Updown generation of Penrose patterns

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1. INTRODUCTION

1.1. Penrose patterns, the surprising non-periodic tilings of the plane, were discovered around 1975, and became generally known through M. Gardner's article [8] and Penrose's paper [11]. Later it was shown ([2]) how they can be considered as topological duals of so-called pentagrids: superpositions of 5 grids, each consisting of equidistant parallel lines, which can be obtained from each other by rotations over multiples of 72° and parallel shifts. This made it clear that Penrose tilings can be completely characterized, at least in the so-called non-singular cases, by a single complex parameter ψ.

1.2. A key role in Penrose's discovery was played by his idea of the operation called deflation, which transforms finite tilings into finite tilings with a larger number of pieces. Applying standard selection principles this led Penrose, in a non-constructive manner, to his infinite tilings. The procedure was described in detail in [6]. There is also a more constructive way, which seems to have been discovered first by J. Conway. We shall call that method updown generation. The present paper is probably the first published account of it. The updown generation method produces the infinite Penrose tilings by means of infinite paths through a simple directed graph (finite automaton).

In [7] updown generation was explained in detail for a very similar case (cf. section 1.6 below). The treatment of that case makes it clear that updown generation applies to rewriting systems in general.
1.3. The main theme of the present paper is to establish the connection between the two very different ways to characterize the Penrose tilings: one by the pentagrid parameter $\xi$, and one by the infinite path in the automaton. A key to this is the dualization of the inflation-deflation idea in the $\xi$-plane (see section 3).

The relation between pentagrid parameter and infinite path will be studied for the non-singular cases only. It seems to be possible but troublesome to extend it to the singular cases.

1.4. The idea to consider tilings as duals of a pentagrid can be generalized at once to other cases of what we can call multigrids. One of them was first published by P. Kramer and R. Neri [10] in the context of crystallography, and that got a sensational actuality when D. Shechtman et al. [12] discovered a new form of matter with a kind of 5-fold symmetry, which was known to be impossible under the assumptions of classical crystallography (based on invariance under groups built up by translations and rotations). For an extensive description of the multigrid method, with further references, we refer to [4], and for corresponding Fourier theory to [3] and [5]. In a vague way one can use the term "quasicrystal" for the topological duals of multigrids, in particular since the Fourier spectra of these duals have the discrete structure we know so well from ordinary crystals.

1.5. As one-dimensional quasicrystals we have to consider the Beatty sequences. These are doubly infinite sequences of zeros and ones (i.e., mappings of $\mathcal{F}$ into \{0,1\}) which have a number of properties in common with Penrose patterns. They are essentially characterized by a real parameter $\gamma$ that is the analog of the complex pentagrid parameter $\xi$. The Beatty sequences show a kind of almost periodicity analogous to the one of the Penrose tilings. The analog of deflation is the rewriting rule 1 $\rightarrow$ 10, 0 $\rightarrow$ 1. A thing we miss in the Beatty sequences, however, is the idea of tilings which are forced upon us by means of fitting rules, so at first sight the Beatty sequences look like solutions without a problem. In [1] a problem is described for which the Beatty sequences form a solution. The problem, quite unrelated to tiling, is to find all sequences of zeros and ones which have infinitely many forefathers in the sense of the rewriting rule. In spite of the fact that the Beatty sequences are not related to tiling, there are various ways to consider them as one-dimensional quasicrystals (see [3]).

In [7] the matter of updown generation was elaborated in great detail for the Beatty sequences. The infinitely-many-forefathers definition of Beatty sequences was shown to be equivalent to a new one: Beatty sequences are exactly those sequences which we get by updown generation.

There is a complete analogy with what was announced in section 1.3. The relation between deflation, rewriting rule and finite automaton is exactly the same, and in both cases the infinite applications of updown generation correspond to infinite paths in the automaton. For the case of the Beatty sequences it was shown in [7] that the two characterizations of an infinite tiling, the one
with the real parameter $\gamma$ and the one with the infinite path, could be related by what was called dual deflation in the space of the $\gamma$'s. In exactly the same way we shall establish in the present paper (section 3.15) that for Penrose tilings the relation between the complex parameter $\xi$ and the infinite path in the automaton is obtained by a similar dual deflation in the $\xi$-plane.

Section 4 will present details of the updown generation for symmetric and for singular Penrose tilings.

Section 5 will treat the question whether different infinite paths can lead to the same tiling.

1.6. In [7] we were able to give an exhaustive treatment Beatty sequences, including the singular ones. The present paper is more modest: if it comes to dual deflation we shall exclude the singular tilings. Our treatment will be directed towards geometrical understanding of the dual deflation, and in the matter of subdividing geometrical figures our intuition prefers to think in terms of interiors rather than in terms of boundaries. In order to add the singular cases it will of course be useful to consult how it was done in the Beatty case, where a certain simplification was obtained by splitting each real number into a left and a right version. In the complex plane such a treatment will certainly be more complicated.

1.7. In this survey we did not mention the Ammann grids as unique description of Penrose tilings. We refer to [9] for this. In particular one might ask how the characteristics of the Ammann grid are related to paths in the automaton, but this question has not been attempted in the present paper.

1.8. The analysis that leads us to the dual deflation in section 3 uses a method that is worth mentioning explicitly here, since it may have many other applications.

The method gives an easy and fast algorithm for deciding whether a given tiling by Penrose rhombs can be extended to a full tiling of the plane. The solution is obtained by normalizing the tiling in the way required for an AR*-pattern (see section 3.1). Then we evaluate, for each one of the finitely many vertices, the polygon in the $\xi$-plane that contains all $\xi$'s for which the corresponding AR*-pattern contains that particular point. If the intersection of all those polygons is empty then the finite tiling can not be infinitely extended. If it is non-empty (possibly just a single point) then every $\xi$ in that intersection provides an AR*-pattern that extends the given finite tiling to a full tiling of the plane.

1.9. Our geometric terminology will use the notion of shape in the plane. A shape is an equivalence class of figures under similarity transform. Similarity includes multiplications (with respect to a point), shifts and rotations, but no reflection with respect to a line. In terms of complex numbers, this similarity transform just means linear transformation.
2. UPDOWN GENERATION OF PENROSE TILINGS

2.1. Figure 1 shows the thick and thin rhombs, the building blocks of the Penrose tilings. The acute angles in the thick rhomb are 72°, those in the thin rhomb are 36°. The edges all have the same length, and are provided with single and double arrows as shown. A Penrose tiling (or, what is the same thing, a Penrose pattern) is a tiling of the plane by thick and thin rhombs satisfying the obvious fitting condition: wherever two edges are pasted together along an edge, they have to have the same kind of arrow along that edge, in the same direction (for a picture of a piece of such a Penrose tiling we refer to [2], figure 2).

One of Penrose's great ideas was the one of deflation, that transforms finite or infinite parts of a tiling into another tiling with smaller pieces, again of the same shape. It is definitely simpler to describe it in terms of rhomb halves, where it becomes a matter of subdivision. We describe this in the next sections. We devote some attention (section 2.3) to systematically discovering the deflation by straightforward inspection of tilings, rather than introducing the deflation as a *deux ex machina*.

2.2. Penrose passed from the arrowed rhombs to the rhomb halves by cutting the thin rhombs along the short diagonal and the thick rhombs along the long diagonal. This leads to four shapes $A$, $A'$, $B$, $B'$ (figure 2).
The edges of the triangles are provided with arrows. In addition to the single and double arrows of the Penrose rhombs, threefold and fourfold arrows have been put along the new edges. Tilings with these $A, A', B, B'$ again have to satisfy the fitting condition that along a common edge two triangles have to have the same kind of arrow in the same direction. This rule implies that the pieces appear in pairs: either an $A$ and an $A'$ are pasted together to form a thin rhomb, or a $B$ and a $B'$ are pasted together to form a thick rhomb. So with the rhomb halves we get exactly the same tilings as with the full rhombs.

2.3. We study how in a tiling of the plane the rhomb halves are connected. For the moment we ignore the threefold and fourfold arrows, and just study the connections by edges with single and double arrows. If an $A$ is pasted to a $B'$ along an edge with single arrow, we denote the situation by $A \leftrightarrow A'$. If an $A$ is pasted to a $B$ along an edge with double arrow, we denote that by $A \leftrightarrow B$. Similarly we use $A' \leftrightarrow B$, etc. Every triangle has exactly two connections, one by $\leftrightarrow$, and one by $\leftrightarrow$. It follows that the triangles are grouped into finite or infinite chains; in each chain the connections $\leftrightarrow$ and $\leftrightarrow$ alternate.

Some combinations are easily seen to be excluded. The cases $B' \leftrightarrow B \leftrightarrow B'$, $A' \leftrightarrow B \leftrightarrow A$, $A \leftrightarrow A'$, shown in figure 3, can never occur, and the same things hold for the mirror images $B \leftrightarrow B' \leftrightarrow B$, $A \leftrightarrow B' \leftrightarrow A'$. In each one of these cases attempts to place more pieces break down in a few steps. So as possible triples we only have to consider $A' \leftrightarrow B \leftrightarrow A'$, $A \leftrightarrow A' \leftrightarrow A$, $B \leftrightarrow A' \leftrightarrow B$ and their mirror images $A \leftrightarrow B' \leftrightarrow B$, $B \leftrightarrow B' \leftrightarrow A'$, $B' \leftrightarrow A \leftrightarrow A'$, $B \leftrightarrow A \leftrightarrow B$.

It easily follows that if we remove all connections $B \leftrightarrow B'$, then the chains fall into pieces $B \leftrightarrow A \leftrightarrow B' \leftrightarrow A \leftrightarrow B'$ and $B \leftrightarrow A' \leftrightarrow A \leftrightarrow B'$ (see figure 4). These pieces have the shape of the thick and the thin rhomb, but a factor $\tau^{-1}$ bigger than the original ones (we use the notation $\tau$ for the golden ratio.

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![Figure 3](image-url)
So a tiling of the full plane into $A$, $A'$, $B$, $B'$ leads to a tiling of the full plane by means of these bigger rhombs.

The big rhombs can again be subdivided into big rhomb halves, to be called big $A$, big $A'$, big $B$, big $B'$. These four big triangles can be provided with arrows again, as shown in figure 5. Figure 6 displays the rule for getting from the arrows on the small pieces to the arrows on the big ones. With these arrowings the big pieces look exactly like the small ones.

We started from a tiling of the plane with small arrowed triangles, and in a unique way this gave rise to a tiling of the plane with big arrowed triangles. And the big triangles obviously satisfy the fitting rule again: adjacent big triangles have the same kind of arrow (with the same direction) on their common side. The tiling with the big arrowed triangles is called the inflation of the original tiling.

Conversely, any tiling of the plane with arrowed rhomb halves can be considered as the inflation of a tiling with smaller pieces, obtained just by dissection. Such dissection will be called deflation. An $A$ (like the first big triangle
in the first row of figure 5) splits into a small $B'$ and a small $A$, a $B$ splits into a small $B'$, a small $A$ and a small $B$, an $A'$ splits into a small $B$ and a small $A'$, and finally a $B'$ splits into a small $B$, a small $A'$ and a small $B'$. On the edges of the small pieces we put arrows: those on the exterior are obtained from the arrowing of the big triangles by using the translation of figure 6 from right to left, and for those in the interior we take the unique possibility to finish the arrowing properly.

2.4. In a tiling with rhomb halves the $A$'s can occur in two different situations, according to their position relative to the inflated tiling. If we deflate a tiling, an $A$ can be obtained in two ways: either as the small $A$ we get by deflation of a big $A$, or as the small $A$ we get by deflation of a big $B$. This gives two possibilities for the position of the big piece if we start from a small $A$. We call these ways $\beta$ and $\delta$.

In total there are 10 such transitions. In figure 5 the greek letters are indicated inside the small triangles. Each letter refers to a way to build a big triangle around a small one. So $\alpha$ refers to the way to pass from the small triangle containing $\alpha$ to the big triangle around it. The symbols $\alpha, \alpha', \beta, \beta', \gamma, \gamma', \delta, \delta'$, $\varepsilon, \varepsilon'$ will be called connectors.

The connectors will also be used in situations obtained by translation, rotation and multiplication (i.e., by linear transformations in the complex plane). The effect of a reflection is that $\alpha$ and $\alpha'$ are interchanged, and similarly $\beta$ and $\beta'$, etc.

Figure 7 presents the finite automaton that describes the transitions. As an example we note that the arrow $\delta$ running from $A$ to $B$ refers to the fact that a small $A$ occurs as a part in the deflation of a big $B$.

In complete analogy with what we had in [7], section 5, the finite automaton can be used to start from a single $A$, $A'$, $B$ or $B'$ in the plane, and to produce tilings of larger and larger pieces of the plane. And again, like in [7], section 6, an infinite path in the automaton corresponds to a tiling of the whole plane (for exceptions see section 2.5 below).

Figure 8 presents an example of updown generation. This will make it clear how it works in general.

2.5. In analogy to what we observed in [7], section 6.4, we have to be aware of the fact that an infinite path in the automaton does not always produce a tiling of the full plane. Sometimes we get a half plane only, and it is even pos-
sible that we only get a 36° sector. In those cases we get to a full Penrose tiling by adding the mirror image of that half plane with respect to its border line, or the repeated mirror images of the sector with respect to its border lines.

Such symmetric tilings can never be obtained by updown generation leading to the full plane at once: the first time that a deflation step would lead to a triangle that is cut into two pieces by an axis of symmetry would give a contradiction.

Details of the generation of symmetric tilings will be given in section 4.

3. THE DUAL FORM OF INFLATION

3.1. In [2] it was explained in full detail how a Penrose rhombus tiling can be obtained from a pentagrid, characterized by a single complex number $\zeta$ as a parameter. If the pentagrid is singular, it corresponds to two different Penrose tilings, and if it is exceptionally singular, there are ten of them. But if we exclude the singular cases, the relation between the parameter $\zeta$ and the Penrose tiling is one to one, assuming that the Penrose tilings are in the standard form called $\text{AR}^*$-patterns in [2], section 15. This normalization amounts to requiring that all edges have unit length, and that all vertices lie in the set $H_{\{1,2,3,4\}}$. With the abbreviation $\zeta$ for the fifth root of unity $\exp(2\pi i/5)$, this $H_{\{1,2,3,4\}}$ is the set of all complex numbers $k_0 + \cdots + k_4 \zeta^4$, with an integral vector $(k_0, \ldots, k_4)$ satisfying $k_0 + \cdots + k_4 \in \{1, 2, 3, 4\}$. The set $H_{\{1,4\}}$, to be mentioned later in this section, is defined in the same way, with $\{1, 4\}$ instead of $\{1, 2, 3, 4\}$.

![Figure 7. The finite automaton for the generation of bigger pieces from small ones.](image)
Figure 8. Example of updown generation. The path in the automaton of figure 7 starts at $A$, then takes step $\delta$ to $B$, from there $y'$ to $B'$, then $y$ to $B$, and finally $\epsilon$ to $B$ again. The starting point of the path corresponds to the small hatched triangle $pqr$. Connector $\delta$ extends it to $pst$, connector $y'$ extends $psf$ to $fuu$, connector $y$ extends $tuw$ to $uxw$, and connector $\epsilon$ extends $uxw$ to $uyw$. The final dissection of the big triangle $uyw$ is obtained by repeated deflation.

The arrows on the various levels are not drawn in the picture. The arrowing of the smallest triangles can be reconstructed by means of the heavy dots: double arrows always point towards a heavy dot.

We recall some of the material of [2]. In [2], Theorem 11.1 there is the description of the set of all complex numbers $\xi$ that correspond to singular pentagrids. The precise form of this set (which is of measure zero) is not important for the present discussion; let us call it $S$.

To any $\xi \in S$ there corresponds a unique $AR^*$-pattern. The question whether a point $k_0 + \cdots + k_4 \xi^4$ occurs as a point in that pattern, can be answered by means of [2], section 8. It uses four regions $V_1$, $V_2$, $V_3$, $V_4$ in the complex plane. The first one is the interior of the regular pentagon with vertices $1, \zeta, \zeta^2, \zeta^3, \zeta^4$. Next, $V_2$ is the interior of the pentagon with vertices $1 + \zeta, \zeta + \zeta^2, \zeta^2 + \zeta^3, \zeta^3 + \zeta^4, \zeta^4 + 1$, whence $V_2 = (\zeta^2 + \zeta^4)V_1 = -V_1/\tau$. Furthermore $V_3 = -V_2$ and $V_4 = -V_1$.

Reformulating [2], theorem 8.1 (by means of [2], formulas (8.2) and (9.1)), we can now describe (assuming $\xi \notin S$) the set of all vertices in the Penrose tiling that corresponds to $\xi$. It is the set of all points $k_0 + \cdots + k_4 \xi^4$ with integers $k_0, \ldots, k_4$ satisfying (with the abbreviation $k = k_0 + \cdots + k_4$) $1 \leq k \leq 4$ and

\begin{equation} \sum_{j=0}^{4} k_j \xi^{2j} - \xi \in V_k. \end{equation}

We denote this set of vertices by $\phi(\xi)$. Since it is the set of vertices in the dual of a non-singular pentagrid, the edges of the rhombs in that dual are obtained by connecting the pairs of points of $\phi(\xi)$ with distance 1.

We recall that in [2], section 5 it was shown how the edges of the rhombs can be provided with single and double arrows in order to become a Penrose tiling.
The procedure is unique, and simply this: at a point of $H_{1,4}$, where $k = 1$ or $k = 4$, all incoming edges get a double arrow pointing towards that point, and then the single arrows can be put on the remaining edges so as to get properly arrowed Penrose rhombs. From now on, “the pattern $\phi(\xi)$” will mean the point set $\phi(\xi)$ with the arrowed edges added to it.

3.2. With the tilings described by means of the $\zeta$’s, the inflation becomes very simple. The inflation of the pattern $\phi(\xi)$ is a pattern of which the vertex set is a subset of $\phi(\xi)$. This point set can be obtained from another tiling by similarity transformation: it is $\tau^{-1}\phi(\eta)$, where $\eta = -\xi/\tau$. This was derived in [2], section 14.

3.3. We shall introduce a particular kind of duality. Let us use the name “z-plane” for the plane in which we draw the Penrose tilings, and “$\zeta$-plane” for the plane where we record the parameter $\xi$. So if $\xi$ is in the $\zeta$-plane, then $\phi(\xi)$ is a point set in the z-plane.

We shall make a correspondence where certain triangles in the z-plane are related to triangles in the $\zeta$-plane. The bigger the triangles in the z-plane, the smaller the corresponding triangles in the $\zeta$-plane will be.

Let us refer to figure 8 for a rough sketch of the situation. Consider the small triangle $pqr$, with the arrows as indicated. We assume that the edges $pr$ and $qr$ have length 1, that $p, q, r$ belong to $H_{1,2,3,4}$, and that $p$ (the endpoint of a double arrow) lies in $H_{1,4}$. To the arrowed triangle $pqr$ we associate the set of all non-singular Penrose tilings that contain this arrowed $pqr$ as one of its half rhombs. Each one of those tilings is a $\phi(\xi)$, with $\xi$ uniquely determined (see [2], Theorem 9.2). The set of all those $\xi$ is a set in the $\zeta$-plane. It will turn out that this set is again a triangular region. But since we excepted the singular cases, we have to say that this set in the $\zeta$-plane is a triangular region with the points of $S$ left out. Forgetting about that exceptional set of measure zero, we just say that triangle $pqr$ corresponds to a triangle in the $\zeta$-plane.

Next we consider all tilings $\phi(\xi)$ with the property that the tiling itself contains the arrowed triangle $pqr$ whereas its inflation contains the arrowed triangle $pst$. In particular those tilings contain the triangles $tpr$, $pqr$ and $qrs$. The new set is a subset of the old one, and the set of $\xi$’s corresponding to the new set is a subset of the old triangle in the $\zeta$-plane. It will turn out that the new set is again a triangle, and of course a sub-triangle of the old one.

The updown generation described in figure 8 leads to a nested set of triangles in the $\zeta$-plane (to be shown in figure 13). What corresponds to the big triangle $vwy$ is the set of all Penrose tilings (with pieces of the original size like $pqr$ and $qrs$) that contain $pqr$ and have the arrowed $vwy$ in their fourth inflation. These tilings contain all points of the sub-division shown in figure 8. We can also say that these are all the tilings obtained by infinite updown generation when we start at the arrowed triangle $pqr$ and apply any infinite path (in the automaton of figure 7) starting with $\delta, \gamma', \gamma, \epsilon$.

3.4. There are some pitfalls in this matter. We have to be careful about the tilings containing the arrowed triangle $pqr$ and having as their inflation
something that contains the arrowed triangle \(pst\). These tilings contain \(p, q, r, s, t\) as vertices, but that does not characterize them. A tiling containing the triangles \(pqr, qrs\) and the vertex \(t\) does not necessarily contain the triangle \(prt\). In a Penrose tiling the word “triangle” means more than just three points: it means arrowed triangle, even if for a moment we do not concentrate on the edges. A further pitfall is the following one: if we apply a connector \(\alpha\) or \(\beta\) we get two-subtriangles that also occur as two of the three sub-triangles we get with connectors \(\gamma\) or \(\delta\) (see figure 5). So if we disregard the arrows on the big \(A\), and just look at the arrows on the small triangles, we do not always characterize the situation properly.

3.5. In order to study the duality for the \(A\)’s we start from a particular position from which all others can be obtained by means of trivial transforms.

As a particular \(A\) we take a triangle in the \(z\)-plane with vertices \(1, 1 + \zeta, 1 + \zeta + \zeta^4\). It is a half thin rhomb. The integral vectors of the points are \((1, 0, 0, 0, 0), (1, 1, 0, 0, 0),\) and \((1, 1, 0, 0, 1),\) so the values of \(k\) are 1, 2, 3 respectively. As mentioned before, edges with an endpoint with \(k = 1\) carry double arrows pointing towards that endpoint, so a double arrow goes from \(1 + \zeta\) to 1. It follows that the half rhomb is an \(A\).

The figure in the \(\xi\)-plane corresponding to this \(A\) can be found by means of (3.1). A pattern \(\phi(\xi)\) containing the point 1 has a \(\xi\) satisfying \(1 - \xi \in V_1\), a pattern containing \(1 + \zeta\) has a \(\xi\) satisfying \(1 + \zeta^2 - \xi \in V_2\), and a pattern containing \(1 + \zeta + \zeta^4\) has a \(\xi\) satisfying \(1 + \zeta^2 + \zeta^8 - \xi \in V_3\). So \(\xi\) has these properties simultaneously if and only if it lies in the intersection of the three pentagons \(1 - V_1, 1 + \zeta^2 - V_2, 1 + \zeta^2 + \zeta^3 - V_3\). This intersection turns out to be the triangle in the \(\xi\)-plane with vertices \(0, 1, 1 - \zeta, 1 - \zeta^4\). These are the points \(P, Q, T\) in figure 9.

![Figure 9](image)

Figure 9. A pentagon in the complex plane showing the shapes of the triangles playing a role in the dual form of inflation. The points \(P, Q, R, S, T\) are \(0, 1, 1 - \zeta, 1 - \zeta^4, 1 - \zeta^3, 1 - \zeta^4\).
3.6. As a particular $A'$ we can take the triangle with vertices $1, 1 + \zeta^4, 1 + \zeta + \zeta^4$. This is just the mirror image (with respect to the real axis) of the previous $A$. Any Penrose tiling containing the one also contains the other, so the corresponding figure in the $\xi$-plane is the same triangle $1, 1 - \zeta, 1 - \zeta^4$.

3.7. As a particular $B$ we take the triangle $1 + \zeta^2 + \zeta^3, 1 + \zeta^2, 1$, with the integral vectors $(1, 0, 1, 1, 0), (1, 0, 1, 0, 0), (1, 0, 0, 0, 0)$. So there is a double arrow from $1 + \zeta^2$ to 1. Note that a pattern containing these three points does not necessarily contain the rhomb of which this triangle is the upper half. It might have been a pattern containing $1 + \zeta^2 + \zeta^3, 1 + \zeta^2, 1$ and $1 + \zeta^2 + \zeta^4$. A necessary and sufficient condition for the tiling $\phi(\xi)$ to contain the arrowed triangle $1 + \zeta^2 + \zeta^3, 1 + \zeta^2, 1$, is that it contains the points $1, 1 + \zeta^2$ and $1 + \zeta^3$. This leads to $\xi \in 1 - V_1, \xi \in 1 + \zeta^4 - V_3, \xi \in 1 + \zeta - V_2$. We conclude that the region in the $\xi$-plane corresponding to our $B$ is the triangle with vertices $0, 1 - \zeta^2, 1 - \zeta^3$. These are the points $P, R, S$ in figure 9.

For $B'$ we can say the same thing as was said for $A'$ in section 3.6. It gives the same region in the $\xi$-plane as $B$.

3.8. Next we study the effect of the connectors. We start with $\beta$ and $\delta$, which act on an $A$. As before, we take the $A$ with vertices $1, 1 + \zeta, 1 + \zeta + \zeta^4$. We consider a pattern $\phi(\xi)$ containing this $A$, and the inflation of that pattern. There are two ways for our $A$ to fit into that inflation, corresponding to $\beta$ and $\delta$, respectively. The two ways are illustrated in figure 10. We shall see that the first case occurs if $\xi$ lies in the region $PTU$, the second case if $\xi$ lies in $PUQ$ (see figures 11 and 12). In order to show this we have to characterize the two possibilities of figure 10 by means of sets of points, independently of arrowing. The points $1, 1 + \zeta, 1 + \zeta + \zeta^4$ occur in both cases. The point $1 + \zeta + \zeta^2$ occurs in the picture on the right, but there is also a possibility to extend the left-hand case by a triangle $1 + \zeta, 1 + \zeta + \zeta^2, -\zeta^4$, so $1 + \zeta + \zeta^2$ does not help to distinguish between $\beta$ and $\delta$. But the points $1 + \zeta + \zeta^3$ and $1 + \zeta^2$ do: they are typical for the left-hand and right-hand case, respectively. So if the pattern $\phi(\xi)$ contains $1, 1 + \zeta, 1 + \zeta + \zeta^4, 1 + \zeta + \zeta^3$, we are in case $\beta$, and if it contains $1, 1 + \zeta, 1 + \zeta + \zeta^4, 1 + \zeta + \zeta^2$, we are in case $\delta$. Applying the method we used before, we see that $\xi$ lies in the intersection of $1 - V_1, 1 + \zeta^2 - V_2, 1 + \zeta^2 + \zeta^3 - V_3, 1 + \zeta + \zeta^2 - V_3$ (the triangular region $PTU$) in case $\beta$, and in the intersection of $1 - V_1, 1 + \zeta^2 - V_2, 1 + \zeta^2 + \zeta^3 - V_3, 1 + \zeta + \zeta^4 - V_2$ (the triangular region $PUQ$) in case $\delta$.

Figure 10. The effect of $\beta$ and $\delta$. 212
3.9. Let us state the result in terms of the automaton of figure 7: If a tiling $\phi(\xi)$ is obtained by updown generation starting from the arrowed triangle $1, 1 + \zeta, 1 + \zeta + \zeta^4$, by means of an infinite path, then $\xi$ lies in $PTU$ if the path starts with $\beta$, and in $PUQ$ if it starts with $\delta$ (see figure 11).

In the case we start with the $A'$ given by $1, 1 + \zeta^4, 1 + \zeta + \zeta^4$, we get a similar result, simply obtained by reflection: if the path starts with $\beta'$ then $\xi$ lies in $PVQ$, if it starts with $\delta'$ then lies in $PVT$ (see figure 11).

3.10. The analysis for $B$ is similar to the one for $A$. The possible connectors are $\alpha'$, $\gamma'$ and $\epsilon$. We start again from a $B$ given by the points $1, 1 + \zeta^2, 1 + \zeta^2 + \zeta^3$. We apply the same method as in case $A$. Under the assumption that we have these three points, we note that having the point $2 + \zeta^2$ is typical for $\epsilon$, having both $1 + \zeta + \zeta^2 + \zeta^3$ and $\zeta + \zeta^2 + \zeta^3$ is typical for $\gamma'$, whereas having both $1 + \zeta + \zeta^2 + \zeta^3$ and $1 + \zeta^2 + \zeta^3 - \zeta^4$ is typical for $\alpha'$. These data lead to the regions $XWS$, $XWP$ and $XRS$ in figure 12.

Figure 11. The dualized inflation for $A$ and $A'$. The big triangle on the left corresponds to an $A$, and the smaller triangles into which it is dissected correspond to the possible connectors $\beta$ and $\delta$ by means of which that $A$ can be inflated. The big triangle on the right corresponds to an $A'$, with similar meaning of the subtriangles.

Figure 12. The dualized inflation for $B$ (on the left) and $B'$ (on the right). The meaning of the greek letters is as in figure 11.
So assuming that $\phi(\xi)$ is obtained by updown generation starting from the arrowed triangle $1, 1 + \xi^2, 1 + \xi^2 + \xi^3$ by means of an infinite path, we conclude that $\xi$ lies in $\mathcal{XWS}$ if the path starts with $\alpha'$, in $\mathcal{XWP}$ if it starts with $\gamma'$, in $\mathcal{XRS}$ if it starts with $\epsilon$.

For $B'$ we get a similar result, just by reflection. Starting from triangle $1, 1 + \xi^3, 1 + \xi^2 + \xi^3$, we get $\xi$ in the region $\mathcal{WXR}$ for paths starting with $\alpha$, in $\mathcal{WXP}$ for paths starting with $\gamma$, and in $\mathcal{WSR}$ for paths starting with $\epsilon'$.

3.11. In sections 3.8-3.10 we obtained triangular regions (the sub-triangles in figures 11 and 12) that have the same shape as the large triangles $\mathcal{PTQ}$ and $\mathcal{PSR}$. It is easy to show that this was to be expected. As an example we take connector $\delta$. Assume that in figure 8 triangle $pqr$ is the $A$ we started from in section 3.5. We want to have the set of all $\xi$ such that the inflation of $\phi(\xi)$ contains the arrowed triangle $pst$. We represent this triangle by $T$. According to section 3.2, the inflation of $\phi(\xi)$ has as its point set $\tau^{-1}\phi(-\xi/\tau)$. The set of all $\xi$ such that $\tau^{-1}\phi(-\xi/\tau)$ contains a triangle $T$ is $-\tau^{-1}\Omega$, where $\Omega$ is the set of all $\eta$ such that $\phi(\eta)$ contains $\tau T$. But $\tau T$ is a $B$, so by section 3.7 the set $-\tau^{-1}\Omega$ has the shape of triangle $\mathcal{PRS}$ in figure 9.

3.12. In the previous sections we studied a single inflation step starting from an arrowed triangle in a particular position. But it is not hard to transfer this to the general case. Wherever in any $\phi(\xi)$ there occurs an $A$, it can be obtained from the particular $A$ of section 3.5 by means of a simple transform. If the $A$ of section 3.5 is called $T$, and the new $A$ is called $T^*$, then we have (according to the end of section 3.1) $T^* = \lambda T + \mu$, with

$$\lambda = (-1)^j \zeta^j, \quad \mu = \sum m_j \zeta^j.$$  

Here $j$ runs from 0 to 4, and $p, q, m_0, \ldots, m_4$ are integers, with $m_0 + \cdots + m_4 = 0$. If $S^*$ is the set of all $\xi$ which are such that $\phi(\xi)$ contains the arrowed triangle $T^*$, and if $S$ is the set of all $\xi$ for which $\phi(\xi)$ contains $T$, then we have (cf. [2], section 10) $S^* = \lambda^* S + \mu^*$, where

$$\lambda^* = (-1)^p \zeta^{2q}, \quad \mu^* = \sum m_j \zeta^{2j}.$$  

If to $T^*$ we apply one of the connectors $\beta$ or $\delta$ then the effect is again obtained from the effect on $T$ by the same linear transformation $z \rightarrow \lambda z + \mu$, and for the dualized connectors $\beta$ or $\delta$ in the $\xi$-plane we get a picture that is obtained from figure 11 by the transformation $\xi \rightarrow \lambda^* \xi^2 + \mu^*$.

3.13. We now study the effect of a finite sequence of inflation steps. Let us start from the little hatched triangle $pqr$ in figure 8. Let us say that this triangle is identical to the $A$ of section 3.5, i.e., $p = 1, r = 1 + \xi, q = 1 + \xi^2 + \xi^4$. Inflation step $\delta$ takes us to $pst$, with a single arrow from $s$ to $p$ and a double arrow from $p$ to $s$. Next step $\gamma'$ takes us to $tuv$, with a single arrow from $t$ to $u$ and a double arrow from $u$ to $v$. This step is carried out with bigger triangles, but that only changes the scale factor. We can use section 3.2 in order to describe what happens on the level of these bigger triangles.
Without going into details about the transformations, we state the result for this case: the set of all $\zeta$ for which $\phi(\zeta)$ contains the arrowed triangle $tuv$ (and therefore its subdivision as shown in figure 8) is the region $QEV$ in figure 13. It is obtained from $PQT$ by first applying transform $\delta$ of figure 11, which produces $PUQ$, and then transformation $\gamma'$ of figure 12 (rotated and on a smaller scale).

Expressing this in terms of paths, we can say that the triangular region $EVQ$ of figure 13 is the set of all $\zeta$ such that $\phi(\zeta)$ is produced from triangle $pqr$ with a path starting with $\delta$, $\gamma'$.

We can of course go on in this way: the set of all $\zeta$ for which $\phi(\zeta)$ contains the whole triangle of figure 8 (obtained by means of $\delta$, $\gamma'$, $\gamma$, $e$), starting from $pqr$, is a triangular region in the $\zeta$-plane which is obtained by the corresponding sequence of operations in the dual, starting from $PTQ$ and leading down to $GHJ$ (figure 13).

3.14. Extending all this to infinity, we get the unique $\zeta$ corresponding to the updown generation produced by any infinite path through the automaton of figure 7.

If we start from the $A$ of section 3.5, and apply updown generation with any infinite path, then we produce $\phi(\zeta)$, where $\zeta$ is obtained by applying the same path, but now with the dual connectors. Starting with the triangle $PTQ$ of

![Diagram](image.png)  

Figure 13. The dualized form of the updown generation of figure 8. The big triangle $PTQ$ corresponds to the hatched triangle $pqr$ in figure 8. $\delta$ (see figure 11) reduces $PTQ$ to $PUQ$, $\gamma'$ reduces $PUQ$ to $EFQ$, $\gamma$ reduces $EFQ$ to $GHQ$, and finally $e$ reduces $GHQ$ to $GHJ$.  

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figure 9, the connectors of the path produce an infinite nested sequence of triangles converging to the \( \xi \).

Corresponding statements can be made for infinite paths starting with \( A', B \) or \( B' \). And if we do not start from one of the particular triangles we used as our point of departure in sections 3.5-3.7, we have to apply section 3.12.

3.15. We repeat that all the time singular patterns have been excluded from the game, which means that in the \( \xi \)-plane the set \( S \) has been excluded. In the case of Beatty sequences ([7]) the singular cases were treated in detail, but from what we saw there it will be clear that a corresponding treatment in the Penrose case can be expected to be quite entangled.

4. SOME SPECIAL CASES OF UPDOWN GENERATION

4.1. In section 2.5 it was stated that an infinite path in the automaton leads to a tiling, but not necessarily to a tiling of the full plane. The process is a matter of a nested sequence of bigger and bigger triangles, and it may happen that during all steps of the updown generation the result stays in one and the same half-plane. In such cases there is a border line that will be reached after a finite number of steps and will never be crossed. Among such cases with border line there are situations where there is a second border line, making a 36° angle with the first one.

In the cases where the updown generation stays in a half-plane, we can restrict ourselves to starting from a triangle (one of \( A, A', B, B' \)) with an edge along the border, and studying how an infinite sequence of connectors can lead to bigger and bigger triangles that still have an edge along that border. The next question is whether the segment along the border will gradually cover the whole line, or a half line only. In the latter case we can restrict ourselves to starting at the moment where the end-point of that half line is reached.

4.2. Let us introduce the notation \( A_1, A_2, A_3, B_1, B_2, B_3, A'_1, A'_2, A'_3, B'_1, B'_2, B'_3 \), where the lower index indicates the kind of arrow on the edge of a triangle along the border. So \( B_4 \) refers to a triangle \( B \) with its fourfold arrow along the border.

From figure 5 we derive all possibilities for deflation steps where a small triangle and its deflation have an edge along one and the same line (a line that is to become a border). The list is

\[
\begin{align*}
A_2 & A_3 & A'_2 & A'_3 & A_3 & A_1 & A'_3 & B_4 & A_2 & B_4 & A'_2 \\
B_2 & A_1 & B_2 & A'_1 & B'_4 & B_4 & B_4 & B'_4 & B_4 & B'_4 & B_4 \\
A_3 & B_1 & A'_3 & B'_1 & B_1 & B_4 & B_4 & B_4 & B_4 & B_4 & B_4 \\
\end{align*}
\]

An entry like \( B'_2 \gamma B_1 \) refers to the fact that \( \gamma \) takes a small \( B'_2 \) into a big \( B_1 \), where the double arrow of the small one falls along the same line as the single arrow of the big one.
4.3. Finite and infinite sequences of such transitions can be studied by drawing a graph with vertices labeled \( A_1, A'_1, \) etc, and representing the list entries by arrows: \( B'_2 \gamma B_4 \) is represented by an arrow \( \gamma \) from vertex \( B'_2 \) to vertex \( B_4 \). We observe that the points \( A_1 \) and \( A'_1 \) are dead ends, and that all infinite paths pass through \( B'_2 \) and \( B_4 \) infinitely often. We get from \( B'_4 \) to \( B_4 \) either by \( \gamma \), or by \( \varepsilon' \gamma e \), or by \( \alpha \beta \delta e \). Put \( C = \{ \gamma, \varepsilon' \gamma e, \alpha \beta \delta e \} \) and \( C' = \{ \gamma', \varepsilon' \gamma' e', \alpha' \beta' \delta' e' \} \). We get 9 ways to pass from \( B_4 \) to \( B'_4 \) itself: taking any item from \( C \) followed by any item from \( C' \). Let \( D \) be the set of these 9: \( D = \{ \gamma \gamma', \alpha \beta \delta \varepsilon \gamma', \ldots \} \).

4.4. We conclude that an infinite path in the automaton of figure 7 keeps the updown generation in a half-plane if and only if it is any finite path followed by an infinite sequence of items taken from \( D \).

The possibility mentioned at the end of section 4.1 is easily seen to occur if and only if the infinite path is, apart from a finite number of initial steps, just a sequence of pieces \( \gamma e \gamma' e' \). In those cases the path leads to a complete tiling of a sector of 36°. In all other cases of paths composed of items from \( D \) we get a complete tiling of a half-plane.

The half-plane tilings immediately lead to full Penrose tilings by adding the mirror image, so that the border line becomes a line of symmetry, and this is the only way to complete the half-plane tiling to a tiling of the full plane. The latter statement can be verified by taking two bordering triangles on opposite sides of the border line, start updown generation with both, and checking that they keep fitting together only if the two paths are each other's mirror image (in the sense that \( \alpha \) and \( \alpha' \) are interchanged, etc.).

In the cases of a 36° sector a reflection along one of the border lines gives a tiling of a 72° sector, and five copies of such a sector fit together to form a tiling of the full plane. And again, this is the only possible extension of the 36° sector tiling to the full plane.

4.5. There are two different shapes arising from paths built by pieces \( \gamma e \gamma' e' \): the infinite star and the infinite sun.

The infinite star is obtained by starting with a \( B \) and applying updown generation with path \( \gamma' e' \gamma \gamma' \gamma' e' \gamma e' \gamma e \ldots \). The value of \( \xi \) can be derived by means of the dual deflations as indicated in section 3.15. With this path (and with the particular \( B \) of section 3.7) the sequence of triangles can be checked to converge to the number 1.

The infinite sun is obtained by starting with a \( B \) and applying updown generation with path \( e \gamma' e' \gamma \gamma' \gamma' e' \gamma e' \gamma e \ldots \). Again, the value of \( \xi \) can be obtained as limit of the nested sequence of triangles obtained by dual deflation: they converge to \( 1 + \xi + \zeta^4 \). In [2], section 13, the value \( \xi = 2 \) was mentioned, but this is only a matter of a shift and a 180° rotation: \( 1 + \xi + \zeta^4 \) is congruent to \( -2 \) modulo the ideal \( P \) (cf. [2], Theorem 9.1 and [2], section 10 (iv)).

For pictures of the infinite star and the infinite sun we refer to [8]. There they are presented with the "kite" and "dart" pieces; by local operations (see [2], section 3) we get to Penrose rhomb tilings.
4.6. Another natural question is which connector sequences generate singular Penrose tilings. They can be analyzed by studying the so-called "bow-tie strings" and their deflations. We refer to [2], section 17 for a description. By careful inspection of the deflations we can find the possible connector sequences. This leads to a result that we state here without proof: A path in the automaton of figure 7 generates a singular Penrose tiling if and only if, apart from a finite number of entries in the beginning, it can be split up into pieces of the forms yy', γγ', αδ, α'δ', γαε, γ'ε'ε'.

4.7. There is essentially only one case where the tiling is both singular and symmetric. It is the cartwheel. We get it with the path γ'yy'yy'y.... For a picture in kite-and-dart form we again refer to [8]. In the Penrose rhomb form it has a "hub" consisting of a tiled decagon (for a picture of that decagon see [2], figure 15). From the sides of that decagon we have 10 "spokes" in the form of bow-tie strings.

If we follow the dual deflation sequence for γ'yy'yy'y... we see at once that the nested sequence of triangles converges to 0. Indeed, the value ζ = 0 belongs to the exceptionally singular pentagrid ([2], section 12).

5. DIFFERENT PATHS LEADING TO ONE AND THE SAME TILING

5.1. We shall use the term elementary tile for any oriented triangle of type A, A', B, B' in the plane with the restriction that the edges with single and double arrows have length 1. An infinite path p in the automaton of figure 7 is said to match an elementary tile t if the type of t corresponds to the point where p begins. It follows from the previous sections that if p matches t then the infinite updown generation based on the pair (p, t) produces a tiling of the full plane or of a half plane or of a 36° sector by means of elementary tiles.

An infinite path can be seen as an infinite sequence of connectors. Two infinite paths will be called similar if their connector sequences differ in at most a finite number of places. If p₁ and p₂ are similar, and if t₁ and t₂ are matching elementary tiles, then there is number k such that after k steps the updown generation with (p₁, t₁) and the one with (p₂, t₂) arrive at big triangles of the same shape, and such that from there on the updown generation works the same for both cases. It follows that the infinite tiling produced by (p₁, t₁) has the same shape as the one produced by (p₂, t₂).

5.2. Conversely, if we start from a Penrose tiling of the plane by elementary tiles, and select a tile t at random, then there is exactly one matching path p such that (p, t) generates that tiling, or at least (in the symmetric cases) half of it or one tenth of it. This path is obtained from the growing sequence of triangles in the successive inflations.

If a tiling has no line of symmetry and if it is generated both by (p₁, t₁) and (p₂, t₂), then the paths t₁ and t₂ differ at most in a finite number of places. This can be argued as follows. Since updown generation with (p₁, t₁) covers the whole plane, it leads us after some k steps to some big triangle T that con-
tains $t_2$ in its interior. This $T$ is a tile in the $k$-th inflation of the given tiling. It follows that the updown generation starting from $t_2$ has $T$ as an intermediate station too, and from there on the successive inflations work the same way for both cases. Consequently $p_1$ and $p_2$ are similar in the sense of section 5.2.

In cases where the updown generation produces a tiled half-plane the resulting full tiling consists of two half-planes, and to each one of them we have a different similarity class of paths. In the cases with the 36° sectors there are also two classes, and each class works for five of the sectors. In the case of the infinite star these are the similarity classes of $\gamma'\varepsilon'\gamma\varepsilon'\gamma\varepsilon'\gamma\varepsilon'...$ and $\gamma\varepsilon'\gamma\varepsilon'\gamma\varepsilon'\gamma\varepsilon'...$; in the case of the infinite sun they are $\varepsilon\gamma'\varepsilon'\gamma\varepsilon'\gamma\varepsilon'\gamma...$ and $\varepsilon'\gamma\varepsilon'\gamma\varepsilon'\gamma\varepsilon'$....

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