence}
\]
(33) \[ u'' \leq 2\left(\alpha + \frac{3}{4}x\right)^{-1}u \quad (0 \leq x \leq \frac{1}{4}) \, . \]

We introduce a function \( v \) as follows: \( v(0) = \alpha \), \( v'(0) = 0 \), \( v \) satisfies (32) with equality. Then \( v(x) \geq u(x) \) on \([0, \frac{1}{4}]\). This follows by standard arguments using the differential (in-)equalities for \( \varphi := \frac{u'}{u}, \Phi := \frac{v'}{v} \). Since \( v \) is convex we have \( v(s) \leq 4sv_1 + \alpha(1 - 4s) \) \((0 \leq s \leq \frac{1}{4})\) where \( v_1 := v\left(\frac{1}{4}\right) \). Hence 
\[ v''(s) \leq 2\left(\alpha + \frac{3}{4}s\right)^{-1} \left(4sv_1 + \alpha(1 - 4s)\right) \text{.} \]
Integrating twice we find
\[ v_1 - \alpha \leq \frac{1}{3}(v_1 - \alpha) - \frac{256}{9}\alpha(v_1 - \alpha - \frac{3}{16})[\left(\frac{1}{4} + \frac{4}{9}\alpha\right) \log(1 + 3(16\alpha)^{-1}) - \frac{1}{4}] \, , \]
from which an upper bound for \( v_1 \) arises, thus also for \( u'(\frac{1}{4}) \). We get
(34) \[ u\left(\frac{1}{4}\right) < \alpha + \alpha \log(1 + 3(16\alpha)^{-1}) \, . \]

Finally we shall show that \( u(\omega)/u\left(\frac{1}{4}\right) < C \), where \( C \) is independent of \( \alpha \) for 
\( \alpha \in \left(0, \frac{1}{3}\right) \). Defining \( \varphi := \frac{u'}{u} \) we get from (32):
\[ \varphi' = 2x(y_0y)^{-1}y_0' - 2xy^{-1}\varphi - \varphi^2 =: F(x, \varphi) \, , \quad \varphi(0) = 0 \, . \]

Let \( \psi(x) := (y_0 + (2xy_0^\frac{1}{2})\frac{1}{2})(y_0)^{-1} \). Then \( \psi'(x) - F(x, \psi(x)) = (2x)^\frac{1}{2}(y_0')^\frac{3}{2}(y_0)^{-2} + (2xy_0^\frac{1}{2})(y_0)^{-1}(x(2y)^{-1} - (y_0)^{-1}) + (2xy_0)^{-1} \). Clearly \( \psi' - F(x, \psi) > 0 \) for 
\( 0 < x < 2^{-\frac{1}{4}} \). Further \( \psi' - F(x, \psi) > 0 \) for \( x \geq 2^{-\frac{1}{4}} \) provided that \( 2(y)^{-1} - (y_0)^{-1} > 0 \) on \([2^{-\frac{1}{4}}, \infty)\). This condition is fulfilled, since by (25) and (28), we have
\[ 2y_0(2^{-\frac{1}{4}}) > (7 - 3\sqrt{2})(3\sqrt{2})^{-1} > (1 + 2\sqrt{\alpha})^2 \frac{1}{2} > (1 + 2\sqrt{\alpha})^2 L_0 > L > y(x) \, , \]
and the second inequality holds if \( 0 < \alpha \leq \frac{1}{3} \).
Further, $F(x,0) > 0 \ (x > 0)$, and $\varphi > 0 \ (x > 0)$. It follows (see [3] p. 177) that $0 < \varphi < \phi$. Therefore,

$$\log(u(\infty)/u(\frac{1}{4})) < \int_{\frac{1}{4}}^{\infty} \psi(x)\,dx = \log(L_0/y_0(\frac{1}{4})) +$$

$$+ \int_{\frac{1}{4}}^{\infty} (2xy_0')^\frac{1}{2}(y_0)^{-1}\,dx = \log(L_0/y_0(\frac{1}{4})) + 2(2y_0(\frac{1}{4}))^{\frac{1}{2}} -$$

$$- 2^{-\frac{1}{2}} \int_{\frac{1}{4}}^{\infty} x^{-3/2}(y_0')^\frac{1}{2}\,dx < \log(L_0/y_0(\frac{1}{4})) + 2(2y_0(\frac{1}{4}))^{\frac{1}{2}}.$$

Using $y_0(\frac{1}{4}) > 37/192$ (from (25)) and $y_0'(\frac{1}{4}) < e^{-\frac{1}{4}}$ (see proof of (26)) we find

$$u(\infty)/u(\frac{1}{4}) < 47L_0. \quad (35)$$

Combining (21), (31), (34) and (35) we can summarize the results of this section by

$$2L_0 \log(1 + L_0\gamma^2) < L(\gamma) - L_0\gamma^2 < 47L_0 \log(1 + 3(16)^{-1}\gamma^2) + 47L_0 \quad (\gamma > 1.8). \quad (36)$$

**Remark.** Numerical experiments suggest that

$$L - L_0\gamma^2 \sim C\log\gamma \quad (\gamma \to \infty).$$
6. Bounds for the limit if \( \gamma < 0 \).

Integrating (1) in the form \( y'' = -2xy^{-1}y' \) over \([1, x]\) we get

\[
y(x) = 1 + \gamma x + 2 \int_0^x (s^2 - xs)(y(s))^{-1} y'(s) \, ds.
\]

Writing \( s^2 - xs = (s - \frac{1}{2}x)^2 - \frac{1}{4}x^2 \) and taking exponentials we arrive at

\[
y(x) = \exp[2x^{-2} + 2\gamma x^{-1} - 2x^{-2} y(x) + I(x)],
\]

where

\[
I(x) := 4x^{-2} \int_0^x (s - \frac{1}{2}x)^2 (y(s))^{-1} y'(s) \, ds.
\]

Since \( I(x) < 0 \) we have

\[
y(x) < \exp[2x^{-2} + 2\gamma x^{-1}] \quad (x > 0).
\]

Obviously, we have the inequality \( L < y(2|\gamma|^{-1}) < \exp[-\frac{1}{4} \gamma^2] \), but we can do much better. First we will show that

\[
L \geq y(2|\gamma|^{-1})(1 - \frac{1}{4} \gamma^2 \exp[-8\sqrt{2} |\gamma|^{-3} \exp[\frac{1}{4} \gamma^2]]).
\]

This inequality states that \( y(2|\gamma|^{-1}) \) is a very good approximation of \( L \) for large \( |\gamma| \).

Secondly, we will obtain sharp inequalities for \( y(2|\gamma|^{-1}) \) which, by (41), will result in sharp inequalities for \( L \).
Utilizing (5) with \( x = a := 2|\gamma|^{-1} - 2\sqrt{2}|\gamma|^{-2} \) and \( x = \beta := 2|\gamma|^{-1} \) respectively, and (40) we have

\[
y'(\beta) > \gamma \exp[-(\beta^2 - \alpha^2)(\gamma(\alpha))^{-1}] > \gamma \exp[-(\beta^2 - \alpha^2) \exp -2\alpha^2 - 2\gamma a^{-1}] > \gamma \exp[-8\sqrt{2}|\gamma|^{-3} \exp[\frac{1}{4} \gamma^2]].
\]

Using this in (8) with \( a = 2|\gamma|^{-1} \) we get (41).

In order to obtain good estimates for \( y(2|\gamma|^{-1}) \) we have, by (38), to find good estimates for \( I(2|\gamma|^{-1}) \). Therefore we put

\[
(42) \quad u(t) := \gamma y(|\gamma|^{-1} t)(y'(|\gamma|^{-1} t))^{-1}.
\]

Then, with \( \varepsilon := 2|\gamma|^{-2} \) we get

\[
(43) \quad \dot{u}(t) = -1 + \varepsilon t u(t) \exp \left[ \int_0^t (u(s))^{-1} ds \right], \quad u(0) = 1,
\]

and

\[
(44) \quad I(2|\gamma|^{-1}) = \int_0^2 (t - 1)^2 (u(t))^{-1} dt.
\]

The solution \( u \) of (43) has the following global behaviour. It starts with the value 1, decreases till \( t = t_0 \), where it attains a minimum \( u_0 := u(t_0) > 0 \) and thereafter it increases very rapidly. Since \( f(t) := u(t) \exp \left[ \int_0^t (u(s))^{-1} ds \right] \) is increasing and \( f(t_0) = (\varepsilon t_0)^{-1} \) we have \( \dot{u}(t) \leq -1 + t/t_0 \) \((0 \leq t \leq t_0)\), from which it follows that \( u(t) \leq 1 - t + \frac{1}{2}t^2/t_0 \), whence \( 0 < u_0 < 1 - \frac{1}{2}t_0 \). Hence \( t_0 < 2 \).
We need the relations

\begin{align}
(45) \quad & t(\dot{u} + 1)^{-1} = \varepsilon^{-1} - \int_0^t s(u(s))^{-1} ds , \\
(46) \quad & \int_0^t s^2(u(s))^{-1} ds = \varepsilon^{-1} + (u + t) \left( \int_0^t s(u(s))^{-1} ds - \varepsilon^{-1} \right) ,
\end{align}

the correctness of both is easily checked by differentiation. Substituting \( t_0 = t_0 \) in (42), (45) and (46) we obtain

\begin{align}
(47) \quad & \int_0^{t_0} (u(s))^{-1} ds = -\log(\varepsilon t_0 u_0) , \\
(48) \quad & \int_0^{t_0} s(u(s))^{-1} ds = \varepsilon^{-1} - t_0 , \\
(49) \quad & \int_0^{t_0} s^2(u(s))^{-1} ds = \varepsilon^{-1} - t_0^2 - u_0 t_0 .
\end{align}

We will now prove that \( t_0 > 1 \) for \( \varepsilon > 0 \) sufficiently small. For suppose that \( t_0 \leq 1 \). Then on the one hand we have

\[ t_0^2 + u_0 t_0 - t_0 = \int_0^{t_0} (s - s^2)(u(s))^{-1} ds \geq \int_0^{t_0} (s - s^2) ds = \frac{1}{2} t_0^2 - \frac{1}{3} t_0^3 , \]

which implies \( u_0 > 1 - \frac{1}{2} t_0 - \frac{1}{3} t_0^2 \geq \frac{1}{6} \). On the other hand we have

\[ \varepsilon^{-1} - 1 \leq \int_0^{t_0} s(u(s))^{-1} ds \leq \int_0^{t_0} (u(s))^{-1} ds = -\log(\varepsilon t_0 u_0) < -\log(\varepsilon u_0) , \]

implying that \( u_0 < \varepsilon^{-1} \exp[1 - \varepsilon^{-1}] \) which contradicts \( u_0 \geq \frac{1}{6} \) for \( 0 < \varepsilon < 0.23 \).
In the sequel we need the inequalities

\[(50) \quad (\varepsilon t_0)^{-1} + \log(\varepsilon t_0) - 1 < \log(u_0^{-1}) < \varepsilon^{-1} + \log(\varepsilon t_0) + 1 - t_0.\]

Using (47), (48) and the inequality \(u(s) \geq 1 - s (0 \leq s \leq 1)\) we can write

\[
(\varepsilon^{-1} - t_0) = \int_0^t s(u(s))^{-1} \, ds < \int_0^t (u(s))^{-1} \, ds = -t_0 \log(\varepsilon t_0 u_0); \\
(\varepsilon^{-1} - t_0) > \int_0^1 s(u(s))^{-1} \, ds + \int_0^t (u(s))^{-1} \, ds = \\
\int_0^1 (s - 1)(u(s))^{-1} \, ds + \int_0^t (u(s))^{-1} \, ds > -1 + \log(\varepsilon t_0 u_0)^{-1},
\]

from which (50) follows.

The next stage in our treatment is the proof of

\[(51) \quad t_0 - 1 < \frac{1}{2} \varepsilon + 4\varepsilon^2 |\log \varepsilon| \quad (0 < \varepsilon \leq 0.05).\]

Since \(\bar{u} = t^{-1}(\dot{u} + 1) + u^{-1}(\ddot{u} + 1)^2 > 0\) on \([0, t_0]\) we have

\[(52) \quad u(t) \leq t_0 + u_0 - t \quad (0 \leq t \leq t_0).\]

Therefore

\[(53) \quad \int_1^{t_0} (s^2 - s)(u(s))^{-1} \, ds \geq \int_1^{t_0} (s^2 - s)(t_0 + u_0 - s)^{-1} \, ds = \\
= (t_0 - 1)[(u_0 + t_0) \log(u_0^{-1}(u_0 + t_0 - 1)) + \frac{1}{2} - \frac{3}{2} t_0 - u_0] > \\
> (t_0 - 1)[t_0 \log(u_0^{-1}(t_0 - 1)) + \frac{1}{2} - \frac{3}{2} t_0 - u_0].\]
Using (48), (49), (50) and the inequality $u(t) \geq 1 - t$ on $[0,1]$ we find

$$t_0(1 - t_0 - u_0) = \int_0^1 (s^2 - s)(u(s))^{-1}ds + \int_1^{t_0} (s^2 - s)(u(s))^{-1}ds >$$

$$> -\frac{1}{2} + \int_1^{t_0} (s^2 - s)(u(s))^{-1}ds ,$$

which with (53) leads to

$$(t_0 - 1)[\varepsilon^{-1} + t_0 \log(\varepsilon t_0(t_0 - 1))) - \frac{3}{2}t_0 + \frac{1}{2} + u_0(t_0 - 1)^{-1}] < \frac{1}{2}$$

from which (51) follows.

Next we show that

$$\int_0^2 (t - 1)^2(u(t))^{-1}dt \leq 3\varepsilon^2 \quad (0 < \varepsilon \leq 0.05) .$$

Since $\bar{u} = 3(\bar{u} + 1)^2(tu)^{-1} + (\bar{u} + 1)^2(\bar{u} + 2)u^{-2} > 0$ we have $\bar{u}(t) > \bar{u}(t_0) = t_0^{-1} + u_0^{-1} > u_0^{-1}$ for $t > t_0$ whence $u(t) > u_0 + \frac{1}{2}u_0(t^2 - t_0^2)$ $(t > t_0)$.

Now, using this latter inequality and (50), (51) we have

$$\int_0^2 (s - 1)^2(u(s))^{-1}ds = \int_0^{t_0} [(t_0 - 1)^2 + 2(t_0 - 1)(s - t_0) +$$

$$+ (s - t_0)^2](u(s))^{-1}ds \leq$$

$$\leq (t_0 - 1)^22^{-\frac{1}{2}} \pi + (2(t_0 - 1)u_0 + u_0^2)\log(1 + \frac{1}{2}u_0^{-2}) < 3\varepsilon^2 \quad (0 < \varepsilon \leq 0.05).$$

Finally, we are able to estimate $\int_0^{t_0} (1 - s)^2(u(s))^{-1}ds$ as follows, using
(48), (51) and (55),

\[
\int_0^{t_0} (1 - s)^2 (u(s))^{-1} ds < \int_0^1 (1 - s)^2 (1 - s)^{-1} ds + \\
+ (t_0 - 1)^2 \int_0^{t_0} s(u(s))^{-1} ds \leq \\
\leq \frac{1}{2} + \left( \frac{1}{4} \varepsilon^2 + 4 \varepsilon^2 |\log \varepsilon| + 16 \varepsilon^4 \log^2 \varepsilon \right) \left( \frac{1}{\varepsilon} - t_0 \right) + \frac{1}{2} + \frac{1}{3} \varepsilon \quad (0 < \varepsilon \leq 0.05),
\]

and, using (52) and (51) and (50)

\[
\int_0^{t_0} (1 - s)^2 (u(s))^{-1} ds > \int_0^1 (1 - s)^2 (u_0 + t_0 - s)^{-1} ds = \\
= \frac{1}{2} - (t_0 - 1) + [(u_0 + 2t_0 - 2)u_0 + (t_0 - 1)^2] \log(u_0 + t_0)(u_0 + t_0 - 1)^{-1} - u_0 > \\
> \frac{1}{2} - (t_0 - 1) - u_0 - \frac{1}{2} \varepsilon - 4 \varepsilon^2 |\log \varepsilon| - \frac{1}{\varepsilon} \frac{1}{\varepsilon} \quad (0 < \varepsilon \leq 0.05).
\]

Combining the last two inequalities with (38), (41), (44), (54) and (55) we can obtain

(56) \quad 1 - \gamma^{-2} \quad L(\gamma) \exp[\frac{1}{2} \gamma^2 + \frac{1}{4}] < 1 + \gamma^{-2} \quad (\gamma < -7).

Remark. Numerical experiments suggest that

\[ L(\gamma) \exp[\frac{1}{2} \gamma^2 + \frac{1}{4}] - 1 \sim C \gamma^{-2} \quad (\gamma \to -\infty). \]
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References.

