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Published: 01/01/1994

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

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$H_2$ Almost Disturbance Decoupling Problem with Internal Stability

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Eindhoven, August 1994
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For both continuous-time and discrete-time systems, we establish the necessary and sufficient conditions under which an $H_2$ almost disturbance decoupling problem with internal stability is solvable by proper controllers and/or strictly proper controllers.
1. Introduction

In classical as well as modern control theory, the problem of disturbance decoupling or almost disturbance decoupling occupies a central part. It plays a central role in several important problems such as $H_2$ and $H_\infty$ optimal control, decentralized control, non-interacting control, model reference or tracking control. Regardless of from where the problem arises, the basic almost disturbance decoupling problem can be stated as follows. Consider a linear time invariant plant (see Fig. 1) with the control input $u$, the measurement output $y$, the output to be controlled $z$ and an exogenous disturbance input $d$. Then the basic problem is to design a linear time invariant compensator or controller accepting $y$ as its input and generating the control signal $u$ as its output such that the controlled output $z$ is either exactly or approximately (in certain sense) decoupled from the disturbance $d$ while guaranteeing the internal stability of the resulting closed-loop feedback system. In the acronym of Willems (\cite{7}), the above problem can be termed ADDPMS, the almost disturbance decoupling problem with measurement feedback and internal stability.

One can cast the ADDPMS problem, for example, into optimization problems where the $H_\infty$ norm or the $H_2$ norm of the transfer function from the disturbance $d$ to the controlled output $z$ is to be reduced to any arbitrarily specified small value. Consequently, these problems are respectively referred to as ($H_\infty$-ADDPMS), and ($H_2$-ADDPMS). The ($H_\infty$-ADDPMS) is now well-studied and well-understood. Necessary and sufficient conditions in terms of geometric subspaces to solve ($H_\infty$-ADDPMS) were given by Weiland and Willems (\cite{6}). Algorithms for the explicit construction of the controllers that solve the ($H_\infty$-ADDPMS) were recently given by Saberi and his co-authors (\cite{3} and \cite{4}). The ($H_2$-ADDPMS), on the other hand, though important to the solutions of $H_2$ optimal control problems (see, for example, \cite{1}), have not been systematically studied. The goal of this paper is to establish the necessary and sufficient conditions under which the ($H_2$-ADDPMS) is solvable. Both continuous-time case and discrete-time case will be considered.

The rest of the paper is organized as follows. We consider the continuous-time case and the discrete-time case in Sections 2 and 3 respectively. In each case, we start with the formal definition of the problem and then present and prove our main results regarding the solvability conditions of the problem.

2. The Continuous-Time Case

In this section, we study the ($H_2$-ADDPMS) for continuous-time systems. We will first formally formulate this problem in Section 2.1 and present our main results regarding the necessary and sufficient conditions for solving this problem in Section 2.2. The proof of these
main results are given in Section 2.3.

2.1. Problem Statement

Consider a linear time-invariant system described by

\[
\begin{align*}
\dot{x} &= Ax + Bu + Ew \\
\Sigma : & \quad y = C_1 x + D_1 w \\
& \quad z = C_2 x + D_2 u
\end{align*}
\]  

(2.1)

where state \( x \in \mathbb{R}^n \), control input \( u \in \mathbb{R}^m \), measured output \( y \in \mathbb{R}^p \), controlled output \( z \in \mathbb{R}^q \) and disturbance \( d \in \mathbb{R}^r \). Here we have assume that the feedthrough terms both from \( w \) to \( z \) and from \( u \) to \( y \) are nonexistent. We note that to achieve the \((H_2-ADDPMS)\) as defined below, it is necessary that the feedthrough term from \( w \) to \( z \) be nonexistent. We also note that assuming that the feedthrough term from \( u \) to \( y \) is nonexistent is without loss of generality.

For notational convenience, throughout this paper, \( \Sigma_1 \) denotes \((A, E, C_1, D_1)\) and \( \Sigma_2 \) denotes \((A, B, C_2, D_2)\).

We have the following formal definition for the \((H_2-ADDPMS)\).

**Definition 2.1.** Consider the system \( \Sigma \). The \( H_2 \) almost disturbance decoupling problem with measurement feedback and stability \((H_2-ADDPMS)\) is solvable if for all \( \eta > 0 \), there exists a linear, proper measurement feedback controller, say \( \Sigma_0 \), such that

1. The closed-loop system comprising of \( \Sigma \) and \( \Sigma_0 \) is internally stable;
2. The \( H_2 \) norm of the closed-loop transfer function from \( w \) to \( z \) is less than or equal to \( \eta \), i.e. \( \|T_{zw}\|_{H_2} \leq \eta \) where

\[
\|T_{zw}\|_{H_2}^2 := \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}T_{zw}(jw)T_{zw}^*(jw) dw.
\]

Moreover, we label an \((H_2-ADDPMS)\) as \((H_2-ADDPMS)\)sp if we restrict our controllers to be strictly proper.

The goal of this part of the paper is to establish the necessary and sufficient conditions, in terms of the original data of the system, under which the \((H_2-ADDPMS)\) as defined above for the continuous-time system \( \Sigma \) is solvable.
2.2. Solvability Conditions for \((H_2\text{-ADDPMS})\)

In this subsection, we develop the necessary and sufficient conditions, in terms of the original data of the system \(\Sigma\), under which the \((H_2\text{-ADDPMS})\) is solvable.

Some of our necessary and sufficient conditions are to be expressed in subspace inclusions. As a matter of fact, the most difficult task in deriving these conditions is the characterization of some geometric subspaces in term of the original system data. The following subspaces are needed in our expressions of the these conditions.

**Definition 2.2.** Consider a linear system \(\Sigma\) characterized by the quadruple \((A, B, C, D)\). Then,

1. The stabilizable weakly unobservable subspace \(V^g(\Sigma)\) is defined as the maximal subspace of \(\mathbb{R}^n\) which is \((A + BF)\)-invariant and contained in \(\text{Ker}(C + DF)\) such that the eigenvalues of \((A + BF)|V^g\) are contained in \(C_g \subseteq C\) for some \(F\).

2. The detectable strongly controllable subspace \(S^g(\Sigma)\) is defined as the minimal \((A + KC)\)-invariant subspace of \(\mathbb{R}^n\) containing in \(\text{Im}(B + KD)\) such that the eigenvalues of the map which is induced by \((A + KC)\) on the factor space \(\mathbb{R}^n/S^g\) are contained in \(C_g \subseteq C\) for some \(K\).

For the case when \(C_g = C\), \(C_g = C^-\) and \(C_g = C^- \cup C^0\) the superscript \(g\) in \(V^g\) and \(S^g\) is replaced by a superscript \(*\), \(-\) and \(\text{-}0\) respectively.

We are now ready to state our necessary and sufficient conditions for the continuous-time \((H_2\text{-ADDPMS})\) to be solvable.

**Theorem 2.1.** Consider the system \(\Sigma\) as given by (2.1). Then the following three statements are equivalent:

1. The \((H_2\text{-ADDPMS})\) is solvable;

2. The \((H_2\text{-ADDPMS})_\text{sp}\) is solvable;

3. The following conditions hold:

   (a) \((A, B)\) is stabilizable;

   (b) \((A, C_1)\) is detectable;

   (c) \(\text{Im}E \subseteq S^*(\Sigma_2) + V^{-0}(\Sigma_2)\)

   (d) \(V^*(\Sigma_1) \cap S^{-0}(\Sigma_1) \subseteq \text{Ker}C_2\)
Remark 2.1. Note that if $\Sigma_2$ is right invertible with no invariant zeros in $C^+$ and $\Sigma_1$ is left invertible with no invariant zeros in $C^+$, then the $H_2$-almost disturbance decoupling problem with measurement feedback and stability is solvable.

Theorem 2.2. Consider the system $\Sigma$ as given by (2.1). Let $C_1 = I$ and $D_1 = 0$. Then the following three statements are equivalent:

1. The $(H_2$-ADDPMS) is solvable via static feedbacks;
2. The $(H_2$-ADDPMS) is solvable via dynamic feedbacks;
3. The following conditions hold:
   (a) $(A, B)$ is stabilizable;
   (b) $\text{Im}E \subseteq S^*(\Sigma_2) + V^{-0}(\Sigma_2)$;

2.3. Proofs of Theorem 2.1 and Theorem 2.2

To carry out the proofs of Theorem 2.1 and Corollary 2.1, we need some preliminaries. For system $\Sigma$ as defined by (2.1), define the following continuous-time linear matrix inequalities (CLMIs),

$$L_r(P) = \begin{bmatrix} A'P + PA + C_2' C_2 & PB + C_1' D_2 \\ B'P + D_2' C_2 & D_2' D_2 \end{bmatrix} \geq 0 \quad (2.2)$$

and

$$L_q(Q) = \begin{bmatrix} AQ + QA' + EE' & QC_1' + ED_1' \\ C_1 Q + D_1 E' & D_1 D_1 \end{bmatrix} \geq 0 \quad (2.3)$$

A real symmetric solution of the CLMI $L_r(P) \geq 0$ is said to be semi-stabilizing if it satisfies

$$\text{Rank} \left[ \begin{bmatrix} sI - A & -B \\ L_r(P) \end{bmatrix} \right] = n + \text{normrank}G(s)$$

where $G(s) := C_2(sI - A)^{-1}B + D_2$.

Similarly, a real symmetric solution of CLMI $L_q(Q) \geq 0$ is said to be semi-stabilizing if it satisfies

$$\text{Rank} \left[ \begin{bmatrix} sI - A & -C_1 \\ L_q(Q) \end{bmatrix} \right] = n + \text{normrank}H(s)$$

where $H(s) := C_1(sI - A)^{-1}E + D_1$.

It is well-known that if $(A, B)$ is stabilizable then there exists a unique semi-stabilizing solution for the CLMI $L_r(P) \geq 0$ and similarly if $(A, C_1)$ is detectable then there exists a
unique semi-stabilizing solutions for the CLMI $L_Q(Q) \geq 0$. Moreover, both of these semi-stabilizing solutions are positive semi-definite.

We are now ready to give the proofs for Theorem 2.1 and Theorem 2.2.

**Proof of Theorem 2.1**: First we observe that $(H_2$-ADDPMS) for the system $\Sigma$ is solvable if and only if the infimum $\gamma^*$ of the $H_2$ optimal control problem for the the system $\Sigma$ is equal to zero. The equivalence between Statements 1 and 2 follows from the fact that, for the continuous-time systems, $\gamma^*$ remains the same if we restrict to the strictly proper controllers (see [5]). We now proceed to show the equivalence between Statements 1 and 3. We note that, under Conditions (a) and (b), the above mentioned $\gamma^*$ is given by

$$\gamma^* = (\text{Trace}(E'PE) + \text{Trace}(A'P + PA + C_2'C_2Q))^\frac{1}{2}$$  \hspace{1cm} (2.4)

We will show that $\gamma^* = 0$ if and only if

(i) $\text{Im}E \subseteq \text{Ker}P$;

(ii) $\text{Im}Q \subseteq \text{Ker}C_2$;

(iii) $\text{Ker}P \subseteq \text{Im}Q$;

It is obvious that $\gamma^* = 0$, together with the fact that $(A'P + PA + C_2'C_2) \geq 0$, immediately implies $PE = 0$. On the other hand we can easily rewrite our expression for $\gamma^*$ as:

$$\gamma^* = (\text{Trace}(C_2QC_2) + \text{Trace}(QA' + AQ + EE')P)^\frac{1}{2}$$  \hspace{1cm} (2.5)

(where we only used that $\text{Trace } S = \text{Trace } S'$ and $\text{Trace } ST = \text{Trace } TS$). Since we also know $QA' + AQ + EE' \geq 0$ we find $C_2Q = 0$. Remains to prove part (iii). Given (i) and (ii) we know from (2.4) that $\gamma^* = 0$ if and only if $Q(A'P + PA) = 0$. On the other hand from (2.5) we find $QA'P + AQ = 0$. Combining the two we find:

$$QPA - AQP = 0.$$

Moreover, from the linear matrix inequality we find $QP = 0$ while we already have $PE = 0$. Hence we have

$$QP(A + BF_1 + EF_2) - AQP = 0$$  \hspace{1cm} (2.6)

Choose an appropriate basis, we find

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \quad E = \begin{pmatrix} E_1 \\ 0 \end{pmatrix}$$

such that $(A_{11}, (B_1, E_1))$ is controllable. Since $(A, B)$ is stabilizable we have that $A_{22}$ is stable and it is then straightforward to check that in this basis we have:

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \quad Q = \begin{pmatrix} Q_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$
Next we study the $(1,1)$ component of (2.6). We get:

\[ Q_{11}P_{11}(A_{11} + B_1F_{11} + E_1F_{21}) - A_{11}Q_{11}P_{11} = 0 \]  

(2.7)

Choose $F_{11}$ and $F_{21}$ such that $A_{11}$ and $A_{11} + B_1F_{11} + E_1F_{21}$ have no eigenvalues in common. Then standard theory tells us the Sylvester equation (2.7) has a unique solution $Q_{11}P_{11} = 0$

However, this implies that:

\[ \text{Trace } PQ = \text{Trace } \begin{pmatrix} 0 & 0 \\ P_{21}Q_{11} & 0 \end{pmatrix} = 0 \]

and, since $PQ$ is the product of two positive semi-definite matrices, this implies $PQ = 0$.

Finally, noting that Ker$P$ and Im$Q$ are characterized in [5] as

\[ \text{Ker}P = S^*(\Sigma_2) + \mathcal{V}^{-0}(\Sigma_2), \quad \text{Im}Q = \mathcal{V}^*(\Sigma_1) \cap S^{-0}(\Sigma_1) \]

This completes our proof.

\textbf{Proof of Theorem 2.2 :} In the case that $C_1 = I$ and $D_1 = 0$, it is known (see [5]) that $Q = 0$ and hence

\[ \gamma^* = \left(\text{Trace}(E'PE)\right)^{\frac{1}{2}} \]

The equivalence between Statements 1 and 2 follows from the well-known fact that the above $\gamma$ remains the same whether static or dynamic feedbacks are used. The equivalence between Statements 1 and 3 follows from the fact, that in this case, $\mathcal{V}^*(\Sigma_1) = \{0\}$ and hence the conditions in Statement 3 of Theorem 2.1 are equivalent to those given in Statement 3 of this theorem.

\textbf{3. The Discrete-Time Case}

In this section, we study the $(H_2$-ADDPMS) for discrete-time systems. As in the continuous-time case, we will first formally formulate this problem in Section 3.1 and present our main results regarding the necessary and sufficient conditions for solving these problems in Section 3.2. The proof of our results is given in Section 3.3.

\textbf{3.1. Problem Statement}

Consider a linear time-invariant system described by

\[ \Sigma : \begin{cases} x(k+1) = Ax(k) + Bu(k) + Ew(k) \\ y(k) = C_1x(k) + D_1w(k) \\ z(k) = C_2x(k) + D_2u(k) \end{cases} \]

(3.1)
where state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, measured output $y \in \mathbb{R}^p$, controlled output $z \in \mathbb{R}^q$ and disturbance $d \in \mathbb{R}^r$. As in the continuous-time case, we have assumed that the feedthrough terms from both $w$ to $z$ and $u$ to $y$ are nonexistent.

For notational convenience, throughout this section, $\Sigma_1$ denotes $(A, E, C_1, D_1)$ and $\Sigma_2$ denotes $(A, B, C_2, D_2)$.

We have the following formal definition for the $(H_2$-ADDPMS).

**Definition 3.1.** Consider the system $\Sigma$. The $H_2$ almost disturbance decoupling problem with measurement feedback and stability $(H_2$-ADDPMS) is solvable if for all $\eta > 0$, there exists a linear, proper measurement feedback controller, say $\Sigma_0$, such that

1. The closed-loop system comprising of $\Sigma$ and $\Sigma_0$ is internally stable;
2. The $H_2$ norm of the closed-loop transfer function from $w$ to $z$ is less than or equal to $\eta$, i.e. $\|T_{zw}\|_{H_2} \leq \eta$ where

$$\|T_{zw}\|_{H_2}^2 := \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Trace}T_{zw}(e^{jw})^*T_{zw}(e^{jw})dw.$$  

Moreover, we label an $(H_2$-ADDPMS) as $(H_2$-ADDPMS)$_{sp}$ if we restrict our controllers to be strictly proper.

The goal of this part of the paper is to establish the necessary and sufficient conditions under which the $(H_2$-ADDPMS) as defined above for the discrete-time system $\Sigma$ are solvable.

### 3.2. Solvability Conditions for $(H_2$-ADDPMS)

In this subsection, we develop the necessary and sufficient conditions under which the $(H_2$-ADDPMS) is solvable.

Before stating our results we have to define some geometric subspaces. We denote by $V^\circ(\Sigma)$ and $S^\circ(\Sigma)$ the subspaces $V^\circ(\Sigma)$ and $S^\circ(\Sigma)$ respectively where $C_2$ is equal to the closed unit disc.

We now state our necessary and sufficient conditions under which the discrete-time $(H_2$-ADDPMS)/$(H_2$-ADDPMS)$_{sp}$ are solvable.

Our first theorem concerns the $(H_2$-ADDPMS)$_{sp}$.

**Theorem 3.1.** Consider the system $\Sigma$ as given by (3.1). The $(H_2$-ADDPMS)$_{sp}$ is solvable if and only if

1. $(A, B)$ is stabilizable;
2. \((A, C_1)\) is detectable;
3. \(\text{Im}E \subseteq \mathcal{V}^\oplus(\Sigma_2)\);
4. \(\mathcal{S}^\oplus(\Sigma_1) \subseteq \text{Ker}C_2\);
5. \(A\mathcal{S}^\oplus(\Sigma_1) \subseteq \mathcal{V}^\oplus(\Sigma_2)\).

**Remark 3.1.** We will see in the proof that 4. and 5. together imply that

\[
\mathcal{S}^\oplus(\Sigma_1) \subseteq \mathcal{V}^\oplus(\Sigma_2) \tag{3.2}
\]

Or, via a dual argument, we can show that 3. and 5. also imply (3.2).

**Remark 3.2.** Note that if \(\Sigma_2\) is right invertible with no invariant zeros outside the unit disc (including infinity) and \(\Sigma_1\) is left-invertible with no invariant zeros outside the unit disc (again including infinity), then the \((H_2-\text{ADDPMS})^p\) is solvable.

We next give the necessary and sufficient conditions under which the \((H_2-\text{ADDPMS})\) is solvable.

**Theorem 3.2.** Consider the system \(\Sigma\) as given by (3.1). The \((H_2-\text{ADDPMS})\) is solvable if and only of

1. \((A, B)\) is stabilizable;
2. \((A, C_1)\) is detectable;
3. \(\text{Im}E \subseteq \mathcal{V}^\oplus(\Sigma_2) + B\text{Ker}D_2\);
4. \((\mathcal{S}^\oplus(\Sigma_1) \cap C_1^{-1}\text{Im}D_1) \subseteq \text{Ker}C_2\)
5. \(\mathcal{S}^\oplus(\Sigma_1) \subseteq \mathcal{V}^\oplus(\Sigma_2)\)

Finally, we discuss the special case when \(C_1 = I\) and \(D_1 = 0\).

**Theorem 3.3.** Consider the system \(\Sigma\) as given by (3.1). If \(C_1 = I\) and \(D_1 = 0\), then the following three statements are equivalent:

1. The \((H_2-\text{ADDPMS})\) is solvable via static state feedback;
2. The \((H_2-\text{ADDPMS})\) is solvable via dynamic state feedback;
3. \((A, B)\) is stabilizable and \(\text{Im}E \subseteq \mathcal{V}^\oplus(\Sigma_2)\);
3.3. Proofs of Theorems 3.1, 3.2 and 3.3

To carry out the proofs of Theorems 3.1 and 3.2, we need to define the following discrete-time linear matrix inequalities (DLMIs),

\[
L_P(P) = \begin{bmatrix} A'PA - P + C_2^T C_2 & A'PB + C_2^T D_2 \\ B'PA + D_2^T C_2 & D_2^T D_2 + B'PB \end{bmatrix} \succeq 0 \tag{3.3}
\]

and

\[
L_Q(Q) = \begin{bmatrix} AQA' - Q + EE' & AQ C_1' + E D_1' \\ C_1 QA' + D_1 E' & D_1 D_1 + C_1 Q C_1' \end{bmatrix} \succeq 0 \tag{3.4}
\]

A real symmetric solution of the CLMI \(L_P(P) \succeq 0\) is said to be semi-stabilizing if it satisfies

\[
\text{Rank} \begin{bmatrix} zI - A & -B \\ L_P(P) \end{bmatrix} = n + \text{normrank}G(z)
\]

where \(G(z) := C_2^T (zI - A)^{-1} B + D_2\).

Similarly, a real symmetric solution of CLMI \(L_Q(Q) \succeq 0\) is said to be semi-stabilizing if it satisfies

\[
\text{Rank} \begin{bmatrix} zI - A & L_Q(Q) \\ -C_1 \end{bmatrix} = n + \text{normrank}H(z)
\]

where \(H(z) := C_1^T (zI - A)^{-1} E + D_1\).

It is well-know that if \((A, B)\) is stabilizable then there exists a unique semi-stabilizing solution for the CLMI \(L_P(P) \succeq 0\) and similarly if \((A, C_1)\) is detectable then there exists a unique semi-stabilizing solutions for the CLMI \(L_Q(Q) \succeq 0\). Moreover, both of these semi-stabilizing solutions are positive semi-definite.

Proof of Theorem 3.1:

Necessity: we have

\[
\gamma^* = (\text{Trace}(E'PE) + \text{Trace}(A'PA - P + C_2^T C_2)Q)^{\frac{1}{2}} \tag{3.5}
\]

Since \(A'PA - P + C_2^T C_2 \succeq 0\) we find that \(\gamma^* = 0\) implies \(PE = 0\). We can easily derive the following alternative expression for \(\gamma^*:\)

\[
\gamma^* = (\text{Trace}(C_2 QC_2') + \text{Trace}(AQA' - Q + EE')P)^{\frac{1}{2}} \tag{3.6}
\]

(where we only used that \(\text{Trace} S = \text{Trace} S' \) and \(\text{Trace} ST = \text{Trace} TS\)). Since we also know that \(AQA' - Q + EE' \succeq 0\) we get that \(\gamma^* = 0\) implies \(C_2 Q = 0\). From (3.5) we then obtain that \(\gamma^* = 0\) if \((A'PA - P)Q = 0\) and from (3.6) we obtain that \((AQA' - Q)P = 0\).

Combining the two yields:

\[
QPA = AQA'PA = AQP
\]
Moreover, we have $PE = 0$ and from (3.3) we obtain that $(A'PA - P + C_2'Q) = 0$ and $C_2Q = 0$ implies $B'PAQ = 0$ and hence $B'PQ = B'PAQA' = 0$. We obtain

$$QP(A + BF_1 + EF_2) - AQ = 0$$

(3.7)

This is the same equation as (2.6) and hence we can use the same arguments to yield that $PQ = 0$. Hence, we get $PAQA'P = PQP = 0$ which implies $PAQ = 0$.

**Sufficiency:** Suppose we have $PAQ = 0$ and $C_2Q = 0$. Then we get that $Q(A'PA - P + C_2C_2)Q = -QPQ$. On the other hand since $P \geq 0$ and $A'PA - P + C_2C_2 \geq 0$ we obtain $QPQ = 0$. In other words $PQ = 0$. It is then straightforward to check that $\gamma^* = 0$.

Finally, recalling the characterization of $KerP = V^\oplus(\Sigma_2)$ and $ImQ = S^\oplus(\Sigma_1)$ from [5], we complete the proof.

**Proof of Theorem 3.2:** Suppose we allow for a non-strictly proper controller with direct feedthrough matrix $N$. Then, for fixed $N$ we have:

$$\gamma^* = \text{Trace}(E_NPE_N') + \text{Trace}(A_NQ'A_N' - Q + E_NE_N'P) + \text{Trace}(D_2N_1'(D_2N_1))^{\frac{1}{2}}$$

where $P$ and $Q$ satisfy the following linear matrix inequalities:

$$\tilde{L}_P(P) = \begin{bmatrix} A_N'PA_N - P + C_{2,N}C_{2,N} & A'_N PB + C_{2,N}D_2 \\ B'PA_N + D_{2,N}C_{2,N} & D_2^2 + B'^2P \end{bmatrix} \geq 0$$

and

$$\tilde{L}_Q(Q) = \begin{bmatrix} A_NQA_N' - Q + E_NE_N' & A_NQC_1' + E_ND_1' \\ C_1QA_N' + D_1E' & D_1D_1 + C_1QC_1' \end{bmatrix} \geq 0$$

where

$$A_N = A + BNC_1, E_N = E + BND_1, C_{2,N} = C_2 + D_2N_1$$

The above is obvious by applying the preliminary feedback $u = Ny + v$. We note that it is necessary for $\gamma^* = 0$ that $D_2N_1 = 0$. We have the following transformation:

$$\begin{pmatrix} I & -C_1'N' \\ 0 & I \end{pmatrix} \tilde{L}(P) \begin{pmatrix} I & 0 \\ -NC_1 & I \end{pmatrix} = L(P).$$

and hence it is easy to check that the stabilizing solution of the DLMI does not depend on $N$. Similarly, it is easy to prove that the preliminary feedback does not change the stabilizing solution of the dual DLMI. From theorem 3.1 we find that for this specific direct feedthrough matrix the $H_2$-ADDPMS is solvable if and only if

- (a) $(A, B)$ is stabilizable;
- (b) $(A, C_1)$ is detectable;
- (c) $P(E + BND_1) = 0$;
- (d) $(C_2 + D_2NC_1)Q = 0$
e. \(P(A + BNC_1)Q = 0\)
f. \(D_2ND_1 = 0\).

We also know from the proof of theorem 3.1 that the above conditions imply \(PQ = 0\). Now observe that, according to [2, Lemma 3.2],

\[
\begin{pmatrix} A & E \\ C_2 & 0 \end{pmatrix} \left[ \text{Im} \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} \right] \subseteq \text{Ker} \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix} + \text{Im} \begin{pmatrix} B \\ D_2 \end{pmatrix} \tag{3.8}
\]

and

\[
\begin{pmatrix} A & E \\ C_2 & 0 \end{pmatrix} \left[ \text{Im} \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} \cap \text{Ker} \begin{pmatrix} C_1 & D_1 \end{pmatrix} \right] \subseteq \text{Ker} \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix} \tag{3.9}
\]

are equivalent to the existence of a matrix \(N\) that satisfies \(-\)f. First consider (3.8) which, given \(PQ = 0\), is easily seen to be equivalent to

\[
\text{Im}E \subseteq \text{Ker}P + B\text{Ker}D_2 \tag{3.10}
\]

and the existence of a matrix \(F\) such that:

\[
P(A + BF)Q = 0, \quad (C_2 + D_2F)Q = 0 \tag{3.11}
\]

On the other hand, we know \(\text{Ker}P = V^{\delta}(\Sigma_1)\) and hence there exists a matrix \(F\) such that \(\text{Ker}P\) is \(A + BF\) invariant and contained in \(\text{Ker}(C_2 + D_2F)\). But then \(PQ = 0\) immediately implies (3.10) for this particular choice of \(F\). In other words, given that \(PQ = 0\), (3.8) is equivalent to (3.11).

Similarly we can show that, given \(PQ = 0\), (3.9) is equivalent to

\[
(\text{Im}Q \cap C_1^{-1}\text{Im}D_1) \subseteq \text{Ker}C_2.
\]

If we remember that \(PQ = 0\) was implied by conditions d. and e., we find, in conclusion that conditions a.-f. are equivalent to \(PQ = 0\) together with (3.10) and (3.11). Combined with our characterizations of \(\text{Ker}P\) and \(\text{Im}Q\) this yields the desired result.

**Proof of Theorem 3.3**: It is known that the infimum for the \(H_2\) optimal control problem using static and/or dynamic state feedback is the same. \((H_2-\text{ADDPS})\) is solvable iff this infimum \(\gamma^* = 0\). This concludes the equivalence of Statements 1 and 2. To establish the equivalence of Statements 2 and 3, we first observe that in this case, \(\gamma^* = (\text{Trace}(E'PE))^\frac{1}{2}\). Hence \(\gamma^* = 0\) if and only if \(\text{Im}E \subseteq \text{Ker}P\). But \(\text{Ker}P = V^{\delta}(\Sigma_2)\) (see [5]) and this concludes the proof. We also note that the equivalence of Statements 2 and 3 also follows from Theorem 3.2. To see this, we note that in the case that \(C_1 = I\) and \(D_1 = 0\), the conditions of Theorem 3.2 reduces to the following conditions

1. \((A, B)\) is stabilizable;
2. $(A, I)$ is detectable;
3. $\text{Im}E \subseteq \mathcal{V}^\oplus(\Sigma_2) + B\text{Ker}D_2$;
4. $\{0\} \subseteq \text{Ker}C_2$;
5. $\mathcal{S}^\oplus(\Sigma_1) \subseteq \mathcal{V}^\oplus(\Sigma_2)$;

Note also that 2. and 4 are automatically satisfied. Moreover, in this case, it is easy to verify that $\mathcal{S}^\oplus(\Sigma_1) = \text{Im}E$ and hence 3 and 5 become second part of condition 3 of this theorem.

4. Conclusions

In this paper, we have studied the $H_2$ almost disturbance decoupling problems with internal stability for both discrete-time and continuous-time systems. Necessary and sufficient conditions are derived under which these problems are solvable. These conditions are expressed in the original data of the given system.

References
