Asymptotic (k-1)-mean significance levels of a nonparametric method for multiple comparisons in the k-sample case
Oude Voshaar, J.H.

Published: 01/01/1977

Citation for published version (APA):
Memorandum COSOR 77-18

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by

J.H. Oude Voshaar

Eindhoven, October 1977

The Netherlands
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1. Introduction and summary.

It is known that the projection argument, by which the Scheffe simultaneous confidence intervals are derived from the \(F\) statistic, can also be applied to the Kruskal - Wallis statistic (see Miller [4], page 165-172). Here we will consider the case of equal sample sizes, say size \(n\), for \(n\) large \((n \to \infty)\). Then we have under the null hypothesis \((F_1 = \ldots = F_k)\):

\[
P \left( \left| \bar{r}_i - \bar{r}_j \right| \leq q_k^{\alpha} \sqrt{\frac{k(n+1)}{12}} \right) \text{ for all } i,j \text{ out of } \{1, \ldots, k\} = 1 - \alpha
\]

where \(\bar{r}_i\) is the mean of the ranks of the \(i\)-th sample and \(q_k^{\alpha}\) is the upper percentile point of the range of \(k\) independent unit normal variables.

We will be concerned with the following problem:

if \(F_1 = \ldots = F_{k-1} = F\) and \(F_k \neq G\), what will be the value (for \(n \to \infty\)) of

\[
\alpha(F,G) := 1 - P \left( \left| \bar{r}_i - \bar{r}_j \right| \leq q_k^{\alpha} \sqrt{\frac{k(n+1)}{12}} \right) \text{ for all } i,j \text{ out of } \{1, \ldots, k-1\}
\]

i.e. what is the probability of concluding incorrectly that some of the \(F_i\)'s among \(F_1, \ldots, F_{k-1}\) are different? \(\alpha(F,G)\) is called the \((k-1)\)-mean significance level (cf. Miller [4]).

The following statement turns out not to be true in general:

\[
\alpha(F,G) \leq \alpha \text{ for all (continuous) } F \text{ and } G.
\]

Even when \(G\) is a shift of \(F\), a counter example can be given. However, if \(G\) is a shift of \(F\) and \(F\) is symmetric and unimodal then: \(\alpha(F,G) \leq \alpha\).

If \(F\) is not symmetric, the above statement remains true if we replace unimodal by strongly unimodal (or more general: log \(F\) and log \((1-F)\) both concave). Our main result will be that in the case of shift alternatives the \((k-1)\)-mean significance level is depending on the skewness of \(F\) (in the sense of the convex order relation introduced by Van Zwet [8]).
2. A general expression for $\alpha(F,G)$

Let $x_{11}, \ldots, x_{1n}; \ldots; x_{k1}, \ldots, x_{kn}$ be independent random variables ($k \geq 3$), where $x_{ij}$ has a continuous distribution function $F_i$. Let $x_{ij}$ denote the rank of $x_{ij}$ among all observations and

$$\bar{x} := \frac{1}{n} \sum_{j=1}^{n} x_{ij}.$$

To determine $\alpha(F,G)$ for $n \to \infty$, we first must know the asymptotic distribution of the range of $\bar{x}_1, \ldots, \bar{x}_{k-1}$ for the case $F_1 = \ldots = F_{k-1}$ and $F_k = G$. If we define

(2.1) \hspace{1cm} p := \int G(x) dF(x)

(2.2) \hspace{1cm} q := \int G^2(x) dF(x)

(2.3) \hspace{1cm} r := \int F(x) G(x) dF(x)

then for $i, j \in \{1, \ldots, k-1\}$ the following relationship can be shown (see Oude Voshaar [6, J]

(2.4) \hspace{1cm} \mathcal{E}(\bar{x}_i) = \frac{1}{2}(kn + 1) + (p - \frac{1}{2})n

(2.5) \hspace{1cm} \text{var}(\bar{x}_i) = \frac{1}{12} k^2 n + (2r - p - \frac{1}{4})kn + (4p - 2p^2 + q - 6r + \frac{1}{6})n +

\hspace{1cm} + \frac{1}{12} k - p + p^2 - q + 2r - \frac{1}{6}

(2.6) \hspace{1cm} \text{cov}(\bar{x}_i, \bar{x}_j) = -\frac{1}{12} kn + (3p - p^2 - 4r + \frac{1}{12})n - \frac{1}{12}.

Remark:

Under $H_0$ (that is $F = G$) we have $p = \frac{1}{2}$ and $q = r = \frac{1}{3}$, so (2.4) to (2.6) reduce to:

$$\mathcal{E}(\bar{x}_i) = \frac{1}{2}(kn + 1)$$

$$\text{var}(\bar{x}_i) = \frac{1}{12} k(k - 1)n + \frac{1}{12} (k - 1)$$

$$\text{cov}(\bar{x}_i, \bar{x}_j) = -\frac{1}{12}(kn + 1)$$
corresponding to the known formulas (cf. Miller [4], page 171).

Using theorem 2.1 of Hajek [2], it is also possible to prove the asymptotic normality of the vector \((\bar{x}_1, \ldots, \bar{x}_{k-1})\) when \(F\) and \(G\) are not identical (see Oude Voshaar [6]).

Hence \(y_n\), defined by:

\[ y_n = (y_{n1}, \ldots, y_{nk-1}) := n^{-\frac{1}{2}}(\bar{x}_1, \ldots, \bar{x}_{k-1}) \]

is asymptotically normally distributed with covariance matrix:

\[
\begin{bmatrix}
a_1 & a_2 & \cdots & a_2 \\
a_1 & a_1 & \cdots & a_1 \\
a_1 & a_1 & \cdots & a_1 \\
a_1 & a_1 & \cdots & a_1
\end{bmatrix}
\]

where

\[
a_1 := \frac{1}{12} k^2 + (2r - p - \frac{1}{4})k + 4p^2 - 2p^2 + q - 6r + \frac{1}{6}
\]

and

\[
a_2 := \frac{1}{12} k^3 + 3p^2 - p^2 - 4r + \frac{1}{12}
\]

If we define \(\bar{y}_n = (\bar{w}_{n1}, \ldots, \bar{w}_{nk-1})\) by:

\[ \bar{w}_{ni} := y_{ni} - \bar{y}_n \]

where

\[ \gamma := 1 \pm \sqrt{\frac{a_1 - a_2}{a_1 + (k - 2)a_2}} \]

and

\[ \bar{y}_n := \frac{1}{k-1} \sum_{j=1}^{k-1} \bar{y}_{nj} \]

then \(\bar{w}_n\) has an asymptotically normal distribution with covariance matrix \((a_1 - a_2)I_{k-1}\). (\(I_{k-1}\) stands for the identity matrix of size \(k - 1\)).

This implies that the range of \((a_1 - a_2)^{-\frac{1}{2}}\bar{w}_{n1}, \ldots, (a_1 - a_2)^{-\frac{1}{2}}\bar{w}_{nk-1}\) has the distribution of a range of \(k - 1\) independent unit normal variables and so the range of \((n(a_1 - a_2)^{-\frac{1}{2}}\bar{r}_i, i = 1, \ldots, k-1)\) also has this distribution (for \(n \to \infty\)).

Substituting this in (1.1) and letting \(s_{k-1}\) denote the range of \(k - 1\) independent unit normal variables, we now have:
(2.9) \[ \alpha(F,G) = P\left[q_k^{-1} < q_k \sqrt{\frac{k^2}{12 a_1 - a_2}}\right] \] for \( n \to \infty \).

where, \( a_1 - a_2 \) depends on \( F \) and \( G \) through (2.1), (2.2), (2.3) and

\[(2.10) \quad a_1 - a_2 = \frac{1}{12} k^2 + (2r - p - \frac{1}{6})(k - 1) + q - p^2 - \frac{1}{12} \cdot \]

For the right interpretation of (2.9) and (2.10) it should be noted that

\[ a_1 - a_2 = \frac{1}{2n} \text{var}(\bar{r}_i - \bar{r}_j) \quad (1 \leq i, j \leq k - 1 \text{ and } n \to \infty) \]

and under \( H_0 \) we have \( a_1 - a_2 = \frac{1}{12} k^2 \).

3. Maximum value of \( \alpha(F,G) \)

In this section we shall compute the maximum value of \( \alpha(F,G) \) and we are anxious to know whether or not \( \alpha(F,G) \) is smaller than \( \alpha \) for all \( F \) and \( G \).

From (2.9) we see that \( \alpha(F,G) \) will be maximized when \( a_1 - a_2 \) assumes its maximum value.

Writing

\[(3.1) \quad 2r - p = \int (2F - 1)GdF = \int_{\{x \mid F(x) < 1\}} (2F - 1)GdF + \int_{\{x \mid F(x) > 1\}} (2F - 1)GdF \]

we see that \( 2r - p \) is maximized for the pairs \( (F,G) \) satisfying the following conditions:

\[ a \quad \text{If } F(x) < \frac{1}{2} \text{ then } G(x) = 0 \]

\[ b \quad \text{If } F(x) > \frac{1}{2} \text{ then } G(x) = 1 \]

that is: \( F = \frac{1}{2} \) on the support of \( G \).

Now it happens that \( q - p^2 \) is maximized by the same pairs \( (F,G) \) (satisfying (3.2)).

For such a pair both \( 2r - p \) and \( q - p^2 \) are equal to \( \frac{1}{4} \), and (by 2.10)

\[ a_1 - a_2 = \frac{1}{12} (k^2 + k + 1). \]
Hence the maximum value of $\alpha(F,G)$ will be

$$\text{P}[q_{k-1} > q_k \sqrt{\frac{k^2}{k^2 + k + 1}}] .$$

With the aid of a table for the c.d.f. of the range of unit normal variables, e.g. Harter [3], we find:

**Table 2.1.**

Maximum values of $\alpha(F,G)$ for $\alpha = 0.01$, $0.025$, $0.05$ and $0.10$

<table>
<thead>
<tr>
<th>$k$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>12</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.01$</td>
<td>0.0153</td>
<td>0.0181</td>
<td>0.0182</td>
<td>0.0178</td>
<td>0.0172</td>
<td>0.0167</td>
<td>0.0162</td>
<td>0.0158</td>
<td>0.0151</td>
<td>0.0143</td>
<td>0.0134</td>
</tr>
<tr>
<td>$0.025$</td>
<td>0.0303</td>
<td>0.0361</td>
<td>0.0386</td>
<td>0.0385</td>
<td>0.0379</td>
<td>0.0372</td>
<td>0.0365</td>
<td>0.0358</td>
<td>0.0347</td>
<td>0.0334</td>
<td>0.0318</td>
</tr>
<tr>
<td>$0.05$</td>
<td>0.0512</td>
<td>0.0643</td>
<td>0.0682</td>
<td>0.0690</td>
<td>0.0688</td>
<td>0.0682</td>
<td>0.0674</td>
<td>0.0667</td>
<td>0.0652</td>
<td>0.0633</td>
<td>0.0612</td>
</tr>
<tr>
<td>$0.10$</td>
<td>0.0877</td>
<td>0.1123</td>
<td>0.1208</td>
<td>0.1240</td>
<td>0.1250</td>
<td>0.1250</td>
<td>0.1245</td>
<td>0.1238</td>
<td>0.1224</td>
<td>0.1202</td>
<td>0.1172</td>
</tr>
</tbody>
</table>

So it may occur that $\alpha(F,G)$ is larger than $\alpha$.

Intuitively it is also clear, that the value of $a_1 - a_2 = \frac{1}{2n} \text{var}(\bar{X}_1 - \bar{X}_2)$ is maximal for the pairs $(F,G)$ satisfying (3.2), since in that case the $k$-th sample is expected to receive the midranks.

In the next chapter we shall see that, even when $G$ is a shift of $F$, $\text{var}(\bar{X}_1 - \bar{X}_2)$ can assume larger values than under $H_0$, large enough to let

$$q_k \sqrt{\frac{1}{12k^2}} \sqrt{\frac{a_1^2 - a_2}{a_1^2}} < q_{k-1} \alpha$$

in formula (2.9) be smaller than $q_{k-1} \alpha$.

4. **Supremum of $\alpha(F,G)$ for shift alternatives**

Now we shall consider only pairs $(F,G)$ for which there exists a real number $c$ such that:

$$G(x) = F(x - c) \text{ for all } x \in \mathbb{R}$$

and ask whether or not $\alpha(F,G) \leq \alpha$ for all such pairs $(F,G)$, that is for all $F$ and $c$.

As in this case $\alpha(F,G)$ depends only on $F$ and the shift $c$, we shall adjust
our notation:
\[ a(F,c) := a(F,G) \text{ where } G \text{ is defined by (4.1)} \, . \]

Furthermore we define \( a(F) \) by the relationship

\[ (4.2) \quad a(F) := \sup_{c \in \mathbb{R}} a(F,c) \, . \]

From this moment also \( p,q,r \) and \( a_1 - a_2 \) should be regarded as functions of \( F \) and \( c \).

First we try to maximize \( 2r - p \) over \( F \) and \( c \).

Suppose \( c > 0 \) (shift to the right). Then \( G(x) \leq F(x) \) for all \( x \in \mathbb{R} \) and consequently

\[
2r - p = \int_{\{x | F(x) < {1\over 2}\}} G(2F - 1) \, dF + \int_{\{x | F(x) > {1\over 2}\}} G(2F - 1) \, dF \\
\leq 0 + \int_{\{x | f(x) > {1\over 2}\}} (2F^2 - F) \, dF = \left[ {2\over 3} F^3 - {1\over 2} F^2 \right]_{F=1}^{F=1} = {5\over 24} 
\]

If \( c < 0 \), then also holds: \( 2r - p < {5\over 24} \) (the proof is analogous to the case of \( c > 0 \)).

The next example shows that \( {5\over 24} \) is the lowest upperbound for \( 2r - p \).

**Example 4.1.**

Let \( F \) be defined by

\[ (4.3) \quad F(x) := \begin{cases} 
  x + {1\over 2} & \text{if } -{1\over 2} \leq x \leq 0 \\
  x + {1\over 2} & \text{if } 0 \leq x \leq {1\over 2} m \\
  \frac{x}{m} + {1\over 2} & \text{if } {1\over 2} m \leq x 
\end{cases} \]

and

\[ G(x) := F(x - {1\over 2}) \]

then for \( m \rightarrow \infty \) one will find:

\[ 2r - p = {5\over 24} - O\left(\frac{1}{m}\right) \, . \]
Maximizing \( q - p^2 \) for shift alternatives means maximizing \( \text{var}(F(x - c)) \) over \( F \) and \( c \), where \( x \) has the distribution function \( F \). This is realized by the same \( F \) of example 4.1 for \( m \to \infty \) (see Statistica Neerlandica, 1977, solution of problem nr. 45), but here the shift \( c \) should be chosen differently, viz.:

\[
c = m\left(\frac{1}{2} \sqrt{5} - 1\right) + \frac{1}{2}.
\]

In that case we have

\[
(4.4) \quad q - p^2 + \frac{5}{24} (3 - \sqrt{5}) \approx 0.159 \quad \text{for} \quad m \to \infty,
\]

so \( \frac{5}{24} (3 - \sqrt{5}) \) turns out to be the supremum of \( q - p^2 \) for shift alternatives.

If \( c = \frac{1}{2} \) (then \( 2r - p \) was maximized) we have

\[
(4.5) \quad q - p^2 + \frac{29}{192} \approx 0.151 \quad \text{for} \quad m \to \infty.
\]

Combination of (4.4) and (4.5) gives:

\[
\frac{1}{12} \left( k^2 + \frac{1}{2} k + \frac{5}{16} \right) \leq \sup_{F,c} (a_1 - a_2) \leq \frac{1}{12} \left( k^2 + \frac{1}{2} k + \frac{5}{2} (3 - \sqrt{5}) - \frac{3}{2} \right)
\]

and after substituting this in (2.9) we obtain the following results:

**Table 4.1.**

Lower and upper bounds for the supremum of \( \alpha(F) \) (for shift alternatives).

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( k = 3 )</th>
<th>( k = 4 )</th>
<th>( k = 5 )</th>
<th>( k = 6 )</th>
<th>( k = 7 )</th>
<th>( k = 8 )</th>
<th>( k = 9 )</th>
<th>( k = 10 )</th>
<th>( k = 12 )</th>
<th>( k = 15 )</th>
<th>( k = 20 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = 0.01 )</td>
<td>0.0079 - 0.0081</td>
<td>0.0175 - 0.0180</td>
<td>0.0230 - 0.0235</td>
<td>0.0263 - 0.0266</td>
<td>0.0268 - 0.0271</td>
<td>0.0271 - 0.0273</td>
<td>0.0273 - 0.0274</td>
<td>0.0273 - 0.0275</td>
<td>0.0273 - 0.0274</td>
<td>0.0272 - 0.0273</td>
<td>0.0270 - 0.0270</td>
</tr>
<tr>
<td>( \alpha = 0.025 )</td>
<td>0.0325 - 0.0333</td>
<td>0.0431 - 0.0439</td>
<td>0.0478 - 0.0483</td>
<td>0.0501 - 0.0506</td>
<td>0.0514 - 0.0518</td>
<td>0.0521 - 0.0525</td>
<td>0.0526 - 0.0529</td>
<td>0.0529 - 0.0531</td>
<td>0.0531 - 0.0533</td>
<td>0.0532 - 0.0533</td>
<td>0.0530 - 0.0531</td>
</tr>
<tr>
<td>( \alpha = 0.05 )</td>
<td>0.0612 - 0.0625</td>
<td>0.0816 - 0.0827</td>
<td>0.0909 - 0.0918</td>
<td>0.0958 - 0.0965</td>
<td>0.0987 - 0.0993</td>
<td>1.005 - 1.010</td>
<td>1.019 - 1.022</td>
<td>1.025 - 1.028</td>
<td>1.034 - 1.037</td>
<td>1.039 - 1.041</td>
<td>1.041 - 1.042</td>
</tr>
<tr>
<td>( \alpha = 0.10 )</td>
<td>0.0612 - 0.0625</td>
<td>0.0816 - 0.0827</td>
<td>0.0909 - 0.0918</td>
<td>0.0958 - 0.0965</td>
<td>0.0987 - 0.0993</td>
<td>1.005 - 1.010</td>
<td>1.019 - 1.022</td>
<td>1.025 - 1.028</td>
<td>1.034 - 1.037</td>
<td>1.039 - 1.041</td>
<td>1.041 - 1.042</td>
</tr>
</tbody>
</table>
In table 4.1 we see that also for shift alternatives \( \alpha(F) \) might be larger than \( \alpha \), however the exceedances, if any, are rather small for the usual values of \( \alpha \).

5. **Conditions on \( F \) such that \( \alpha(F) < \alpha \) (for shift alternatives)**

We now investigate which "type" of distribution functions causes \( \sup_F \alpha(F) \) to be larger than \( \alpha \) and which conditions on \( F \) are required to guarantee \( \alpha(F) \leq \alpha \). The first theorem we prove is:

**Theorem 5.1.**

If \( F \) is symmetrical and unimodal (and continuous) then

\[
2r - p \leq \frac{1}{6}
\]

and hence \( \alpha(F) < \alpha \).

**Proof.** (due to prof. R. Doornbos)

As the problem is translation invariant, it is no restriction to let \( F \) be symmetrical in \( x = 0 \). Write:

\[
2r - p = \int_{-\infty}^{0} F(x - c)(2F(x) - 1)dF(x) + \int_{0}^{\infty} F(x - c)(2F(x) - 1)dF(x)
\]

then substitution in the first term of

\[
x = -x'
\]

and

\[
F(-x' - c) = 1 - F(x' + c)
\]

gives:

\[
\int_{-\infty}^{0} F(x - c)(2F(x) - 1)dF(x) = \int_{0}^{\infty} (F(x' + c) - 1)(2F(x') - 1)dF(x')
\]

so:

\[
2r - p = \int_{0}^{\infty} (F(x + c) + F(x - c) - 1)(2F(x) - 1)dF(x)
\]
Now for \( x > 0 \), the relationship

\[
F(x + c) + F(x - c) \leq 2F(x)
\]

holds.

If \( x \geq |c| \), this follows from the unimodality of \( F \) (then \( F \) concave for \( x > 0 \)). For \( 0 < x < |c| \) symmetry and unimodality both are necessary to prove (5.3).

Putting (5.3) in (5.2) we find:

\[
2r - p \leq \int_{0}^{\infty} (2F(x) - 1)^2 dF(x) = \left[ \frac{1}{\delta} (2F(x) - 1)^3 \right]_{F(x) = 1}^{\infty} = \frac{1}{6}.
\]

As \( q - p^2 \leq \frac{2}{\sqrt{4} - \sqrt{3}} < \frac{1}{6} \) for all shift alternatives we have \( a_1 - a_2 < \frac{1}{12} k^2 + 1 \), so the square root in (2.9) is larger than \( \sqrt{\frac{k^2}{k^2 + 1}} \) which is not small enough to compensate the difference between \( q_k^\alpha \) and \( q_k^{\alpha_k} \), so \( \alpha(F) < \alpha \), as shown in the next table.

**Table 5.1.** Upper bound for \( \alpha(F) \), when \( F \) is symmetrical and unimodal.

<table>
<thead>
<tr>
<th>( k=3 )</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>12</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = .01 )</td>
<td>.0057</td>
<td>.0071</td>
<td>.0077</td>
<td>.0081</td>
<td>.0083</td>
<td>.0085</td>
<td>.0086</td>
<td>.0087</td>
<td>.0089</td>
<td>.0091</td>
</tr>
<tr>
<td>( .025 )</td>
<td>.0135</td>
<td>.0173</td>
<td>.0190</td>
<td>.0200</td>
<td>.0206</td>
<td>.0211</td>
<td>.0214</td>
<td>.0217</td>
<td>.0222</td>
<td>.0227</td>
</tr>
<tr>
<td>( .05 )</td>
<td>.0262</td>
<td>.0339</td>
<td>.0375</td>
<td>.0396</td>
<td>.0410</td>
<td>.0421</td>
<td>.0429</td>
<td>.0435</td>
<td>.0445</td>
<td>.0455</td>
</tr>
<tr>
<td>( .10 )</td>
<td>.0516</td>
<td>.0674</td>
<td>.0749</td>
<td>.0793</td>
<td>.0823</td>
<td>.0845</td>
<td>.0860</td>
<td>.0872</td>
<td>.0893</td>
<td>.0913</td>
</tr>
</tbody>
</table>

Now we want to relax the conditions on \( F \) in theorem 5.1, especially the symmetry, a requirement which is often not fulfilled in practice. But unimodality alone is not sufficient to ensure \( \alpha(F) \leq \alpha \), since \( F \) in example 4.1 also is unimodal. As we shall see later, strong unimodality will be sufficient and also a more general theorem will be proved.

As \( F \) in example 4.1 is extremely skewed for \( m \to \infty \), one would guess that \( \alpha(F) \) is larger when \( F \) is more skewed. We shall see that this guess is the right starting point to come to a more general theory.
To describe the skewness of distribution functions, we shall use the convex order relation, introduced by Van Zwet [8]. During the remaining part of this section we shall restrict ourselves to a class $F$ on which that weak order relation will be defined.

**Definition 5.1.**

Let $F$ be the class of distribution functions $F$ for which there exists an interval $I_F = (x_1, x_2)$ ($x_1$ and $x_2$ may be not finite) such that the following three conditions are satisfied:

(5.4) $F(x_1) = 0$ and $\lim_{x \to x_2} F(x) = 1$

(5.5) $F$ is differentiable on $I_F$

(5.6) $F' > 0$ on $I_F$

Note that $F$ differs somewhat from the class $F$ defined by Van Zwet.

**Definition 5.2.**

For $F, G \in F$ we say:

$F \preceq G$ if and only if $G^{-1} F$ is convex on $I_F$

$F \prec G$ should be interpreted as: $G$ is more skewed to the right then $F$.

If the densities of $F$ and $G$ are called $f$ and $g$, then we also can say (lemma 4.1.3, Van Zwet [8]):

$F \preceq G$ if and only if

(5.7) $\frac{(G^{-1})'(u)}{(F^{-1})'(u)} = \frac{f(F^{-1}(u))}{g(G^{-1}(u))}$ is nondecreasing for $u \in (0,1)$.

Furthermore we define for $F \in F$ (where $x$ has distribution $F$)

(5.8) $2r - p(F) := \sup_{c \in \mathbb{R}} \int_0^1 (2u - 1)(F^{-1}(u) - c))du = 2 \sup_{c \in \mathbb{R}} \text{cov}(F(x), F(x - c))$
and

\[(5.9) \quad q - p^2(F) := \sup_{c \in \mathbb{R}} \text{var } F(x - c)\]

and let \(F^*\) be defined by

\[(5.10) \quad F^*(x) := 1 - F(-x)\]

Note that \(F^*\) is also an element of \(F\) and \(-x\) has the distribution function \(F^*\).

"\(F\) is less skew than \(G\)" can now be expressed as:

\[(5.11) \quad G^* \leq F \leq G \quad \text{or} \quad G \leq F \leq G^* .\]

We want to prove that \((5.11)\) implies \(2r - p(F) \leq 2r - p(G)\) and \(q - p^2(F) \leq q - p^2(G)\) and hence \(a(F) \leq a(G)\), since \(a(F)\) is an increasing function of \(2r - p(F)\) and \(q - p^2(F)\).

But first we have to state some lemmas.

\textbf{Lemma 5.1.}

\(F \leq G\) if and only if \(G^* \leq F^*\).

\textbf{Proof.}

\(G^{-1}F(-x)\) is convex in \(x\), so \(F^{-1}G(-x)\) is concave and hence
\((F^*)^{-1}G^*(x) = -F^{-1}(G(-x))\) is convex.

\(\Rightarrow\) Note that \(F^{**} = F\).

\textbf{Lemma 5.2.}

Let \(f_1\) and \(f_2\) be functions on an interval \(I \subset \mathbb{R}\), where \(f_2\) is positive and \(f_1/f_2\) is nondecreasing on \(I\).

If furthermore \(x_1, x_2, x_3, x_4 \in I\) such that \(x_1 \leq x_3\) and \(x_2 \leq x_4\) then:

\[
\int_{x_1}^{x_2} f_1(x)dx \int_{x_1}^{x_3} f_2(x)dx \leq \int_{x_1}^{x_4} f_1(x)dx \int_{x_1}^{x_4} f_2(x)dx .
\]
This lemma can be proved by elementary calculus.

With the aid of these lemmas we now prove the following theorem:

**Theorem 5.2.**

\[(5.13) \quad \text{If } F, G \in F \text{ and } G \leq F \leq C \text{ then } 2r - p(F) \leq 2r - p(G)\]

**Proof.**

First we show:

\[(5.14) \quad \text{if } F \leq G \text{ then } \sup_{c \in (0, \infty)} \int_0^1 (2u - 1)F(F^{-1}(u) - c)du \leq \sup_{c \in (0, \infty)} \int_0^1 (2u - 1)G(G^{-1}(u) - c)du\]

which has been proved, if for any \(c > 0\) there exists a \(c' > 0\) such that:

\[(5.15) \quad F(F^{-1}(u) - c) \geq G(G^{-1}(u) - c') \quad \text{for } u \in (0, \frac{1}{4})\]

and

\[(5.16) \quad F(F^{-1}(u) - c) \leq G(G^{-1}(u) - c') \quad \text{for } u \in (\frac{1}{4}, 1)\]

Take \(c'\) such that it satisfies:

\[F(F^{-1}(\frac{1}{4}) - c) = G(G^{-1}(\frac{1}{4}) - c')\]

hence

\[(5.17) \quad c' = G^{-1}(\frac{1}{4}) - G^{-1}(F(F^{-1}(\frac{1}{4}) - c))\]

If \(\inf I_F > -\infty\) and \(c \geq F^{-1}(\frac{1}{4}) - \inf I_F\), then take

\[c' = G^{-1}(\frac{1}{4}) - \inf I_G\]

(If \(\inf I_F > -\infty\) then also \(\inf I_G > -\infty\), as \(G^{-1}F\) is convex).

For \(u > \frac{1}{4}\) we use lemma 5.2 with:

\[f_1 = (G^{-1})', \quad f_2 = (F^{-1})', \quad x_1 = F(F^{-1}(\frac{1}{4}) - c), \quad x_2 = \frac{1}{4}, \quad x_3 = F(F^{-1}(u) - c), \quad x_4 = u\]
Then $f_1/f_2$ is nondecreasing because of 5.7, so we can conclude:

$$\frac{G^{-1}(\frac{1}{2}) - G^{-1}(F(F^{-1}(\frac{1}{2}) - c))}{F^{-1}(\frac{1}{2}) - F^{-1}(F(F^{-1}(\frac{1}{2}) - c))} \leq \frac{G^{-1}(u) - G^{-1}(F(F^{-1}(u) - c))}{F^{-1}(u) - F^{-1}(F(F^{-1}(u) - c))}$$

and with (5.17):

$$\frac{c' - c}{c} \leq \frac{G^{-1}(u) - G^{-1}(F(F^{-1}(u) - c))}{c}$$

hence (as $c > 0$)

$$F(F^{-1}(u) - c) \leq G(G^{-1}(u) - c') \quad \text{for } u > \frac{1}{2}.$$ 

The proof of (5.16) is identical, except for the interchange of $x_1$ and $x_2$ and also of $x_3$ and $x_4$.

Thus (5.14) has been proved.

If $c < 0$ we have to make use of $G^* \leq F$.

By lemma 5.1 this is equivalent to $F^* \leq G$, hence with (5.14) we have:

$$\sup_{c \in \mathbb{R}} \int_0^1 (2u - 1)F^*(F^{-1}(u) - c)du \leq \sup_{c \in \mathbb{R}} \int_0^1 (2u - 1)G(G^{-1}(u) - c)du$$

By (5.10):

$$(F^*)^{-1}(u) = -F^{-1}(1 - u)$$

so

$$F^*(F^{-1}(u) - c) = 1 - F(F^{-1}(1 - u) + c)$$

and hence

$$\int_0^1 (2u - 1)F^*(F^{-1}(u) - c)du = -\int_0^1 (2u - 1)F(F^{-1}(1 - u) + c)du =$$

$$= \int_0^1 (2u - 1)F(F^{-1}(u) + c)du.$$ 

Substituting this in (5.18), we find:
if $G^* < F$ then

\[
(5.19) \sup_{c \in \mathbb{R}} \int_0^1 (2u - 1)F(F^{-1}(u) - c) \leq \sup_{c \in \mathbb{R}} \int_0^1 (2u - 1)G(G^{-1}(u) - c) du.
\]

Combining (5.14) and (5.19), the proof is completed.

\textbf{Remark.}

Also (5.11) implies $2r - p(F) \leq 2r - p(G)$, as $2r - p(G^*) = 2r - p(G)$.

To prove the analogon for $q - p^2$, we will need the following lemma:

\textbf{Lemma 5.3.}

If $f_1$ and $f_2$ are functions on $(0,1)$, such that

(i) $\int_0^1 f_1(x) dx = \int_0^1 f_2(x) dx < \infty$

(ii) there exists a $x_0 \in (0,1)$ such that $f_1(x) \leq f_2(x)$ for $x \in (0,x_0)$ and $f_1(x) \geq f_2(x)$ for $x \in (x_0,1)$

then

\[
\int_0^1 xf_1(x) dx \geq \int_0^1 xf_2(x) dx.
\]

This lemma is a special case of a theorem due to J.F. Steffenson (see Mitrinović [5], page 114, theorem 13).

\textbf{Theorem 5.3.}

If $F, G \in F$ and $G^* \preceq F \preceq G$ then $q - p^2(F) \leq q - p^2(G)$.

\textbf{Proof.} Let $x$ and $y$ have distribution function $F$ and $G$. As $u$, defined by $u := F(x - c)$, has the distribution function $H(u) = F(F^{-1}(u) + c)$, we have:
First we prove that for any $c > 0$ there exists a $c' \in \mathbb{R}$ such that

$$\text{var } F(x - c) \leq \text{var } G(y - c').$$

We take $c'$ such that $\mathcal{E}F(x - c) = \mathcal{E}G(y - c')$, that is

$$\int_0^1 F(F^{-1}(u) + c) du = \int_0^1 G(G^{-1}(u) + c') du.$$

There exists such a $c'$, since $F$ and $G$ are continuous and hence the integrals are continuous functions of $c$ and $c'$.

By (5.20), (5.21) and (5.23) we have that (5.22) is fulfilled if

$$\int_0^1 uF(F^{-1}(u) + c) du \geq \int_0^1 uG(G^{-1}(u) + c') du.$$

This follows from lemma 5.3, if we substitute

$$f_1(u) = F(F^{-1}(u) + c) \quad \text{and} \quad f_2(u) = G(G^{-1}(u) + c').$$

Condition (i) of lemma 5.3 is satisfied by (5.23) and condition (ii) is satisfied because:

1. By (5.23) there exists $u_0 \in (0,1)$ such that $F(F^{-1}(u_0) + c) = G(G^{-1}(u_0) + c')$ ($F$, $F^{-1}$, $G$ and $G^{-1}$ are continuous).

2. As $F \leq G$ we can use lemma 5.2 in the same way as in the proof of theorem 5.2 with $\frac{1}{2}$ replaced by $u_0$. This gives $F(F^{-1}(u) + c) \leq G(G^{-1}(u) + c')$ for $u \in (0,u_0)$ and the reverse inequality for $u \in (u_0,1)$.

Hence now we have:
If $c < 0$ we use $G^* < F$, which is equivalent to $F^* < G$.

As $-\bar{x}$ has distribution function $F^*$, (5.25) gives:

$$\sup_{c \in \mathbb{R}} + \text{var } F(-\bar{x} - c) \leq \sup_{c \in \mathbb{R}} \text{var } G(y - c)$$

and because

$$\text{var } F^*(-\bar{x} - c) = \text{var } F(x + c)$$

we find

$$G^* < F \Rightarrow \sup_{c \in \mathbb{R}} - \text{var } F(x - c) \leq \sup_{c \in \mathbb{R}} \text{var } G(y - c).$$

Together with (5.25) this completes the proof. 

**Application:**

Now we take $G$ defined by (the negative exponential distribution):

$$G(x) = 1 - e^{-x} , \quad x \in (0, \infty).$$

Then $G^* < F < G$ is equivalent to:

$$\log F \text{ and } \log(1 - F) \text{ both concave}.\,$$

Since for $G$ defined by (5.26) the following relation holds:

$$2r - p(G) = \frac{3}{16} \quad \text{and} \quad q - p^2(G) = \frac{1}{9}$$

we may conclude that if $F$ satisfies (5.27), then by (2.10)

$$a_1 - a_2(F) \leq \frac{1}{12}(k^2 + 4k + \frac{1}{12})$$

and hence by (2.9) we can construct the following table:
Table 5.2.

Upper bounds for $\alpha(F)$ where $\log F$ and $\log(1 - F)$ both concave (and $F \in F$)

<table>
<thead>
<tr>
<th>$k$=</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>12</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$=.01</td>
<td>.0053</td>
<td>.0073</td>
<td>.0083</td>
<td>.0088</td>
<td>.0092</td>
<td>.0094</td>
<td>.0095</td>
<td>.0097</td>
<td>.0098</td>
<td>.0099</td>
<td>.0100</td>
</tr>
<tr>
<td>.025</td>
<td>.0127</td>
<td>.0176</td>
<td>.0200</td>
<td>.0214</td>
<td>.0223</td>
<td>.0229</td>
<td>.0234</td>
<td>.0237</td>
<td>.0241</td>
<td>.0245</td>
<td>.0248</td>
</tr>
<tr>
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<td>.0249</td>
<td>.0345</td>
<td>.0393</td>
<td>.0422</td>
<td>.0440</td>
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<td>.0462</td>
<td>.0468</td>
<td>.0478</td>
<td>.0486</td>
<td>.0493</td>
</tr>
<tr>
<td>.10</td>
<td>.0496</td>
<td>.0682</td>
<td>.0777</td>
<td>.0834</td>
<td>.0870</td>
<td>.0895</td>
<td>.0914</td>
<td>.0928</td>
<td>.0947</td>
<td>.0965</td>
<td>.0979</td>
</tr>
</tbody>
</table>

Corollary.

Table 5.2 is also valid if $F$ is strongly unimodal.

Proof.

If $F$ is strongly unimodal, that is $\log f$ is concave, then $\log F$ and $\log(1 - F)$ both concave.

If $F$ is twice differentiable, this can be proved with the aid of lemma 5.2, but here a more general proof (due to F.W. Steutel) is given:

\[
\log f \text{ concave } \iff \log f(u + a) - \log f(u) \geq \log f(x + a) - \log f(x)
\]

(5.28)

\[
\text{for all } u < x \text{ and } a > 0
\]

\[
\log F \text{ concave } \iff \frac{F(x + a)}{F(x + a)} \leq \frac{F(x)}{F(x)}
\]

(5.29)

\[
\text{for all } x \in \mathbb{R} \text{ and } a > 0 .
\]

Now (5.28) implies (5.29) because

\[
\frac{F(x + a)}{F(x + a)} = \int_{-\infty}^{x} \frac{f(u + a)}{f(x + a)} du \geq \int_{-\infty}^{x} \frac{f(u)}{f(x)} du = \frac{F(x)}{F(x)} .
\]

In a similar way one can prove the log-concavity of $1 - F$ from (5.28). \qed
6. Some final remarks

Of course \( \alpha(F,G) \) (or \( \alpha(F) \) for shift alternatives) can only exceed \( \alpha \) if the distribution of \( \bar{x}_1, \ldots, \bar{x}_{k-1} \) depends on the distribution of the \( k \)-th sample.

This is not the case with the multiple comparisons procedures for normal models (e.g. the methods of Tukey and Scheffé). Consequently, if not all samples come from the same distribution, the probability of concluding those samples to be different, which in fact are identical, is smaller than \( \alpha \) for these methods.

The same holds true for the nonparametric method proposed by Steel [7], since here the rank means of the \( i \)-th and the \( j \)-th sample are computed from those two samples only and not from all \( k \) samples together. Besides the fact, that for Steel's procedure the outcome of the comparison of two samples is not influenced by the other observations, it also has in general a larger power than the method based on the Kruskal-Wallis test (see: de Boo [1]).

So we may conclude that Steel's method is more suited for nonparametric simultaneous inference in the \( k \)-sample case.

Finally it should be remarked that the method derived from Friedman's test (see Miller 4, page 172-178) suffers from the same defect and consequently the \((k-1)\)-mean significance level may also be larger than \( \alpha \). But this subject will be treated in a forthcoming paper.

References


