Anticipating synchronization of chaotic Lur'e systems

Huijberts, H.J.C; Nijmeijer, H.; Oguchi, T.

Published in:
Chaos

DOI:
10.1063/1.2710964

Published: 01/01/2007

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the author's version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

Citation for published version (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal ?

Take down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Download date: 04. Dec. 2018
Anticipating synchronization of chaotic Lur’e systems

Henri Huijberts
School of Engineering and Materials Science, Queen Mary, University of London, Mile End Road, London E1 4NS, United Kingdom

Henk Nijmeijer
Department of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Toshiki Oguchi
Department of Mechanical Engineering, Tokyo Metropolitan University, 1-1 Minami-Osawa, Hachioji-shi, Tokyo 192-0397, Japan

(Received 8 December 2006; accepted 1 February 2007; published online 21 March 2007)

In this paper we consider the anticipating synchronization of chaotic time-delayed Lur’e-type systems in a master-slave setting. We introduce three scenarios for anticipating synchronization, and give sufficient conditions for the existence of anticipating synchronizing slave systems in terms of linear matrix inequalities. The results obtained are illustrated on a time-delayed Rössler system and a time-delayed Chua oscillator. © 2007 American Institute of Physics. [DOI: 10.1063/1.2710964]

In their ground breaking 1990 paper, Pecora and Carroll\(^1\) showed that, in spite of sensitive dependence on initial conditions, coupled chaotic systems can synchronize. Following this work, in the last decades considerable interest in the notion of synchronization of complex or chaotic systems has arisen. Among others, perhaps the main motivation so far seems to lie in potential applications regarding secure or private communications, though the number of reported applications is still limited to date. On the other hand synchronization appears in numerous different ways, most notably in biological applications (heart beat, brain activity, neural activity, walking, etc.), nature (planetary motion), physics (organ pipes, etc.) and engineering (coordinated robot motion, etc.).\(^2-4\) More recently, it was observed by Voss\(^5\) that coupled chaotic Ikeda time-delay systems can exhibit what is called anticipating synchronization in that the state of one of the systems synchronizes with the future state of the other system, which allows one to predict the state of the latter system in spite of the inherent unpredictability of chaotic systems. In many applications where synchronization seems to be essential, the coupling between systems naturally contains a small amount of delay that nevertheless does not prevent synchronization from occurring. In this regard one may think about transport delay (e.g., chemical transport between cells in a living organism) or other forms of delay like optical delay in the synchronized movement of swarms of birds. Another example occurs when multiple agents, like, e.g., robots, share a communication network, which in principle will lead to a natural delay in the execution of (shared) tasks. It is therefore of interest to investigate how anticipating synchronization can be achieved for general classes of complex or chaotic systems. In this paper we will consider a wide class of chaotic time-delay systems, so-called Lur’e systems, and discuss when and how one can design anticipating synchronization schemes for these systems.

I. INTRODUCTION

In the last decades, following the work of Pecora and Carroll,\(^1\) considerable interest in the notion of synchronization of complex or chaotic systems has arisen. Among others, perhaps the main motivation so far seems to lie in potential applications regarding secure or private communications, though the number of reported applications is still limited to date. On the other hand synchronization appears in numerous different ways, most notably in biological applications (heart beat, brain activity, neural activity, walking, etc.), nature (planetary motion), physics (organ pipes, etc.) and engineering (coordinated robot motion, etc.).\(^2-4\) Most of the examples mentioned in essence deal with the synchronized motion of—possibly complex—oscillators.

In the present paper we focus on what is sometimes called anticipating synchronization or predicted synchronization. Voss\(^5\) describes the remarkable phenomenon that it is possible that synchronization may occur between a time-delayed Ikeda master system and a coupled nondelayed Ikeda slave system. That is, synchronization is exhibited between drive and time delayed response system, so that in this manner the slave dynamics act as a predictor of the master dynamics. Following Voss,\(^5\) these anticipating synchronization phenomena have been observed experimentally in an electronic circuit\(^6\) and in chaotic semiconductor diode lasers\(^7\) and numerically in chaotic external cavity semiconductor lasers.\(^8\) Further, recently several other papers on synchronization of systems containing delays have appeared. We will discuss a few, without claiming to be complete. Pyragas\(^9\) considered the synchronization of two directionally coupled
Mackey-Glass systems. Yalcın et al.\textsuperscript{10} discussed the (nonanticipative) master-slave synchronization of time-delayed Lur’e systems, and the results obtained were later generalized.\textsuperscript{11,12} Senthilkumar and Lakshmanan\textsuperscript{13} discussed the transitions from anticipating synchronization to lag synchronization via complete synchronization of time-delay systems with two time delays. Cherrier et al.\textsuperscript{14} and Cruz\textsuperscript{15} used a time-delayed Chua-type oscillator for studying secure communication schemes. Finally, Senthilkumar et al.\textsuperscript{16} discussed phase synchronization in time-delay systems.

It is our belief that the simple but nonetheless ground breaking observation of Voss,\textsuperscript{5} as well as the more recent other contributions regarding synchronization of delay-differential equations mentioned above, may form an indicator of an important underlying phenomenon. Namely, in many applications where master-slave synchronization seems to be essential, the coupling between master and slave naturally contains a small amount of delay that nevertheless does not prevent synchronization from occurring. In this regard one may think about transport delay (e.g., chemical transport between cells in a living organism) or other forms of delay like optical delay in the synchronized movement of swarms of birds. Another example occurs when multiple agents, like, e.g., robots, share a communication network, which in principle will lead to a natural delay in the execution of (shared) tasks.

Given the practical importance of the phenomenon underlying anticipating synchronization as indicated above, it is of great interest to investigate how anticipating synchronization can be achieved for general classes of complex or chaotic systems. All examples of anticipating synchronization mentioned above\textsuperscript{5-8} essentially dealt with one-dimensional systems with delay in the state and an asymptotically stable linear part. Oguchi and Nijmeijer\textsuperscript{17} recently generalized this by considering anticipating master-slave synchronization where the master is a higher-dimensional chaotic time-delay system with time-delayed states entering linearly and a linear part that is not necessarily asymptotically stable. The purpose of the present paper is to further generalize this latter work in several ways. The main objectives are to show that so-called Lur’e-type systems, i.e. systems having nonlinearities depending on a particular output function, form a natural class for considering anticipating synchronization between delayed master systems and coupled slave systems, and that using state observers considerably enlarges the class of time-delayed master systems for which an anticipating synchronizing slave system can be obtained. The results derived in this manner extend the work of Oguchi and Nijmeijer\textsuperscript{17} in a substantial way, but again rely on stability theory of delay-differential equations and so-called Lyapunov-Krasovskii functionals.\textsuperscript{18} In order to find verifiable conditions guaranteeing anticipating synchronization, sufficient conditions are derived in terms of sets of linear matrix inequalities (LMIs). The solvability of such LMIs can be tested using software like the Matlab LMI Toolbox.

The organization of the rest of this paper is as follows: In Sec. II we first briefly describe the anticipating synchronization scheme of Voss.\textsuperscript{5} After this we introduce three anticipating synchronization schemes that generalize this anticipating synchronization scheme. More specifically, we first consider Lur’e master systems with time-delayed nonlinearity, for which the slave system is based on a nondelayed copy of the master system. After this, we consider the same type of master system, but now with a slave system that is based on a combination of an observer for the master system and a non-delayed copy of the master system. A third generalization we consider after this is that of a Lur’e system with a time-delayed nonlinearity as well as a sector-bounded nondelayed nonlinearity. For each of the scenarios we also give sufficient conditions in terms of LMIs for the anticipating synchronization between master and slave systems. In Sec. III we illustrate our results on a time-delayed Rössler system and a time-delayed Chua oscillator. Among others, we illustrate a method to find a lower bound on the maximum time-delay for which anticipating synchronization can be achieved. In Sec. IV, conclusions are drawn.

Throughout the paper, one extensively used result regarding LMIs will be what is called the Schur complement equivalence, which for symmetric matrices \(P, S\) and matrix \(Q\) of appropriate dimensions states that\textsuperscript{19}

\[
\begin{pmatrix}
 S & Q \\
 Q^T & P 
\end{pmatrix} > 0 (\leq 0) \iff 
\begin{pmatrix}
 P > 0 (\leq 0), \\
 S - QP^{-1}Q^T > 0 (\leq 0).
\end{pmatrix}
\]

II. SCENARIOS FOR ANTICIPATING SYNCHRONIZATION

Voss\textsuperscript{5} considers a chaotic time-delayed Ikeda equation of the form

\[
\dot{x} = -\alpha x - \beta \sin x^2,
\]

where \(x(i, t) := x(t - \tau)\) and \(\alpha, \beta, \tau > 0\). The system (3) is used to drive a copy of (2) of the form

\[
\dot{y} = -\alpha y - \beta \sin x.
\]

Define the error \(e(t) = x(t) - y(t)\). It is then straightforwardly checked that \(e\) satisfies

\[
\dot{e} = -\alpha e,
\]

and hence the fact that \(\alpha > 0\) implies that \(e(t) \to 0\) for \(t \to \infty\).

Thus, the response system (3) synchronizes with the future state of (2) at time \(t + \tau\), and hence (3) anticipates the dynamics of (2).

In the rest of this section, we will discuss three scenarios for anticipating synchronization that generalize this anticipating synchronization scheme, and give sufficient conditions under which anticipating synchronizing dynamics exist.

A. Lur’e systems with time-delayed nonlinearity

Consider a chaotic driver system of the form

\[
\dot{x} = Ax + \Phi(\eta^2), \quad x \in \mathbb{R}^n,
\]

\[
\eta = Cx, \quad \eta \in \mathbb{R}^m
\]

where \(A\) and \(C\) are matrices of appropriate dimensions such that the pair \((C, A)\) is observable.\textsuperscript{[20]} That a matrix pair \((C, A)\) is called observable if the rank of the matrix

\[
C^T \quad C^T A^T \quad \cdots \quad C^T (A^{n-1})^T 
\]

equals \(n\). Further, \(\Phi : \mathbb{R} \to \mathbb{R}^m\)

Downloaded 07 Jul 2008 to 131.155.151.29. Redistribution subject to AIP license or copyright; see http://chaos.aip.org/chaos/copyright.jsp
is such that solutions of (5) exist and are unique. We further consider a receiver of the form

\[ \dot{\hat{y}} = Ay + \Phi(\eta) + K(\dot{\eta} - \eta), \]

where \( K \in \mathbb{R}^{n \times 1} \) is to be determined. Defining the error \( e(t) = x(t) - \hat{y}(t) \), we then obtain

\[ \dot{e} = (Ax + \Phi(\eta)) - (Ay + \Phi(\eta') - K(\eta' - \dot{\eta}^2)) \]

\[ = Ae + KCe^T, \quad (7) \]

and hence (6) will anticipate the dynamics of (5) if and only if \( K \) is such that the linear differential-difference equation (7) is asymptotically stable. The following result gives conditions for the existence of such a \( K \) in terms of so-called linear matrix inequalities (LMIs).^19

**Theorem 1:** Let \( \tau > 0 \) be given. Assume that there exists a scalar \( \alpha > 0 \) and matrices \( P > 0, Q > 0, X, Y, \) and \( W \) such that the following LMI holds:

\[ \begin{pmatrix} PA + A^T P + Y + Y^T + Q & XC - Y + W^T - \tau Y & \alpha \tau^T P \\ C^T X^T - Y^T + W & Q - W - W^T - \tau W & \alpha \tau^T C^T X^T \\ - \tau Y^T & - \tau W^T & - \alpha \tau P & 0 \end{pmatrix} < 0. \quad (8) \]

For these \( \alpha, P, Q, X, Y, W \), define the matrices

\[ N := \begin{pmatrix} Y & 0 \\ 0 & C^T X^T \end{pmatrix}, \quad \Pi := \alpha \text{ diag}(P,P), \]

\[ \Gamma := \begin{pmatrix} PA + A^T P + Y + Y^T + Q & XC - Y + W^T - \tau Y \\ C^T X^T - Y^T + W & Q - W - W^T \end{pmatrix}, \quad \Delta := \Pi \Gamma^{-1} N^T, \]

and let \( \tau^* \) be the minimum eigenvalue of the matrix pencil \( \Gamma, -\Delta \), i.e.,

\[ \tau^* := \min\{\tau | \det(\Gamma + \tau \Delta) = 0\}. \quad (10) \]

Then \( \tau^* > \tau \) and for \( K = P^{-1} X \) the error dynamics (7) are asymptotically stable for every \( 0 < \tau < \tau^* \).

**Proof:** See Appendix A.

**Remark 1:** The LMI (8) is obtained by means of a Lyapunov-Krasovskii functional* that was recently proposed by Xu and Lam. This Lyapunov-Krasovskii functional is more general than the more “classical” Lyapunov-Krasovskii functionals employed in other work. As a consequence of this greater generality, the result of Theorem 1 is less conservative than results that would have been obtained using the more classical functionals.

### B. Observer-based anticipation for Lur’e systems with time-delayed nonlinearity

Note that the receiver (6) contains \( n \) free parameters. One can increase the number of free parameters and hence the chances of being able to design anticipating dynamics as follows. Recall that observability of the pair \((C,A)\) implies the existence of a matrix \( L \in \mathbb{R}^{n \times 1} \) such that all eigenvalues of \( A + LC \) have real parts that are strictly smaller than zero. Given such \( L \), consider the dynamics

\[ \dot{\hat{y}} = Ay + \Phi(\eta) + M(y^T - z), \]

\[ \dot{z} = Az + \Phi(\eta') + L(\dot{\eta} - \eta), \]

\[ \dot{\eta} = Cz, \]

where \( M \in \mathbb{R}^{n \times n} \) still needs to be determined. Define error functions \( e(t) = x(t) - y(t), \) \( e(t) = x(t) - z(t) \). We then have that

\[ \dot{e} = Ax + \Phi(\eta') - (Ay + \Phi(\eta') + M(y^2 - z^2)) \]

\[ = Ae + M((x^T - e^T) - (x^T - e^T)) = Ae + M(e^T - e^T) \quad (12) \]

and

\[ \dot{e} = (A + LC)e. \quad (13) \]

Since we have constructed \( L \) in such a way that all eigenvalues of \( A + LC \) have real parts that are strictly smaller than zero, we have that the zero-solution of (13) is exponentially stable. This then implies that we will have that the prediction error \( e \) tends to zero if the zero-solution of (12) with \( \epsilon = 0 \) is asymptotically stable. Thus, we have that the \( y \) dynamics in (11) will anticipate the dynamics of (5) if the zero-solution of the following linear delay-differential equation is asymptotically stable:

\[ \dot{e} = Ae + Me^T. \quad (14) \]

Also note that (14) has the same form as (7) with \( K = M, \) \( C = I \). We therefore have the following immediate consequence of Theorem 1.
Theorem 2: Let $\bar{\tau} > 0$ be given, and assume that there exist a scalar $\alpha > 0$ and matrices $P > 0, Q > 0, X, Y, \text{and } W$ such that (8) holds with $C=I$. Define matrices $\Gamma, \Delta$ as in (9) with $C=I$, and let $\tilde{\tau}$ be the minimum eigenvalue of the matrix pencil $(\Gamma, -\Delta)$. Then $\tau^* > \bar{\tau}$ and for $M= P^{-1} X$ the error dynamics (14) are asymptotically stable for every $0 < \tau < \tau^*$. □

C. Observer-based anticipation for Lur’e systems with time-delayed and sector-bounded nonlinearity

Consider a generalization of the transmitter (5) of the following form:

$$\dot{x} = Ax + B\psi(\eta) + \Phi(\eta), \quad x \in \mathbb{R}^n, \quad \psi: \mathbb{R} \to \mathbb{R},$$

$$\eta = Cx, \quad \eta \in \mathbb{R},$$

where $A, B, C$ are matrices of appropriate dimensions, the pair $(C, A)$ is observable, and there exists $\gamma > 0$ such that $\psi$ satisfies the following sector condition:

$$0 \leq (\psi(x+h) - \psi(x))h \leq \gamma h^2, \quad \forall x, h \in \mathbb{R}. \quad (16)$$

Let $L$ be such that the real parts of all eigenvalues of $A + LC$ are strictly smaller than zero, and consider the following receiver for (15):

Define the matrices

$$N := \begin{pmatrix} -Y & \alpha A^T P & 0 \\ -W & \alpha X^T & 0 \\ 0 & 0 & \alpha B^T P \end{pmatrix}, \quad \Pi := -\alpha \text{diag}(P, P, P),$$

$$\Gamma := \begin{pmatrix} PA + A^T P + Y + Y^T + Q & X - Y + W^T & PB + \gamma A C^T - \tilde{\tau} Y \\ X^T - Y^T + W & -Q - W - W^T & 0 \\ B^T P + \gamma A C & 0 & -\tilde{\tau} W \end{pmatrix} \begin{pmatrix} \alpha \tilde{\tau} A^T P \\ 0 \\ 0 \end{pmatrix} = 0,$$ \quad (20)

Define error signals $e(t) := x(t) - y(t), \quad \hat{e}(t) := x(t) - z(t)$. We then obtain the following error dynamics:

$$\dot{\hat{e}} = (A + LC)\hat{e},$$ \quad (18)

$$\dot{e} = A e + M e^T + B [\psi(C x) - \psi(C(x - e))] - M e^T.$$

As in the previous subsection we will then have that the $y$ dynamics in (17) anticipate the dynamics of (5) if and only if $M$ is such that the following dynamics are asymptotically stable:

$$\dot{\hat{e}} = A e + M e^T + B [\psi(C x) - \psi(C(x - e))].$$ \quad (19)

We now have the following result.

Theorem 3: Let $\bar{\tau} > 0$ be given. Assume that there exist scalars $\lambda \geq 0$ and matrices $P > 0, Q > 0, X, Y, \text{and } W$ such that the following LMI holds:

$$\Gamma := \begin{pmatrix} PA + A^T P + Y + Y^T + Q & X - Y + W^T & PB + \gamma A C^T - \tilde{\tau} Y \\ X^T - Y^T + W & -Q - W - W^T & 0 \\ B^T P + \gamma A C & 0 & -\tilde{\tau} W \end{pmatrix} \begin{pmatrix} \alpha \tilde{\tau} A^T P \\ 0 \\ 0 \end{pmatrix} = 0,$$ \quad (20)

$$(\Gamma, -\Delta).$$ Then $\tau^* > \bar{\tau}$ and for $M= P^{-1} X$ the dynamics (19) are asymptotically stable for every $0 < \tau < \tau^*$. □

Proof: See Appendix B.

III. EXAMPLES

A. Example 1: Time-delayed Rössler system

Consider the following transmitter dynamics:

$$\dot{\bar{x}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0.4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bar{x} + \begin{pmatrix} -\exp(\eta^2) \\ 0 \\ 2 \exp(-\eta^2) - 4 \end{pmatrix},$$

$$\eta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \bar{x}.$$ \quad (22)

It is straightforwardly checked \cite{22} that when $\tau=0$, the coordi-
nate transformation \( \xi_1 = x_1, \xi_2 = x_2, \xi_3 = e^3 \) transforms (22) into the following chaotic Rössler system:

\[
\begin{align*}
\dot{\xi}_1 &= -\xi_2 - \xi_3, \\
\dot{\xi}_2 &= \xi_1 + 0.4\xi_2, \\
\dot{\xi}_3 &= 2 + (\xi_1 - 4)\xi_3.
\end{align*}
\]

As is illustrated in Fig. 1, the transmitter (22) exhibits chaotic behavior for \( 0 \leq \tau \leq 0.35 \).

In the rest of this subsection, we illustrate the application of Theorems 1 and 2 to the design of anticipating synchronizing dynamics for (22).

1. **Anticipation without use of an observer**

We first consider receiver dynamics of the form (6). Solving the LMI (8) for \( \bar{\tau} = 0.35, \alpha = 1 \) gives the solution in Appendix C. From this, we obtain

\[ K = P^{-1}X = \begin{bmatrix}
-0.9783 \\
-1.1114 \\
-1.7839
\end{bmatrix}. \] (24)

For the transmitter, we take the initial conditions \( x(0) = \text{col}(0.212, -0.53, -1) \), and we take \( \gamma(\theta) = 0 \) for \( \theta < 0 \). The behavior of the prediction error is then given in Fig. 2.

Figure 3 gives the behavior of the prediction error if \( K \) is taken as in (24) and \( \tau = 0.1 \). This illustrates the fact that according to Theorem 1 we have that the error dynamics are asymptotically stable for every \( 0 < \tau \leq \bar{\tau} = 0.35 \).

Calculation of the matrices \( \Gamma \) and \( \Delta \) according to (9) gives that the eigenvalues of the matrix pencil \( (\Gamma, -\Delta) \) are given by \{368.5639, 7.8619, 2.6104, 1.3172, 0.4059, 0.3588\}. Hence according to Theorem 1 we have that for the anticipating dynamics we have designed in fact the error dynamics are asymptotically for all \( 0 < \tau \leq \bar{\tau} = 0.3588 \).

Now note that the values of \( P \) and \( X \) found when solving the LMI for \( \tau = 0.35 \) might not be the values of \( P \) and \( X \) that would have been found if the LMI would have been solved for \( \tau = 0.3588 \) in the first place. It is therefore conceivable that repeated iterations of solving the LMI for a given value of \( \tau \) followed by using Theorem 1 to find a higher upper bound of \( \tau \) for which one still has stability, may in the end...
lead to a considerably higher bound for \( \tau \). Figure 4 gives the results of two such experiments, one with an initial value of \( \tau \) = 0.35, the other with an initial value of \( \tau \) = 10\(^{-4} \). It appears that in both cases the values of \( \tau ^* \) obtained converge to a value that is approximately equal to 0.4739. In fact, it may be proven that indeed the procedure described above will converge to the maximal value of \( \tau ^* \) for which the LMI is feasible. However, note that due to the fact that our conditions are sufficient conditions for the existence of anticipating dynamics, the procedure described above will in general not yield the maximal time delay for stability of the anticipation scheme. Rather, it gives a lower bound on the maximum time delay for which anticipating synchronization can be achieved.

We see in Fig. 4 that the procedure described above needs at least 100 iterations to get close to the maximum value of \( \tau ^* \). However, using standard bisection-like techniques, one can modify the algorithm in such a way that the convergence is sped up, as is illustrated in Fig. 5.

So far, we have only considered the case that \( \alpha = 1 \). Obviously, we will have that \( \tau ^* \) depends on \( \alpha \), so it may be that the procedure described above gives higher values of \( \tau ^* \) for \( \alpha \neq 1 \). Figure 6 illustrates that this is indeed the case. From this figure, we see that the “optimal” value of \( \alpha \) is 1.3, which leads to \( \tau ^* = 0.4805 \).

2. Observer-based anticipation

We next consider receiver dynamics of the form (11). Solving the LMI for \( \tau = 0.35, \alpha = 1 \) with \( C = I \) gives the solution in Appendix D. From this, we obtain

\[
M = P^{-1}X = \begin{pmatrix}
-0.6763 & 0.4112 & -0.1841 \\
-0.1755 & -0.8629 & -0.1902 \\
-0.5320 & -0.1954 & -0.6875
\end{pmatrix}.
\] (25)

Furthermore, if we require that the eigenvalues of \( A + LC \) are at \(-1, -2, -3\), we find that
\[ L = \begin{pmatrix} -12.5600 \\ 4.6240 \\ -6.4000 \end{pmatrix}. \] (26)

For the transmitter we take the same initial conditions as above, and we take \( y(\theta) = 0, z(\theta) = 0 \) for \( \theta < 0 \). The behavior of the prediction errors is then given in Fig. 7. Figure 8 gives the prediction errors when \( M \) and \( L \) are taken as in (25) and (26), respectively, and \( \tau = 0.1 \).

By using the procedure described in the previous subsection, we obtain the lower bounds of the maximum time delay for anticipating synchronization as a function of \( \alpha \) as depicted in Fig. 9. Note that this figure seems to suggest that \( \tau' \) converges to 1 as \( \alpha \to \infty \).

**B. Example 2: Chua oscillator with time delayed nonlinearity**

Consider the following Chua oscillator with time delayed nonlinearity: 15, 17, 23

\[
\begin{align*}
\dot{x}_1 &= \alpha(x_2 - x_1 - h(\eta)), \\
\dot{x}_2 &= x_1 - x_2 + x_3, \\
\dot{x}_3 &= -\beta x_2 - \gamma x_3 - \beta \epsilon \sin(\nu \eta), \\
\eta &= x_1,
\end{align*}
\] (27)

where

\[
h(\eta) = b \eta + \frac{1}{2}(a-b)(|\eta + 1| - |\eta - 1|),
\] (28)

and \( \alpha = 10, \beta = 19.53, \gamma = 0.1636, a = -1.4325, b = -0.7831, \nu = 0.5, \epsilon = 0.2, \) and \( \tau = 0.019 \). Defining
A = \begin{pmatrix} -\alpha(1 + b) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & -\gamma \end{pmatrix}, \quad B = \begin{pmatrix} -\alpha(a - b) \\ 0 \\ 0 \end{pmatrix}.

C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.

\psi(\eta) = \frac{1}{2}(|\eta + 1| - |\eta - 1|), \quad \Phi(\eta') = \begin{pmatrix} 0 \\ 0 \\ -\beta e \sin(\nu \eta') \end{pmatrix}, \quad M = P^{-1}X = \begin{pmatrix} -11.4986 & -7.5093 & 0.9322 \\ -0.6253 & -0.1285 & 0.2127 \\ 3.0420 & 4.9385 & -3.5010 \end{pmatrix}.

If we further require the eigenvalues of \(A+LC\) to be at \(-4, -5, -6\) we obtain that

\begin{align*}
M &= P^{-1}X = \begin{pmatrix} -11.4986 & -7.5093 & 0.9322 \\ -0.6253 & -0.1285 & 0.2127 \\ 3.0420 & 4.9385 & -3.5010 \end{pmatrix}, \\
L &= \begin{pmatrix} -11.6674 \\ -4.8206 \\ 15.8739 \end{pmatrix}.
\end{align*}

As initial conditions for (27) we take \(x(0) = \text{col}(1,-2,0)\), \(x(\theta) = 0\) for \(\theta < 0\), while for the response system (17) we take \(y(\theta) = 0\), \(z(\theta) = 0\) for \(\theta < 0\). The behavior of the prediction...
error \( e(t) = x(t) - y(t) \) and observer error \( \epsilon(t) = x(t) - z(t) \) are then given in Figs. 11 and 12, respectively.

Using Appendix E and (21), \( \Gamma \) and \( \Delta \) can be calculated, and one finds that \( \tau^* = 0.0266 \). Hence by Theorem 3 we have that in fact the prediction error will tend to zero asymptotically for all \( 0 < \tau < 0.0266 \).

In Fig. 13 we have depicted the lower bounds of time delay for anticipating synchronization as a function of \( \alpha \). From this figure, we see that the “optimal” value of \( \alpha \) is 0.92, which leads to \( \tau^* = 0.0688 \).

Solution of the LMI (20) with \( \alpha = 0.92, \tau = 0.0688 \) gives the results in Appendix F, which on its turn results in

\[
M = P^{-1}X = \begin{pmatrix}
-13.7446 & -13.3845 & -1.7307 \\
-0.4706 & 2.2930 & 1.3367 \\
-0.6389 & -9.1402 & -10.8548
\end{pmatrix}
\]

Using the same initial conditions and value of \( L \) as above, the behavior of the prediction error \( e(t) \) is then given in Figs. 14 and 15 for the cases that \( \tau = 0.0688 \) and \( \tau = 0.019 \), respectively.

IV. CONCLUSIONS

In this paper we considered the design of anticipating synchronizing dynamics for complex or chaotic Lur’e-type delay systems. We have introduced three types of anticipating synchronization schemes that generalize the schemes of Voss and Oguchi and Nijmeijer, and have given sufficient conditions in terms of the solvability of linear matrix inequalities (LMIs) for the anticipating synchronization between master and slave systems. The anticipating synchronization scenarios were illustrated on the examples of a time-delayed Rössler system and a time-delayed Chua oscillator. In the examples we also illustrated a method to find a lower bound on the maximum time delay for which anticipating synchronization can be achieved.

All conditions for anticipating synchronization obtained in the paper are sufficient conditions. As a consequence, they may be too conservative, and anticipating synchronizing slave systems of the forms proposed in the paper may still
exist even though the conditions are not satisfied. On the other hand, using LMI software it is straightforward to check the conditions and to design anticipating slave systems, so one might argue that conservativeness is a reasonable price to be paid for ease of design as compared to possibly more advanced, less restrictive methods. Further, as also indicated in the paper, the solution of a LMI is nonunique in general. In the design of anticipating dynamics, this could be explored for optimization of the anticipation scheme, e.g., in terms of transient behavior. We have not explored this avenue in this paper, but this certainly remains a topic for future research. Finally, we have assumed throughout that the structure and parameters of the master system, including the time delay, are known exactly. Obviously, in practice this will not be the case in general, and hence a next step in our research will be to derive robust versions of the results obtained in this paper, possibly along similar lines as in the work of Oguchi and Nijmeijer \cite{oguchi} where robust versions of earlier work \cite{Bhat} were obtained. Since generally controllers based on the solution of LMIs come equipped with a certain degree of robustness, \cite{willems} it is to be expected that a LMI approach will also be the most promising approach for deriving these robust versions.

**APPENDIX A: PROOF OF THEOREM 1**

The proof of this theorem uses the following result. \cite{prabhu}

Lemma 1: Consider a linear delay-differential equation of the form

\[ \dot{e} = A_0 e + A_1 e^\tau. \]  

(A1)

For \( \tau > 0 \), assume that there exist matrices \( P > 0 \), \( Q > 0 \), \( Z > 0 \), \( Y \), and \( W \) such that the following LMI is satisfied:

\[
\begin{pmatrix}
PA_0 + A_0^T P + Y + Y^T + Q & PA_1 - Y + W^T - \tau Y - \tau A_1^T Z \\
A_1^T P - Y^T + W & -Q - W - W^T - \tau W - \tau A_1^T Z \\
- \tau Y^T & - \tau W^T - \tau Z \\
\tau Z A_0 & \tau Z A_1 \\
0 & - \tau Z
\end{pmatrix} < 0.
\]

(A2)

Recall \cite{zaletel} that a matrix is negative definite if and only if all its eigenvalues are strictly negative. Due to continuity and the fact that a matrix has a zero eigenvalue if and only if it is not invertible, it then follows that (A4) holds for every \( 0 < \tau < \tau^* \), where \( \tau^* \) is defined in (9). This establishes our claim.

**APPENDIX B: PROOF OF THEOREM 3**

The proof of this theorem relies on the following result.

Lemma 2: Consider the dynamics

\[ \dot{e} = A_0 e + A_1 e^\tau + B[\psi(Cx) - \psi(C(x - e))], \]

(B1)

where the function \( \phi \) satisfies the sector condition (16). Let \( \tilde{\tau} \) be given and assume that there exists matrices \( P > 0 \), \( Q > 0 \), \( Z > 0 \), \( Y \) and \( W \) and a constant \( \lambda > 0 \) such that the following LMI is satisfied:

\[ \Gamma + \tau N P^{-1} N^T < 0. \]

(A4)
\[
\begin{pmatrix}
PA_0 + A_0^T P + Y + Y^T + Q & PA_1 - Y + W^T & PB + \gamma \lambda C^T - \tau Y & \tau A_0^T Z & 0 \\
A_1^T P - Y^T + W & -Q - W - W^T & 0 & -\tau W & \tau A_1^T Z & 0 \\
B^T P + \gamma \lambda C & 0 & -2\lambda & 0 & 0 & -\tau B^T Z \\
-\tau Y^T & -\tau W^T & 0 & -\tau Z & 0 & 0 \\
\tau Z A_0 & \tau Z A_1 & 0 & 0 & -\tau Z \\
0 & 0 & \tau Z B & 0 & 0 & -\tau Z
\end{pmatrix} < 0.
\] (B2)

Then the origin is an asymptotically stable equilibrium point for (B1) for every \( 0 < \tau \leq \bar{\tau} \).

Proof: Assume that there exists matrices \( P > 0 \), \( Q > 0 \), \( Z > 0 \), \( Y \), and \( W \) and a constant \( \lambda > 0 \) such that the LMI (B2) holds. Consider a Lyapunov-Krasovskii functional of the following form:

\[
V(e_i) = V_1(e_i) + V_2(e_i) + V_3(e_i),
\] (B3)

where \( e_i := \{e(a) \mid t - \tau \equiv a \leq t \} \) and

\[
V_1(e_i) = e(t)^T Pe(t),
\]

\[
V_2(e_i) = \frac{1}{\tau} \int_{t-\tau}^t [e(t)^T A_0^T Z A_0 e(t) + 2 e(t)^T A_0^T Z A_1 e(t - \tau) + 2 e(t - \tau)^T A_1^T Z A_0 e(t) - 2 e(t - \tau)^T A_1^T Z A_1 e(t - \tau) - \tau \dot{e}(a)^T Z \dot{e}(a) + \tau \ddot{e}(x(t), e(t))^T B^T Z B] da,
\] (B5)

\[
V_3(e_i) = \frac{1}{\tau} \int_{t-\tau}^t [e(t)^T Q e(t) - e(t - \tau)^T Q e(t - \tau)] da,
\]

where

\[
\ddot{e}(x(t), e(t)) = \dddot{e}(x(t), e(t)) - \gamma C e(t).
\] (B6)

Note that it follows from the sector condition (16) that

\[
\ddot{e}(x(t), e(t)) - \gamma C e(t) \leq 0.
\] (B7)

Combining (B5) and (B7) we then obtain for \( \lambda > 0 \) that

\[
\dot{V}(e_i) \leq \hat{V}_1(e_i) + \hat{V}_2(e_i) + \hat{V}_3(e_i) - \frac{2}{\tau} \int_{t-\tau}^t [e(t)^T P e(t) + e(t - \tau)^T W e(t - \tau) - \tau \dot{e}(a)^T W \dot{e}(a) - \tau \ddot{e}(a)^T W \ddot{e}(a) + PB \ddot{e}(x(t), e(t))] da,
\]

where

\[
\ddot{\psi}(x(t), e(t)) = \dddot{\psi}(x(t), e(t)) - \gamma C e(t)
\]

\[
\lambda(t) = \Lambda_0 + \tau \Lambda_1,
\]

\[
\Lambda_0 = \begin{pmatrix}
PA_0 + A_0^T P + Y + Y^T + Q & PA_1 - Y + W^T & PB + \gamma \lambda C^T - \tau Y & \tau A_0^T Z & 0 \\
A_1^T P - Y^T + W & -Q - W - W^T & 0 & -\tau W & \tau A_1^T Z & 0 \\
B^T P + \gamma \lambda C & 0 & -2\lambda & 0 & 0 & -\tau B^T Z \\
-\tau Y^T & -\tau W^T & 0 & -\tau Z & 0 & 0 \\
\tau Z A_0 & \tau Z A_1 & 0 & 0 & -\tau Z \\
0 & 0 & \tau Z B & 0 & 0 & -\tau Z
\end{pmatrix},
\]

\[
\Lambda_1 = \begin{pmatrix}
A_0^T Z A_0 & A_0^T Z A_1 & 0 & -Y \\
A_1^T Z A_0 & A_1^T Z A_1 & 0 & -W \\
0 & 0 & B^T Z B & 0 \\
-\tau Y^T & -\tau W^T & 0 & -\tau Z
\end{pmatrix}.
\]

Now assume that the LMI (B2) holds. We then have that \( Z > 0 \). Further, define the matrix
\[ \Delta(\tau) = \Delta_0 + \tau \Delta_1, \]  

(B9)

where

\[
\Delta_0 = \begin{pmatrix}
  PA_0 + A_0^T P + Y + Y^T + Q & PA_1 - Y + W^T & PB + \gamma \lambda C^T \\
  A_1^T P - Y^T + W & -Q - W - W^T & 0 \\
  B^T P + \gamma \lambda C & 0 & -2\lambda
\end{pmatrix},
\]

\[
\Delta_1 = \begin{pmatrix}
  -Y & A_0^T Z & 0 \\
  -W & A_1^T Z & 0 \\
  0 & 0 & B^T Z
\end{pmatrix}
\begin{pmatrix}
  Z^{-1} & 0 & 0 \\
  0 & Z^{-1} & 0 \\
  0 & 0 & Z^{-1}
\end{pmatrix}
\begin{pmatrix}
  -Y^T & -W & 0 \\
  ZA_0 & ZA_1 & 0 \\
  0 & 0 & ZB
\end{pmatrix}.
\]

Since \( Z > 0 \), we then have that

\[ \Delta(\tau) \leq \Delta(\bar{\tau}) < 0 \text{ (for all } 0 < \tau \leq \bar{\tau}), \]  

(B10)

where the last inequality follows by applying the Schur complement equivalence (1) to (B2). Applying the Schur complement equivalence to (B10) then on its turn gives that \( \Lambda(\tau) < 0 \) for all \( 0 < \tau \leq \bar{\tau} \), which then establishes that the origin is an asymptotically stable equilibrium point for (B1).

Proof of Theorem 3:

The first part of the proof follows from Lemma 2 by setting \( M = A_1 = P^{-1} X, Z = aP \). The second part of the proof is analogous to the second part of the proof of Theorem 1 in Appendix A.

APPENDIX C: SOLUTION OF LMI (8) FOR \( \alpha = 1, \tau = 0.35 \) IN SEC. III A

\[
P = \begin{pmatrix}
  0.9734 & 0.9063 & -1.1351 \\
  0.9063 & 1.6488 & -1.3781 \\
  -1.1351 & -1.3781 & 1.8331
\end{pmatrix},
\]

\[
Q = \begin{pmatrix}
  0.2703 & 0.2383 & -0.4055 \\
  0.2383 & 0.4551 & -0.5157 \\
  -0.4055 & -0.5157 & 0.7694
\end{pmatrix},
\]

\[
W = \begin{pmatrix}
  0.1103 & -0.0531 & 0.2021 \\
  -0.1023 & 0.0500 & 0.1926 \\
  0.1298 & 0.2362 & 0.1894
\end{pmatrix},
\]

\[
X = \begin{pmatrix}
  0.0655 \\
  -0.2605 \\
  -0.6279
\end{pmatrix},
\]

\[
Y = \begin{pmatrix}
  -0.2592 & -0.0699 & -0.1550 \\
  -0.0252 & -0.3269 & -0.1462 \\
  0.1557 & 0.1194 & -0.7068
\end{pmatrix}.
\]

APPENDIX D: SOLUTION OF LMI (8) FOR \( \alpha = 1, \tau = 0.35, C = I \) IN SEC. III A

\[
P = \begin{pmatrix}
  1.2360 & 0.2174 & -0.1916 \\
  0.2174 & 1.2254 & -0.3373 \\
  -0.1916 & -0.3373 & 1.0236
\end{pmatrix},
\]

\[
Q = \begin{pmatrix}
  0.4857 & 0.0406 & -0.1161 \\
  0.0406 & 0.4301 & -0.0532 \\
  -0.1161 & -0.0532 & 0.5282
\end{pmatrix},
\]

\[
W = \begin{pmatrix}
  0.2461 & -0.0653 & 0.0854 \\
  0.0118 & 0.2918 & -0.0130 \\
  0.1325 & 0.0736 & 0.1759
\end{pmatrix},
\]

\[
X = \begin{pmatrix}
  -0.7721 & 0.3581 & -0.1372 \\
  -0.1826 & -0.9021 & -0.0413 \\
  -0.3558 & 0.0122 & -0.6043
\end{pmatrix},
\]

\[
Y = \begin{pmatrix}
  -0.9333 & 0.0822 & -0.2895 \\
  0.0223 & -1.0983 & 0.0432 \\
  -0.3243 & -0.0257 & -0.6226
\end{pmatrix},
\]

APPENDIX E: SOLUTION OF LMI (20) FOR \( \alpha = 1 \) IN SEC. III B

\[
P = \begin{pmatrix}
  0.1531 & -0.0592 & 0.2021 \\
  -0.0592 & 2.1789 & -0.0551 \\
  0.2021 & -0.0551 & 0.2471
\end{pmatrix},
\]

\[
Q = \begin{pmatrix}
  0.5436 & -0.1049 & 0.0211 \\
  -0.1049 & 0.8037 & 0.0996 \\
  0.0211 & 0.0996 & 0.6863
\end{pmatrix},
\]

\[
W = \begin{pmatrix}
  0.1531 & -0.0592 & 0.0765 \\
  -0.0592 & 2.1789 & -0.0551 \\
  0.0765 & -0.0551 & 0.2471
\end{pmatrix},
\]

\[
X = \begin{pmatrix}
  -0.7721 & 0.3581 & -0.1372 \\
  -0.1826 & -0.9021 & -0.0413 \\
  -0.3558 & 0.0122 & -0.6043
\end{pmatrix},
\]

\[
Y = \begin{pmatrix}
  -0.9333 & 0.0822 & -0.2895 \\
  0.0223 & -1.0983 & 0.0432 \\
  -0.3243 & -0.0257 & -0.6226
\end{pmatrix},
\]

\[
W = \begin{pmatrix}
  0.2461 & -0.0653 & 0.0854 \\
  0.0118 & 0.2918 & -0.0130 \\
  0.1325 & 0.0736 & 0.1759
\end{pmatrix},
\]
\( X = \begin{pmatrix} -1.4905 & -0.7641 & -0.1377 \\ -0.8498 & -0.1078 & 0.6012 \\ -0.0936 & 0.6529 & -0.8056 \end{pmatrix}, \)

\( Y = \begin{pmatrix} -1.4011 & -0.9766 & -0.1855 \\ -0.7825 & -0.1528 & 0.7506 \\ -0.1185 & 0.9117 & -0.7606 \end{pmatrix}, \)

\( \lambda = 0.8768. \)

### APPENDIX F: SOLUTION OF LMI (20)

FOR \( \alpha = 0.92, \tau = 0.0688 \) IN SEC. III B

\( P = \begin{pmatrix} 0.0136 & 0.0055 & 0.0011 \\ 0.0055 & 0.0997 & 0.0171 \\ 0.0011 & 0.0171 & 0.0062 \end{pmatrix}, \)

\( Q = \begin{pmatrix} 0.0004 & -0.0055 & -0.0009 \\ -0.0055 & 0.0979 & 0.0176 \\ -0.0009 & 0.0176 & 0.0037 \end{pmatrix}, \)

\( W = \begin{pmatrix} 0.1825 & 0.1817 & 0.0321 \\ -0.0118 & 0.1208 & 0.0078 \\ -0.0009 & 0.0250 & 0.0443 \end{pmatrix}, \)

\( \lambda = 0.0426. \)