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A Negative Bendixson-Like Criterion for a Class of Hybrid Systems

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Abstract—A condition which ensures the absence of periodic orbits for nonsmooth dynamical systems is presented. The condition is a higher dimensional generalization of Bendixson's criterion applicable to differential inclusions that are useful in the description of hybrid systems. The main argument is based on contraction analysis of the d-measured volume along the system trajectories. A connection to methods for estimating the Hausdorff dimension is emphasized. For a class of hybrid systems described by a linear system with relay feedback the conditions are presented in the form of linear matrix inequalities. A simple but illustrative example is analyzed.

Index Terms—Differential inclusions, periodic solutions.

I. INTRODUCTION

DISCONTINUOUS dynamical systems and, particularly, relay systems have attracted considerable attention over the last decades. While the mathematics of smooth dynamical systems still produces new and interesting discoveries, in applied disciplines it has been realized that for many applications discontinuities should be taken into account. For example, discontinuities can be used to simplify modeling of friction in mechanical systems, to design disturbance tolerant sliding mode controllers, to deal with a switching control strategy in manufacturing systems, and so on. A hot topic in research in the control community is formed by the class of so called hybrid dynamical systems, which combine continuous and discrete dynamics. Although the existing literature on this subject includes a growing number of monographs and papers (see, e.g., [1]–[5], to mention a few), those systems are far from being well understood.

Hybrid systems, being nonlinear dynamical systems, can have a very rich behavior and one of the main theoretical problems is to predict and to understand them without explicitly solving the equations describing the system. In this paper we study oscillatory behavior in nonlinear hybrid dynamical systems that recently receive lots of attention in the control community [6], [7]. The class of systems we study is

described by nonlinear differential equations with a discontinuous right-hand side. A particular result that is obtained here is a generalization of Bendixson's negative divergency test for that class of systems. This simple test gives a sufficient condition for the nonexistence of periodic orbits for smooth planar systems. This classical result claims that if in a simply connected domain the divergency of a vector field does not change sign, then this domain does not contain a periodic orbit. A classical proof of this statement is based on the divergence theorem and cannot be generalized to the higher dimensional case. The main purpose of this paper is to present a possible generalization of the Bendixson result to arbitrary dimension taking into account the possible discontinuity of the right hand side. There are several higher dimensional generalizations of this criterion, see, e.g., [8]–[12]. Muldowney and Li [9]–[11] used an approach based on compound matrices to prove a negative Bendixson-like criterion. In this paper we investigate this question by a method which allows to estimate the Hausdorff dimension of invariant compact sets [13]–[17]. In doing so, we first present a generalization for an estimate of the Hausdorff dimension formula for nonsmooth systems and then, based on that result we prove a negative criterion for the nonexistence of periodic orbits. It is also worth mentioning that although we present only sufficient conditions for nonexistence of periodic solutions of particular hybrid systems our approach is also applicable to disprove some bifurcations that can occur in such systems.

From a practical point of view a design based on global stability of a system can be too restrictive and conservative. A possible weaker criterion is that all trajectories tend to a set consisting of equilibrium points; that is the system can not exhibit oscillatory behavior. This fact indicates the importance of Bendixson-like criteria for the design and control of dynamical systems. A similar motivation can be found, for example in a recent paper [18], where for smooth systems a condition was presented which guarantees that almost all trajectories tend to an equilibrium. Another approach to simplify the stability analysis of discontinuous systems is based on the generalization of the Invariance Principle for differential inclusions, see, e.g., [19].

A general idea behind the proof of the nonexistence criterion is relatively simple. If one is able to show that in a simply connected positively invariant domain the (semi)flow generated by the system contracts the area of some initial surface, it is sufficient to claim that no periodic orbits can lie inside this domain. By reversing time, the same holds true for area-expanding systems. Together with Liouville's theorem this argument gives another proof of Bendixson's criterion that can be generalized to arbitrary dimensions. To characterize the area of a surface one can use the so called Hausdorff 2-measure, so area-contracting
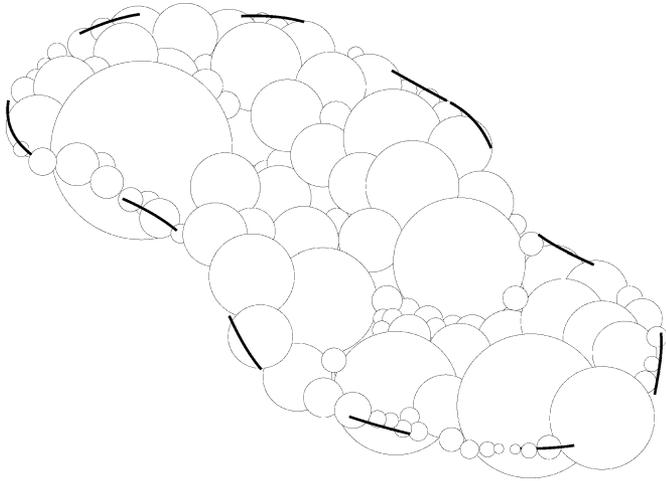
Fig. 1. Hausdorff 2-measure is approximated by the cross-sections of little balls with the surface.

Fig. 2. Properties of the Hausdorff d-measure.

systems are those for which the Hausdorff 2-measure of any initial measurable set vanishes with time.

The main method employed in our study is based on stability/dichotomy-like properties of solutions with respect to each other rather than with respect to some invariant sets. The first results in this direction were developed by Demidovich [20], see also [21], [22] and Yoshizava [23]. Methods based on similar ideas are appreciated now in the control community [24]–[26]. A natural way to investigate those properties is based upon linearization of the dynamical system along any given trajectory which excludes the consideration of non smooth systems. In this paper, instead of linearization we investigate the behavior of some quadratic forms defined for a pair of trajectories of the system, and which allows to consider discontinuous systems. The conditions presented in this paper are formulated in terms of inequalities involving two eigenvalues of some matrix pencil.

The paper is organized as follows. In Section II, we present some necessary background material. Section III contains some result on the estimation of the Hausdorff dimension of invariant sets. Based on these results in Section IV we present a new version of a generalized Bendixson's criterion. Particular attention is then drawn to LMI-based results for linear systems with relay feedback.

II. HAUSDORFF DIMENSION

Consider a compact subset K of $\mathbb{R}^n$. Given $d \geq 0$, $\varepsilon > 0$, consider a covering of K by open spheres B_i with radii $r_i \leq \varepsilon$ (see Fig. 1). Denote by

$$\mu(K, d, \varepsilon) = \inf \sum_i r_i^d \qquad (1)$$

the d-measured volume of covering of the set K. Here, the infimum is calculated over all ε-coverings of K. There exists a limit, which may be infinite

$$\mu_d(K) := \sup_{\varepsilon > 0} \mu(K, d, \varepsilon) = \lim_{\varepsilon \to 0} \mu(K, d, \varepsilon).$$

It can be proved that μ_d is a Borel regular measure on $\mathbb{R}^n$ (see [34]).

Definition 1: The measure μ_d is called the *Hausdorff d-measure.*

Some properties of the measure μ_d can be summarized as follows. There exists a single value $d = d_*$, such that for all $d < d_*$, $\mu_d(K) = +\infty$ and for all $d > d_*$, $\mu_d(K) = 0$, with

$$d_* = \inf\{d : \mu_d(K) = 0\} = \sup\{d : \mu_d(K) = +\infty\}.$$

(see [16, Prop. 5.3.2]).

Definition 2: The value d_* is called the *Hausdorff dimension* of the set K.

In the sequel, we will use the notation $\dim_H K$ for the Hausdorff dimension of the set K.

For the control community the notions of Hausdorff measure and Hausdorff dimension are not common and we like to clarify the above definitions.

Suppose we have a two-dimensional bounded surface S with area $m(S)$. We cover this surface by open spheres as required in the definition of the Hausdorff measure. Then, for $d = 1$ and $d = 3$ we have

$$\mu_1(S) = \lim_{\varepsilon \to 0} \mu(S, 1, \varepsilon) = +\infty$$
$$\mu_3(S) = \lim_{\varepsilon \to 0} \mu(S, 3, \varepsilon) = 0$$

while for $d = 2$ we have

$$\mu_2(S) = \frac{m(S)}{\pi}.$$

This example illustrates the behavior of $\mu_d(K)$ for a given K as a function of d. Namely, for values of d less than $\dim_H K$, $\mu_d(K)$ is infinite and for all values of d greater than $\dim_H K$ $\mu_d(K)$ is zero (see [16, Prop. 5.3.2]). This situation is schematically presented in Fig. 2.

For "good" sets such as a piece of an arc, a piece of a smooth surface, etc. the Hausdorff dimension can be used as the dimension in a normal sense (i.e., in the sense of Brouwer, or Lebesgue). This follows from the result (see, e.g., [16, Prop. 5.3.5]) which claims that for a set K of positive n-dimensional Lebesgue measure the Hausdorff dimension of K is n. However for other sets such as the Cantor set the value of the Hausdorff dimension can be fractional. Sets of such type are often encountered as invariant sets of "chaotic" systems, that makes the Hausdorff dimension of invariant sets an important characteristic for "chaotic" systems.

III. UPPER ESTIMATES FOR THE HAUSDORFF DIMENSION OF INVARIANT COMPACT SETS

The extremal property of the Hausdorff dimension suggests an idea of how to estimate it for invariant sets of dynamical systems. Namely, if one is able to prove that for a given set K its Hausdorff d-measure is zero, then it follows that d is an upper estimate of the Hausdorff dimension of K. A possible way to show that the Hausdorff d-measure is zero is to prove the following inequality:

$$\mu_d(\varphi(K)) \leq \nu \mu_d(K) \tag{2}$$

where $\nu < 1$, φ is some mapping and K an invariant set under φ, i.e., $\varphi(K) = K$, and hence $\mu_d(\varphi(K)) = \mu_d(K)$. This identity together with (2) implies that $\mu_d(K) = 0$.

When φ is a flow generated by a system of differential equations inequality (2) follows from the fact that *the d-measured volume of an open neighborhood of the invariant set K decreases along the system trajectories.* This observation suggests to employ a Lyapunov-like technique to estimate the Hausdorff dimension of invariant sets.

As has been mentioned in the Introduction, a generalization of Bendixson's criterion can be derived if one is able to show that in some simply connected region there are no invariant sets with Hausdorff dimension larger than or equal 2. This result can be obtained if one takes a 2-measured volume as a Lyapunov-like function candidate. However, in this section we present a more general result which holds for an arbitrary d-measure and in the next section we present a higher dimensional generalization of Bendixson's criterion.

Consider a system of differential equations

$$\dot{x} = f(x) \quad x \in \Omega \subset \mathbb{R}^n \quad x_0 \in \Omega \tag{3}$$

where $f : \Omega \to \mathbb{R}^n$ is a (possibly) discontinuous vector field defined on some open positively invariant set Ω, and which satisfies conditions guaranteeing the existence of solutions $x(t, x_0)$ in Ω in some reasonable sense, that is, if the function f is discontinuous and satisfies some mild regularity assumptions, one can construct a set-valued function $\mathbf{f}$ according to numerous possible definitions (e.g., Filippov convex definition, Utkin's equivalent control, etc.) such that an absolutely continuous solution of the differential inclusion

$$\dot{x} \in \mathbf{f}(x)$$

is called a solution for system (3). We assume that the set-valued function $\mathbf{f}$ is bounded, upper semicontinuous with closed convex values.

The parameterized mapping $x_0 \mapsto x(t, x_0)$, $t \geq 0$, or the semi-flow will be denoted as $\varphi^t : \Omega \to \Omega$.

Consider a scalar differentiable function $V : \Omega \times \Omega \to \mathbb{R}$, $V(x, x) = 0$.

Define the time derivative of the function V along *two* solutions $x(t, x_1)$, $x(t, x_2)$ of (3) as follows:

$$\dot{V}(x_1, x_2) := \frac{\partial V(x_1, x_2)}{\partial x_1} \dot{x}_1 + \frac{\partial V(x_1, x_2)}{\partial x_2} \dot{x}_2.$$

Since V is Lipschitz continuous and the solutions $x(t, x_i)$ are absolutely continuous functions of time, the derivative

$$\dot{V}(x(t, x_1), x(t, x_2))$$

exists almost everywhere in $[0, \min_i \bar{T}_i)$, where $\bar{T}_i$ is the maximal interval of existence of the solution $x_i(t, x_{i0})$ in Ω.

For the function V, we can also define its upper derivative as follows:

$$\dot{V}^*(x_1, x_2) = \sup_{\xi_i \in \mathbf{f}(x_i)} \left(\frac{\partial V(x_1, x_2)}{\partial x_1} \xi_1 + \frac{\partial V(x_1, x_2)}{\partial x_2} \xi_2 \right).$$

Then, for almost all $t \geq 0$ it follows that

$$\dot{V}(x(t, x_1), x(t, x_2)) \leq \dot{V}^*(x(t, x_1), x(t, x_2)).$$

By the same token, the lower derivative of the function V is defined as

$$\dot{V}_*(x_1, x_2) = \inf_{\xi_i \in \mathbf{f}(x_i)} \left(\frac{\partial V(x_1, x_2)}{\partial x_1} \xi_1 + \frac{\partial V(x_1, x_2)}{\partial x_2} \xi_2 \right).$$

and satisfies

$$\dot{V}(x(t, x_1), x(t, x_2)) \geq \dot{V}_*(x_1(t, x_1), x_2(t, x_2)).$$

almost everywhere.

We formulate the following hypothesis.

H1: There exists a $n \times n$ symmetric positive–definite matrix P, such that the function

$$V(x_1, x_2) = (x_1 - x_2)^\top P (x_1 - x_2) \tag{4}$$

satisfies the following inequality:

$$\dot{V}^*(x_1, x_2) \leq (x_1 - x_2)^\top Q(x_1)(x_1 - x_2) + o(\|x_1 - x_2\|^2)$$

for all $x_1, x_2 \in \Omega$ with a symmetric differentiable matrix valued function Q, bounded on Ω and with the function of higher order terms o obeying

$$\frac{o(\|x_1 - x_2\|^2)}{\|x_1 - x_2\|^2} \to 0, \quad \text{as} \ \|x_1 - x_2\| \to 0$$

uniformly over x_1, x_2 from any compact subset of Ω.

H1a: The lower derivative of the function $W(x_1, x_2) = (x_1 - x_2)^\top Q(x_1)(x_1 - x_2)$ satisfies the following condition: For any compact subset of Ω there is a number M that

$$\dot{W}_*(x_1, x_2) \geq M V(x_1, x_2).$$

Note that assumption H1a is always satisfied if f is locally Lipschitz continuous.

H2: All solutions starting in Ω are defined for all $t \geq 0$.

We begin with the following preliminary result.

Lemma 1: Suppose Assumptions H1 and H2 are satisfied. Then any solution $x(t, x_0)$ to (3), with $x_0 \in \Omega$ is right-unique

(for the definition of right-uniqueness see [1, Ch. 2, p. 106]) and depends continuously on the initial conditions.

Proof: We prove the statement under the weaker assumption that for all x_1, x_2 from any compact subset of Ω there is a positive L such that

$$\dot{V}^* \leq L(x_1 - x_2)^\top P(x_1 - x_2).$$

Then

$$\frac{d}{dt}\left(V(x_1(t), x_2(t))e^{-Lt}\right) \leq 0$$

almost everywhere. The absolutely continuous function Ve^{-Lt} does not increase, and if $x_1 = x_2$ it follows that $V(x_1(t), x_2(t)) = 0$ for $t \geq 0$ where $x_i(t) = x(t, x_i)$. Thus the right-uniqueness is proved. The local Lipschitz continuous dependence on the initial conditions follows from the inequalities:

$$\|x_1(t) - x_2(t)\|^2 \leq \frac{1}{\lambda_{\min}(P)} V(x_1(t), x_2(t))$$
$$\leq \frac{1}{\lambda_{\min}(P)} V(x_1, x_2)e^{Lt}$$
$$\leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} e^{Lt} \|x_1 - x_2\|^2$$

where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the lower and upper bounds for the eigenvalues of P. ∎

The previous lemma shows that the Cauchy problem (3) is well-posed and continuous dependence on initial conditions follows.

Let $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_n(x)$, $x \in \Omega$ be the ordered solutions of the following generalized eigenvalue problem:

$$\det(Q(x) - \lambda P) = 0$$

which are real since both Q and P are symmetric.

Consider a compact set S of finite Hausdorff d-measure for some $d = d_0 + s$, $d \leq n$, where $d_0 \in \mathbb{N}$ and $s \in [0, 1)$. Suppose that $S \subset \Omega$, then $\varphi^t(S) \subset \Omega$ for all positive t. Now, we formulate the following result.

Theorem 1: Suppose hypotheses H1, H1a, and H2 are satisfied. If for some $d = d_0 + s$, $0 < d_0 \leq n$, $0 \leq s < 1$ it follows that

$$\sup_{x \in \Omega} \left(\lambda_1(x) + \cdots + \lambda_{d_0}(x) + s\lambda_{d_0+1}(x)\right) < 0. \qquad (5)$$

Then

$$\lim_{t \to \infty} \mu_d(\varphi^t(S)) = 0.$$

The proof is based on the construction of a finite set of affine maps $\mathbb{R}^n \to \mathbb{R}^n$ which locally approximate φ^t. Then, using the linear part of those maps we approximate how the d-measured volume is changed under those maps to compute the change of $\mu_d(\varphi^t(S))$. The proof of this theorem is presented in the Appendix .

The main result of this section is the following theorem.

Theorem 2: Suppose hypotheses H1, H1a, and H2 are satisfied, and there exist positive integer d_0 and real $s \in [0, 1)$ such that

$$\sup_{x \in \Omega} \left(\lambda_1(x) + \cdots + \lambda_{d_0}(x) + s\lambda_{d_0+1}(x)\right) < 0. \qquad (6)$$

Suppose that there is an invariant compact set $K \in \Omega$.

Then, $\dim_H K \leq d_0 + s$.

Proof: From the previous result it follows that $\mu_d(\varphi^t(K)) \to 0$ as $t \to \infty$. Since K is invariant $\mu_d(K) = \mu_d(\varphi^t(K))$. Therefore, $\mu_d(K) = 0$. ∎

For the continuously differentiable right-hand side of system (3) the previous theorem can be formulated in infinitesimal form.

Corollary 1: [17] Suppose that hypothesis H2 is satisfied and there is an invariant compact set $K \in \Omega$. Assume also that there is a symmetric positive definite $n \times n$ matrix P such that the solutions of the following generalized eigenvalue problem:

$$\det(Q(x) - \lambda P) = 0$$

satisfy the inequality (6), with the matrix Q defined as

$$Q = P\frac{\partial f(x)}{\partial x} + \frac{\partial f^\top(x)}{\partial x}P.$$

Then, $\dim_H K \leq d_0 + s$.

In [17], this result was derived using the linearization of the flow φ^t with an approach close to that due to Douady–Oesterlé [13] and Leonov[16]. It is now seen that this result can also be obtained from a more general argument which is applicable to discontinuous systems.

We have presented local conditions that are easier to verify analytically for particular examples. One can derive a further generalization via integral conditions of the form

$$\sup_{x_{10}} \int_0^\tau \left(\lambda_1(x(t, x_{10})) + \cdots + s\lambda_{d_0+1}(x(t, x_{10}))\right) dt < 0$$

which can be useful for numerical methods.

A. Example: The Lorenz System

It is well known that all trajectories of the Lorenz system

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases} \qquad (7)$$

are ultimately bounded for arbitrary positive σ, r, b; that is, for fixed σ, r, b there is an invariant compact set K. Let us estimate its Hausdorff dimension.

Proposition 1: Suppose the parameters of the system are such that the following inequality:

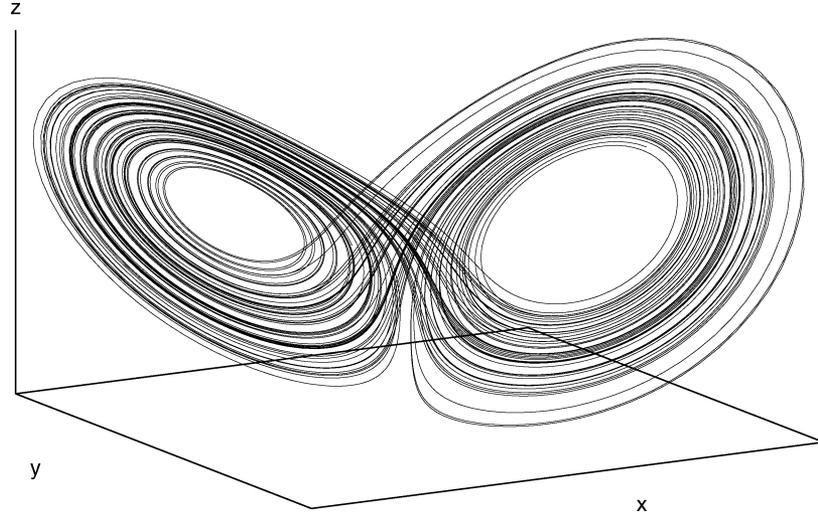$$\limsup_{t \to \infty} \left[\frac{y(t)^2}{b} + (z(t) - 2r)^2\right] \leq 4r^2. \qquad (8)$$

Fig. 3.   Typical trajectory of the Lorenz system.

is satisfied for all initial conditions. Then

$$\dim_H K \le 3 - \frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma - 1)^2 + 4r\sigma}}. \qquad (9)$$

Proof: The Jacobian of the Lorenz system is

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}$$

and let P be

$$P = \begin{pmatrix} \frac{r}{\sigma} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then

$$PJ + J^\top P = \begin{pmatrix} -2r & 2r - z & y \\ 2r - z & -2 & 0 \\ y & 0 & -2b \end{pmatrix}.$$

We know that $\lambda_1 + \lambda_2 + \lambda_3 = -2(\sigma + 1 + b)$, and from $-2(\sigma + 1 + b) < 0$ it follows that $\lambda_3 < 0$ for all x, y, z. Hence, by Corollary 1, to find an upper estimate of s it is sufficient to find a lower estimate of $\bar{\lambda}_3$. Given this estimate, we then evaluate the upper bound on the Hausdorff dimension as

$$\dim_H K \le 3 - \frac{2(\sigma + b + 1)}{|\bar{\lambda}_3|}.$$

The smallest negative eigenvalue of some symmetric matrix Q is a number $\bar{\lambda}$ such that the matrix $Q - \lambda I$ is positive definite as long as $\lambda < \bar{\lambda}$. For all nonpositive λ the following matrix inequality:

$$PJ + J^\top P - \lambda P \ge PJ + J^\top P - \lambda P_1$$

is satisfied, where $P_1 = \operatorname{diag}\{r/\sigma, 1, 0\}$. Therefore, to find a lower estimate for λ_3 it is sufficient to find a lower estimate of the smallest nonpositive solution of the following equation:

$$\det(PJ + J^\top P - \lambda P_1) = 0$$

or

$$r\left(2 + \frac{\lambda}{\sigma}\right)(2 + \lambda) = \frac{y^2}{b} + (2r - z)^2 + \lambda\frac{y^2}{2b}.$$

Using inequality (8) and neglecting the term $\lambda y^2/2b$ (since we are looking for a negative lower estimate of the smallest solution for λ) it is then straightforward to complete the proof. ∎

In [27], it has been proven that for the standard values of parameters of the Lorenz system $\sigma = 10$, $r = 28$, $b = 8/3$, the inequality (8) is satisfied. Thus, in this case, $\dim_H K \le 2.4013$. A typical trajectory of the Lorenz system for those values of the parameters is presented in Fig. 3.

The inequality (9) claims that the Hausdorff dimension of the invariant set of the (chaotic) Lorenz system is bounded by the Lyapunov dimension of the origin. This statement is known as the Eden conjecture [28] which has been proved by Ljashko [29] for a certain set of parameters. The method proposed by Ljashko does not depend on the estimates of the ultimate bounds for the system trajectories but leads to very cumbersome calculations. Recently, Leonov found a set of parameters of the Lorenz system for which the Lyapunov dimension of the Lorenz attractor equals to the local Lyapunov dimension of the origin [30]. Our result can be proved with a much simpler derivation.

IV. A HIGHER DIMENSIONAL GENERALIZATION OF BENDIXSON'S CRITERION

We begin with some definitions.

Definition 3: [33] A set $S \subset \mathbb{R}^n$ is called d-dimensional rectifiable set, $d \in \mathbb{N}$ if $\mu_d(S) < \infty$ and μ_d-almost all of S is contained in the union of the images of countably many Lipschitz functions from $\mathbb{R}^d$ to $\mathbb{R}^n$.

The rectifiable sets are generalized surfaces of geometric measure theory. Any one-dimensional closed rectifiable contour

γ bounds some two-dimensional rectifiable set, for example the cone over γ.

A set is said to be simply connected if any simple closed curve can be contracted to a point continuously in the set.

Theorem 3: Suppose that Assumptions H1, H1a, and H2 are satisfied, let Ω be a simply connected set. Suppose that

$$\sup_{x \in \Omega} \lambda_1(x) + \lambda_2(x) < 0. \qquad (10)$$

Then, Ω does not contain whole periodic orbits.

Proof: The proof of Theorem 3 follows an idea used in the proof of the Leonov theorem [32], see also [31, Th. 8.3.1]. Suppose (10) holds but there is a periodic orbit γ passing through a point x_0 $\gamma := \{x \mid \exists t \geq 0, \ x = x(t, x_0)\}$ which lies entirely in Ω.

Since as assumed the function $\mathbf{f}$ is bounded on any compact set it follows that there is a positive constant $L > 0$ such that for all $x_0 \in \gamma$

$$\|x(t_1, x_0) - x(t_2, x_0)\| \leq L|t_1 - t_2|$$

and thus the set γ is the image of a Lipschitz continuous function. Therefore, the set γ is a rectifiable one-dimensional set. From the theorem on existence of area-minimizing surfaces (see [34, Th. 5.6]) it follows that there exists a two-dimensional rectifiable set $\bar{S} \in \mathbb{R}^n$ such that its boundary is γ and it has minimal Hausdorff 2-measure.

Let S be a rectifiable two-dimensional set $S \subset \Omega$ with boundary γ. The existence of such set follows from the fact that Ω is simply connected. As before, we denote by φ^t the flow of system (3). Let $\mu(S)$ be the Hausdorff 2-measure of a 2-dimensional surface S. Since γ is invariant under φ^t and $\varphi^t(S) \subset \Omega$ for any $t \geq 0$ (Ω is positively invariant) we have

$$\inf_{t \geq 0} \mu\left(\varphi^t(S)\right) \geq \mu(\bar{S}) > 0. \qquad (11)$$

At the same time, using (10) from Theorem 1 it follows that

$$\lim_{t \to \infty} \mu\left(\varphi^t(S)\right) = 0 \qquad (12)$$

which contradicts (11). Therefore, (10) ensures the absence of periodic trajectories lying in Ω. ∎

It is worth noting that this theorem being applied to smooth systems together with its time reversed version (for smooth systems we have local right and left uniqueness) gives the classical Bendixson divergency condition.

The main idea of the proof (see [32]) is based on the existence of a surface with minimal area given its boundary. Although the mathematical problem of proving existence of a surface that has minimal area and is bounded by a prescribed curve, has a long defied mathematical analysis, an experimental solution is easily obtained by a simple physical device. Plateau, a Belgian physicist, studied the problem by dipping an arbitrarily shaped

wire frame into a soap solution. The resulting soap film corresponds to a relative minimum of area and thus produces a minimal surface spanned by that wire contour. A classical solution to Plateau's problem can be found, for example, in [35] with some regularity assumptions on the contour γ that can be violated if γ is a closed orbit corresponding to a periodic solution of a system of differential equations with discontinuous right hand sides. Fortunately, the argument based on geometric measure theory allows to overcome this difficulty.

A final remark in this section is that the theorem can disprove not only the existence of periodic orbits, but also one-dimensional rectifiable invariant sets (see the proof). Therefore the main results of this section can be also used to disprove the existence of invariant sets of more general nature like homoclinic and/or heteroclinic orbits.

A. Example

Consider the following system:

$$\dot{x} = Ax + Bu \quad u = -\mathrm{sign}(y) \quad y = Cx \qquad (13)$$

where $x \in \mathbb{R}^3$, $u, y \in \mathbb{R}^1$ and the matrices A, B, C are given as follows:

$$A = \begin{pmatrix} \alpha & 1 & 1 \\ -1 & \beta & -1 \\ -1 & 1 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix} \quad C = (0\ 0\ 1)$$

with positive b. Consider the smooth function (4) in the form

$$V = (x_1 - x_2)^\top (x_1 - x_2).$$

For this system the corresponding solution according to the Filippov convex definition coincides with the Utkin solution [1]. At the discontinuity points of the right-hand side, the corresponding set valued function in the differential inclusion is obtained by the closure of the graph of the right hand side and by passing over to the convex hull. As shown in [1, p. 155], these procedures do not increase the upper value of $\dot{V}^*$ and hence it is sufficient to compute the derivative of V only in the area of continuity of the right-hand side. The derivative of V in this area satisfies

$$\dot{V} \leq 2(x_1 - x_2)^\top \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -1 \end{pmatrix} (x_1 - x_2).$$

The condition *H1a* is satisfied since $b > 0$. The previous theorem suggests that if $\min\{\alpha, \beta\} \geq -1$, a sufficient condition for the absence of periodic solutions is

$$\alpha + \beta < 0. \qquad (14)$$

To demonstrate that violation of the condition (14) can result in oscillatory behavior we perform a computer simulation for the following parameter values: $\alpha = 1$, $\beta = -1/2$, $b = 1$. The
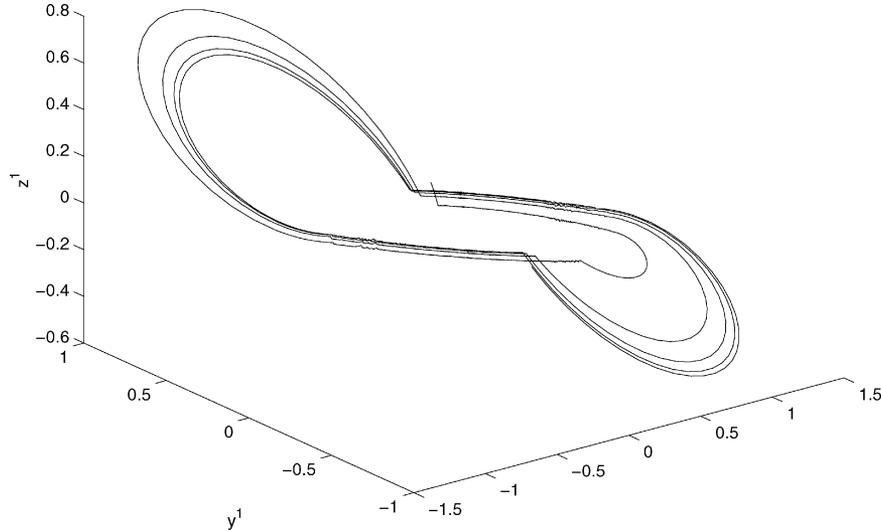
Fig. 4.   Oscillatory behavior of (13) for $\alpha + \beta > 0$.

results of the simulation are presented in Fig. 4. It is seen that the system possesses an orbitally stable limit cycle.

B. An LMI Based Criterion for Lur'e Systems With Discontinuous Right-Hand Side

In the previous example, the matrix A was chosen as a sum of a diagonal and a skew-symmetric matrix that made all necessary calculations trivial. Next we present an LMI based criterion which ensures the absence of periodic solutions for the following system:

$$\dot{x} = Ax + Bu \quad u = -b\operatorname{sign}(y) \quad y = Cx \qquad (15)$$

where $x \in \mathbb{R}^n$, $n \geq 2$, $u, y \in \mathbb{R}^1$, $b > 0$ and the matrices A, B, C are of corresponding dimensions.

Theorem 4: Suppose there exists μ and positive definite matrix P such that the following inequality:

$$\begin{pmatrix} P(A - \mu I_n) + (A - \mu I_n)^\top P & * & * \\ B^\top P - C & 0 & 0 \\ B^\top(PA + A^\top P) + \gamma C & 0 & 0 \end{pmatrix} \geq 0 \qquad (16)$$

is satisfied for some $\gamma \geq 0$. Then if

$$\operatorname{tr} A - (n - 2)\mu < 0$$

the system (15) does not have periodic solutions.

Proof: According to (16) the matrix P satisfies the following equation $PB = C^\top$. Thus, taking the derivative of the following function:

$$V = (x_1 - x_2)^\top P(x_1 - x_2)$$

yields (as in the previous example it is sufficient to compute the derivative in the area of continuity of the right-hand side)

$$\begin{aligned} \dot{V} &= (x_1 - x_2)^\top (PA + A^\top P)(x_1 - x_2) \\ &\quad - 2b(Cx_1 - Cx_2)(\operatorname{sign} Cx_1 - \operatorname{sign} Cx_2) \\ &\leq (x_1 - x_2)^\top (PA + A^\top P)(x_1 - x_2). \end{aligned} \qquad (17)$$

Let us verify the condition H1a with the function

$$W = (x_1 - x_2)^\top (PA + A^\top P)(x_1 - x_2).$$

With similar calculations as above the function W satisfies the condition H1a since $(PA + A^\top P)B = -\gamma C^\top$.

Now consider the smallest solution λ_n of the following equation:

$$\det(PA + A^\top P - \lambda P) = 0. \qquad (18)$$

From the hypothesis it follows that $\lambda_n \geq 2\mu$. On the other hand if λ_i, $i = 1, \ldots, n$ are the solutions of (18) then

$$\lambda_1 + \cdots + \lambda_n = 2\operatorname{tr} A.$$

Since

$$\lambda_i \geq \lambda_n \qquad (19)$$

it follows that

$$\lambda_1 + \lambda_2 \leq 2(\operatorname{tr} A - (n - 2)\mu) < 0$$

and according to Theorem 3 the system (15) has no periodic solutions.   ∎

V. CONCLUSION

In this paper we presented a new *discontinuous version of a Bendixson's like criterion*. The criterion is based on a new result on the estimation of the Hausdorff dimension of invariant sets for (possibly) discontinuous systems. The new criterion can be applied for the design and control of discontinuous systems when the requirement of global stability is too restrictive. Our study is based on dichotomy-like properties of solutions of dynamical systems with respect to each other rather than with respect to some invariant sets. We hope that further development

of this approach will allow better understanding bifurcations in nonsmooth dynamical systems [37].

APPENDIX
PROOF OF THEOREM 1

We prove the theorem under condition that $P = I_n$ since this condition can be met under appropriate linear change of coordinates.

For a linear operator T in $\mathbb{R}^n$ let $\sigma_1 \geq \cdots \geq \sigma_n$ be its singular values and introduce the following notation for some $d = d_0 + s$, $d \leq n$, $s \in [0, 1)$ $\omega_d(T) = \sigma_1 \sigma_{d_0} \sigma_{d_0+1}^s$. If $d = 0$ we put $\omega_d(T) = \sigma_1$.

For similarity denote

$$x(t, x_1) = x_1(t), \quad x(t, x_2) = x_2(t), \quad \Delta(t) = x_1(t) - x_2(t).$$

Since $V(x_1(t), x_2(t))$ is an absolutely continuous function of time

$$
\begin{aligned}
V(x_1(t), x_2(t)) - V(x_1, x_2) &= \int_0^t \dot{V}(x_1(\tau), x_2(\tau)) d\tau \\
&\leq \int_0^t \dot{V}^*(x_1(\tau), x_2(\tau)) d\tau \\
&\leq \int_0^t \Delta^\top(\tau) Q(x_1(\tau)) \Delta(\tau) d\tau \\
&+ \int_0^t o(\|x_1(\tau) - x_2(\tau)\|^2) d\tau
\end{aligned}
$$

and, therefore

$$\Delta^\top(t) \left(I_n - tQ(x_1(t))\right) \Delta(t) \leq \Delta(0)^\top \Delta(0) + h \qquad (20)$$

where the higher order function h is given by $h = h_1 + h_2$ with

$$h_1 = \int_0^t \Delta^\top(\tau) Q(x_1(\tau)) \Delta(\tau) d\tau - t\Delta^\top(t) Q(x_1(t)) \Delta(t)$$
and
$$h_2 = \int_0^t o(\|x_1(\tau) - x_2(\tau)\|^2) d\tau.$$

Let us estimate h_1. According to the mean value theorem

$$h_1 = t \left[\Delta^\top(\theta t) Q(x_1(\theta t)) \Delta(\theta t) - \Delta^\top(t) Q(x_1(t)) \Delta(t)\right]$$

for some $0 < \theta < 1$. Let $\mathcal{B}$ be a compact subset of Ω such that $\forall \tau, 0 \leq \tau \leq t, x_1(\tau), x_2(\tau) \in \mathcal{B}$. Denoting

$$q := t \left[\Delta^\top(\theta t) Q(x_1(\theta t)) \Delta(\theta t) - \Delta^\top(t) Q(x_1(t)) \Delta(t)\right]$$

it follows from assumption H1a that

$$
\begin{aligned}
q &\leq -tM(\mathcal{B}) \int_{\theta t}^t \Delta^\top(\tau) \Delta(\tau) d\tau \\
&= -t^2(1 - \theta) M(\mathcal{B}) \Delta^\top(\xi) \Delta(\xi)
\end{aligned}
$$

with $\theta t < \xi < t$. With the same L as in the proof of Lemma 1

$$\Delta^\top(\xi) \Delta(\xi) \leq e^{L(\mathcal{B})\xi} \Delta^\top(0) \Delta(0)$$

and, therefore

$$|h_1| \leq t^2(1 - \theta) |M(\mathcal{B})| e^{L(\mathcal{B})\xi} \Delta^\top(0) \Delta(0).$$

Hence, if x_2 belong to the ball E_{x_1} of radius ε centered at x_1

$$E_{x_1} = \{x \in \mathbb{R}^n \mid (x - x_1)^\top (x - x_1) \leq \varepsilon^2\}$$

then $|h_1| = o(t)O(\varepsilon^2)$. Using the same ideas to estimate h_2 yields

$$|h_2| = O(t)o(\varepsilon^2)$$

and, finally

$$|h| = o(t)O(\varepsilon^2) + O(t)o(\varepsilon^2).$$

According to (20) the mapping φ^t maps the ball E_{x_1} into the ellipsoid E'_{x_1}

$$
\begin{aligned}
E'_{x_1(t)} = \{x \in \mathbb{R}^n | (x - x_1(t))^\top (I - tQ(x_1(t)))(x - x_1(t)) \\
\leq \varepsilon^2 + |h|\}.
\end{aligned}
$$

To estimate the singular values of the linear part T_x of φ^t, given $t > 0$, consider any linear operator that maps E_{x_1} into E''_{x_1} that can be rewritten as

$$
E''_{x_1} = \{x \in \mathbb{R}^n | (x - x_1)^\top \mu(I - tQ(x_1(t)))(x - x_1) \leq \varepsilon^2\}
$$
$$
\mu = \frac{\varepsilon^2}{\varepsilon^2 + |h|} = \frac{1}{1 + \frac{|h|}{\varepsilon^2}}.
$$

This linear operator T, should satisfy the following relation for $\eta \in \mathbb{R}^n$

$$\mu \eta^\top T^\top (I - tQ) T \eta \leq \eta^\top \eta.$$

Since for any square matrix X the matrices $X^\top X$ and $XX^\top$ are orthogonally congruent, the last inequality can be rewritten as

$$\mu S T T^\top S^\top \leq I$$

where $S^\top S = I - tQ$ is a positive definite matrix provided $0 < t < 1/|\lambda_1|$, with $\lambda_1 \geq \cdots \geq \lambda_n$ being the eigenvalues of Q. The last inequality is equivalent to

$$T T^\top \leq \frac{1}{\mu} S^{-1} S^{-\top} = \frac{1}{\mu} (I - tQ)^{-1}$$

and using simultaneous diagonalization of the right- and left-hand sides of this inequality one gets the following estimate:

$$
\omega_d(T) \leq \left(1 + \frac{|h|}{\varepsilon^2}\right)^d
$$
$$
\times \frac{1}{\sqrt{(1 - t\lambda_1)\cdots(1 - t\lambda_{d_0})\cdot(1 - t\lambda_{d_0+1})^s}}
$$
$$
= \left(1 + d\frac{|h|}{\varepsilon^2} + o\left(\frac{|h|}{\varepsilon^2}\right)\right)
$$
$$
\times \left(1 + \frac{t}{2}(\lambda_1 + \cdots + \lambda_{d_0} + s\lambda_{d_0+1}) + o(t)\right).
$$

Passing to the limit for $\varepsilon \rightarrow 0$ and using the condition imposed on λ's one concludes that there is a $t \geq 0$ such that $\omega_d(T_x) \leq \varrho < 1$. Now, the proof can be completed along the same lines as in the proof of Douady and Osterlé result (see, e.g., [16, Th. 5.4.1]): As before, via the chain rule, the estimation of the singular values of the linear part T_x^m of φ^{mt}, $m = 1, 2\ldots$, satisfies

$$
\omega_d(T_x^m)
$$
$$
= \omega_d(T_{\varphi_x^{(m-1)t}} \ldots T_{\varphi^t x} T_x) \leq \omega_d(T_{\varphi_x^{(m-1)t}}) \cdots \omega_d(T_x) \leq \varrho^m
$$

and the Hausdorff d-measure of S decays with time.

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