Processing networks : introduction and basis structure

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PROCESSING NETWORKS: INTRODUCTION AND BASIS STRUCTURE:

by

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Abstract

A processing network problem is a network flow problem with the following characteristics:

1. conservation of flow in nodes and on arcs.
2. capacity bounds on arcs.
3. there is a non empty subset PN of the node set, such that for each node in PN the flows on the outgoing arcs (or on the incoming arcs) are proportional to each other.

A mathematical formulation of the min cost flow problem in a processing network will be given. The structure of a basis will be discussed. This structure can be used in specifications of the primal simplex algorithm for the min cost flow problem, in analogy with other type of network flow problems (pure, generalized and multicommodity networks).
1. Introduction

Besides the usual requirements of a pure network flow problem - conservation of flow and capacity bounds on arcs - a processing network has an additional property: there is a non empty subset \( PN \) of the node set, such that for each node in \( PN \) the flows on the outgoing arcs (or on the incoming arcs) are proportional to each other.

The concept of processing networks comes from process industry where refining processes (some commodity splits up in several components in given proportions) and blending processes (several components are blended in given proportions) play an important role. Processing networks can be used to model input-output type of problems [17], but in fact, since any LP-problem can be transformed to a processing network problem [14], processing networks could prove to be a good modelling tool for a large class of practical problems.

A mathematical formulation of the min cost flow problem in a processing network is given in the next chapter.

In network flow problems a basis (in the common sense of linear programming) can be characterized as a subgraph with special structure of the original network. For pure, generalized and multicommodity networks the basis structure is well known and has been exploited in a number of specifications of the primal simplex algorithm [1,3,4,5,7,8,9,10,11,16] yielding great advantages both in solution times and memory requirements. In chapter 3 a summary is given of the basis structure in pure and generalized networks. The basis structure for processing networks will be discussed in chapter 4. A specification of the primal simplex method which exploits this structure is in development.

2. Mathematical formulation

A directed and connected graph \( G(N,A) \) is considered. \( N \) is the set of nodes, \( A \) the set of arcs \( (i,j), i,j \in N \). The number of nodes in \( N \) is given by \( n \). Each arc \( (i,j) \) has capacity bounds \( l_{ij} \) and \( u_{ij} \), the flow level is given by
For each node \( i \in N \) the following sets are defined:

\[
A(i) := \{ j \in N | (i,j) \in A \}
\]

\[
B(i) := \{ j \in N | (j,i) \in A \}.
\]

The set \( N \) is partitioned into three subsets:

- \( RN \) : refining nodes
- \( BN \) : blending nodes
- \( TN \) : transportation nodes

A refining node is a node with two or more outgoing arcs. The flow on each outgoing arc \((i,j)\), \( j \in A(i) \) of such a node \( i \) is required to be a given fraction \( \alpha_{ij} \) \((0 < \alpha_{ij} < 1)\) of the total flow entering node \( i \) (fig. 1a)

\[
\frac{\alpha_{ij}}{\alpha_{ir}} x_{ir} - x_{ij} = 0 \quad i \in RN, r \in A(i), j \in A(i) - \{r\}.
\]

Fig. 1. A refining node

Let \( r \) be some node in \( A(i) \), then the proportionality demands can be formulated as:

(1) \[
\frac{\alpha_{ij}}{\alpha_{ir}} x_{ir} - x_{ij} = 0 \quad i \in RN, r \in A(i), j \in A(i) - \{r\}.
\]
It is assumed that:

1. \[ a_{ij} > 0 \quad j \in A(i), \; i \in RN \]
2. \[ \sum_{j \in A(i)} a_{ij} = 1 \quad i \in RN \]

Arc \((i, r)\) is called the representative arc of processing node \(i\) (or process \(i\)) because if the flow in \((i, r)\) is known according to (1) flows in all \((i, j), j \in A(i)\) are known.

A blending node is a node with two or more incoming arcs. The flow on each incoming arc \((j, i), j \in B(i), i \in BN\) is required to be a given fraction \(\alpha_{ji}\) of the total flow leaving node \(i\) (fig. 2a).

![Diagram](image)

**fig. 2.** A blending node.

The blending requirements are:

\[ \frac{a_{ji}}{a_{ri}} \alpha_{ri} x_{ri} = x_{ji} = 0 \quad i \in BN, \; r \in B(i), \; j \in B(i) - \{r\} \]

in which \((r, i)\) is the representative arc of blending node (process) \(i\). Furthermore \(\alpha_{ji} > 0, \forall j \in B(i), i \in BN\) and \[ \sum_{j \in B(i)} \alpha_{ji} = 1, \forall i \in BN. \]

The set of nodes:

\[ PN := \{i | i \in RN \cup BN\} \]
is called the set of processing nodes. It consists of \( P \) nodes of which \( P_R \) are refining nodes and \( P_B \) are blending nodes. Every node in \( N - PN \) is called a transportation node. In all \( i \in N \) conservation of flow is assumed.

The set of arcs \( A \) is partitioned into:

\[
\begin{align*}
RA: & \text{ refining arcs} \\
BA: & \text{ blending arcs} \\
TA: & \text{ transportation arcs}
\end{align*}
\]

A refining arc is an arc \((i,j) \in A\) in which \( i \in RN \), a blending arc \((j,i)\) is one in which \( i \in BN \). All other arcs are called transportation arcs.

The set:

\[
PA := \{(i,j), (i,j) \in RA \cup BA\}
\]

is called the set of processing arcs. With \( PA(i) \) will be denoted the set of processing arcs leaving \( i \in RN \) or entering \( i \in BN \). The number of arcs in \( PA(i) \) is given by \( m_i \) and is called the order of process \( i \). The set \( N(i) \) is defined as:

\[
N(i) := \{j | (i,j) \cup (j,i) \in PA(i)\} + \{i\}, \ i \in PN.
\]

A directed graph with one or more processing nodes will be called a processing network.

Throughout it is assumed that a node can not be a refining node and a blending node at the same time. If some node \( i \) has blending requirements on incoming arcs and refining requirements on outgoing arcs this can be established by introduction of an extra node \( i' \) and a transportation arc \((i,i')\), taking all outgoing arcs of \( i \) as outgoing arcs of \( i' \). Similarly it is assumed that some arc \((i,j)\) can not be both a refining arc of node \( i \) and a blending arc of node \( j \). The remedy in this situation would be to introduce an additional transportation node \( i' \) and replace arc \((i,j)\) by arcs \((i,i')\) and \((i',j)\).

In drawing diagrams of processing networks it is convenient to distinguish refining nodes, blending nodes and transportation nodes. Refining nodes and blending nodes will be presented as in fig. 1b and 2b, a transportation node is given by a simple circle.
The minimal cost flow problem can be formulated as follows:

(3) minimize \[ \sum_{(i, j) \in A} c_{ij} x_{ij} \]

(4) \[ \sum_{j \in J} x_{ij} - \sum_{j \in J} x_{ji} = b_i \quad i \in N \]

(5) \[ a_{ij} x_{ir} - x_{ij} = 0 \quad i \in RN, r \in A(i), j \in A(i) - \{r\} \]

(6) \[ a_{ij} x_{ri} - x_{ji} = 0 \quad i \in BN, r \in B(i), j \in B(i) - \{r\} \]

(7) \[ l_{ij} \leq x_{ij} \leq u_{ij} \quad (i, j) \in A \]

Constraints (4) are the conservation of flow equations, in which \( b_i \) denotes the (external) demand or supply in node \( i \), (5) and (6) are the refining and blending requirements; capacity bounds are given by (7).

The coefficient matrix of the left hand sides of (4) - (6) is called the technology matrix \( T \). The structure of this matrix is illustrated in fig. 3.

fig. 3. Structure of technology matrix.
In this matrix $R_i$ ($i = 1, \ldots, P_R$) and $B_i$ ($i = 1, \ldots, P_B$) are $(m_i - 1) \times m_i$ matrices with the following structure:

$$R_i = \begin{bmatrix}
\frac{a_{ii}^1}{\alpha_{ir}} & -1 \\
\frac{a_{ii}^2}{\alpha_{ir}} & -1 \\
\vdots & \vdots \\
\frac{a_{ii}^{m_i-1}}{\alpha_{ir}} & -1 \\
\end{bmatrix} \quad B_i = \begin{bmatrix}
\frac{a_{ji}^1}{\alpha_{ri}} & -1 \\
\frac{a_{ji}^2}{\alpha_{ri}} & -1 \\
\vdots & \vdots \\
\frac{a_{ji}^{m_i-1}}{\alpha_{ri}} & -1 \\
\end{bmatrix}$$

It is possible to give a more compact formulation for the min cost flow problem. This formulation is achieved by substituting the expressions for $x_{ij}$ in (5) and $x_{ji}$ in (6) into (4). Doing so each refining process and each blending process is represented by a single column in the technology matrix $T^*$. $T^*$ has $n$ rows and each row $i$ of $T^*$ can be identified by node $i$ of the network. In this setting it seems natural to say that each column $(i,j)$ of $T^*$ describes some kind of process. Three types of processes (columns) can be distinguished:

1. **transportation processes.** Column $(i,j)$ has a +1 in row $i$ and a -1 in row $j$. All other elements in the column are zero.
2. **refining processes.** The elements in column $(i,j)$ are $1/a_{ij}$ in row $i$ and $-a_{ik}/a_{ij}$ in row $k$ for every $k \in A(i)$. All other elements are zero.
3. **blending processes.** There is a $-1/a_{ji}$ in row $i$ of column $(j,i)$ and $a_{ki}/a_{ji}$ in row $k$, $k \in B(i)$. Other elements in the column are zero.

It is observed that matrix $T^*$ has the following qualities:

- the column sum of each column in $T^*$ is zero.
- if there are more than one negative (positive) elements in some column of $T^*$ there is only one positive (negative) element.
Note that the columns can be scaled such that the only positive element is +1 (or the only negative element is -1). As an illustration for the processing network of fig. 4 the two discussed types of technology matrices are presented in fig. 5 and 6.

**fig. 4. Example of processing network**

3. **Basis structure in pure and generalized networks**

For pure and generalized networks the basis structure is well described in graphical terms [2,6,7,9,16]. A summary is given here.

A basis in a pure network is a (rooted) spanning tree. The reverse is true too. Every spanning tree is a (not necessarily feasible) basis [2].

In a generalized network the basis structure is somewhat more complex: A quasi-tree is a tree with one additional arc. This arc may be a self-loop. To put it in other words: a quasi-tree is a connected graph with exactly one cycle. A basis in a generalized network is a set of quasi-trees, such that every node of the network is contained in some quasi-tree.
fig. 5. Technology matrix, first version.

fig. 6. Technology matrix, compact version, representative arcs A3, B6, 9C.
(a so called forest of quasi-trees). The reverse is in general not true. Suppose a quasi-tree has a cycle C consisting of at least two arcs and nodes (so it is not a self-loop). Let the product of gains of all clockwise oriented arcs in C be given by $k_1$ ($k_1 = 1$ if no such arcs are present in C) and let the product of all gains of all counter clockwise oriented arcs be given by $k_2$ ($k_2 = 1$ if none are present). A forest of quasi-trees is a basis if and only if in the cycles of the quasi-trees consisting of two or more arcs: $k_1 \neq k_2$. The proof can easily be established by considering theorem 1 (iv) and (v) of [18]. The conclusion can be drawn that the basis structure in a generalized network does not only depend on the topology of the network but also on the values of the gains.

4. Basis structure in processing networks

For the purpose of describing a basis in terms of the processing network formulation (3) - (7) is most adequate. This formulation will be used here. However the results and the concepts defined in the sequel can easily be adapted to suit the compact way of describing.

A subgraph of G(N,A) whose arcs can be identified with a basis will be called a basis graph.

Before discussing the basis structure one important assumption is made. Consider the graph $G^*(N,TA)$ consisting of all nodes $i \in N$ and all transportation arcs. It is assumed that $G^*(N,TA)$ is connected. This assumption makes it easier to describe a basis, but is not really restrictive. If $G^*(N,TA)$ would consist of several connected components $C_1, C_2, \ldots, C_k$ it can be made connected by introducing "bridge" arcs $(s,j)$, where $s$ is some node in $C_1$, $j$ some node in $C_i$ ($i = 2, \ldots, k$) and $\ell_{sj} = 0$, $u_{sj} = 0$, $c_{sj} = 0$.

4.1. Some properties

A cycle consisting of only transportation arcs is called a transportation cycle. A first observation follows directly from the theory of pure networks:
**Lemma 1:** The basis graph cannot contain any transportation cycle.

**Proof:** Suppose on the contrary there is such a transportation cycle. Then dependency between columns in the coefficient matrix corresponding to the arcs in the cycle is easily established, see for instance [2].

The main reason for assuming $G^*(N,TA)$ connected is that it renders an easy expression for the number of arcs in a basis graph.

**Lemma 2:** The number of arcs in a basis graph is

$$(n - 1) + \sum_{i \in PN} (m_i - 1).$$

**Proof:** The technology matrix $T$ consists of $n + \sum_{i \in PN} (m_i - 1)$ rows and at least as many columns since $G^*(N,TA)$ is connected. Adding up the first $n$ rows of $T$ leaves the zero row, so the rank of $T$ is smaller than or equal to $(n - 1) + \sum_{i \in PN} (m_i - 1)$. It is easy to see that the rank must be equal to this number by constructing a spanning tree in $G^*(N,TA)$ and taking all non-representative arcs of the refining and blending processes.
The matrix which results can be written as $P = \begin{bmatrix} B & C \\ O & D \end{bmatrix}$, where $B$ is an $(n - 1) \times (n - 1)$ matrix which corresponds to a spanning tree and $D$ is a non singular diagonal matrix of order $\sum_{i \in PN} (m_i - 1)$. Since both $B$ and $D$ are non singular $P$ is non singular and thus represents a basis.

An other straight-forward result is:

**Lemma 3**: Either all $m_i$ or $(m_i - 1)$ arcs of each set $PA(i)$, $i \in PN$ belong to the basis graph.

**Proof**: Suppose for some set $PA(i)$, $i \in PN$ two or more arcs are omitted. If (at least) the representative arc and some non representative arc would be left out a zero row would appear in the last $\sum_{i \in PN} (m_i - 1)$ rows. If (at least) two non representative arcs are omitted two rows of the last $\sum_{i \in PN} (m_i - 1)$ rows would be linearly dependent.

Both cases are in contradiction with lemma 2.

At this stage several concepts are introduced. A processing node $i$ (or process $i$) is called *active* if all $m_i$ arcs of the set $PA(i)$ are contained in the basis graph. Otherwise (only $(m_i - 1)$ arcs are present in the basis graph) it is called *inactive*. If we would consider the basis graph in which all processing arcs contained in it are left out, this graph would contain several connected components consisting of only transportation arcs. Such a connected component is called a *transportation tree* since it can not contain any cycle. A transportation tree could consist of a single node (including processing nodes).

The next lemma gives a relation between the number of transportation trees and the number of active processes in the basis.

**Lemma 4**: If there are $(q + 1)$ transportation trees in the basis graph ($q \geq 0$) then there must be $q$ active processes and the other way round.
Proof: Every node \( i \in N \) belongs to some transportation tree. If the number of transportation trees is \( (q + 1) \), \( q \geq 0 \) these trees contain \( n - (q + 1) \) transportation arcs. From Lemma's 2 and 3 it follows that the number of active processes must be \( q \). The second part of the proof is obtained by reversing the argument.

A more hidden property of a basis is the following. Denote by \( APN \) the set of active processing nodes. Suppose \( APN \neq \emptyset \) and let \( S \) be a non empty subset of \( APN \). The number of elements of \( S \) is \( |S| \). Further let \( T(S) \) be the set of transportation trees containing at least one node of \( N(i), i \in S \). The number of those trees is given by \( |T(S)| \).

**Lemma 5:** \( |T(S)| \geq |S| + 1 \).

Proof: Suppose that the transportation trees in \( T(S) \) contain in total \( a \) nodes. This means that the set \( T(S) \) contains \( a - |T(S)| \) arcs. The set \( T(S) \cup \bigcup_{i \in S} P(i) \) contains \( a - |T(S)| + |S| + \sum_{i \in S} (m_i - 1) \) arcs. In matrix terms these arcs have entries unequal to zero in exactly \( a + \sum_{i \in S} (m_i - 1) \) rows.

Adding up the \( a \) rows corresponding to the conservation of flow equations leaves the zero row. Given the fact that the columns in a basis matrix are linearly independent this means that:

\[
a - |T(S)| + |S| + \sum_{i \in S} (m_i - 1) \leq a - 1 + \sum_{i \in S} (m_i - 1)
\]

or

\[
|T(S)| \geq |S| + 1.
\]

Equality in lemma 5 holds in any case for \( S = APN \) according to lemma 4, but there can be real subsets of \( APN \) for which the equality holds too. A consequence of lemma's 4 and 5 is that every transportation tree is incident to at least one active processing node, that is to say every transportation tree contains some node \( j \in N(i), i \in APN \). Note worthy is that lemma 5 also applies to pure networks in the following way. If in a pure network \( S \) is defined as a subset of basis arcs and \( T(S) \) the set
of nodes incident to those arcs this is evident. Here lemma 5 can be formulated as: there can not be a cycle in a basis graph.

Suppose that a sub graph of $G(N,A)$ satisfies lemma's 1 - 5 then it is not said that it characterizes a basis. This is illustrated in fig. 7.

\[ \text{fig. 7. Not a basis.} \]

Let $a_{ij}$ denote the column in the technology matrix corresponding to arc $(i,j)$. It is easily checked that:

\[
2a_{s1} - 2a_{s2} + 2a_{1A} - 2a_{2B} + a_{A3} + a_{A4} - a_{B5} - a_{B6} + \\
+ a_{37} - a_{57} + a_{48} - a_{68} = 0
\]

showing that these columns are linearly dependent. As in the case of generalized networks the basis does not only depend on the topology of the network but also on the coefficients $a_{ij}$.

For practical purposes lemma 5 is inadequately formulated. In the next section it will be shown with the help of lemma 5 that the representative arcs of active processes can be chosen in a convenient way.

4.2. Representative arcs of active processes

The main point to be made in this section is:
Lemma 6: The representative arcs of active processes can be chosen in such a way that the arcs contained in the transportation trees plus these representative arcs form a spanning tree in $G(N,A)$. This spanning tree will be called a representative spanning tree.

Proof: Suppose that a basis contains $(q + 1)$ transportation trees $T_1, T_2, \ldots, T_{q+1} \ (q \geq 0)$. If $q = 0$ there is a spanning tree of transportation arcs. Consider the case that $q \geq 1$ and let the representative arcs of the active processes be chosen in an arbitrary way.

It is helpful to consider the following derivated graph: nodes $1, 2, \ldots, q+1$ (node $i$ corresponds to transportation tree $T_i$) and edges (unordered pairs) $(u,v)$ for every representative arc of active processes which has endpoints in $T_u$ and $T_v$. Notice that one of the nodes incident to a representative arc is the processing node attached to it, which, given a basis belongs to one and the same transportation tree. If it can be proved that the representative arcs of active processes can be chosen in such a way that the derivated graph is connected then also the graph consisting of the transportation trees plus these representative arcs is connected and thus a representative spanning tree since it contains $n$ nodes and $(n - 1)$ arcs.

Suppose that the derivated graph is not connected and consists of $k \geq 2$ connected components. This means that at least one component must contain a cycle. Consider such a cycle $C_1$ consisting of nodes $1, 2, \ldots, \ell$ (numbered so only for convenience) and edges $(1,2), (2,3), \ldots, (\ell-1,\ell), (\ell,1)$. It will be proved that the number of connected components can be reduced to $(k - 1)$ as long as $k \geq 2$. By induction it follows that in the end a connected derivated graph should result.

The edges in $C_1$ correspond to $\ell$ active processes which have "entrances" in at least $(\ell + 1)$ transportation trees according to lemma 5. So there must be at least one node $\ell \in C_1$, say $(\ell + 1)$ in the derivated graph, such that edge $(i, i + 1)$ can be replaced by $(i, \ell + 1)$. If node $(\ell + 1)$ belongs to an other connected component the number of components is reduced by one.
Otherwise \((\ell + 1)\) belongs to the same connected component as cycle \(C_1\); when the replacement is carried out a new cycle \(C_2\) is formed with at least one arc which was not contained in \(C_1\). The \(\ell\) edges of \(C_1\) plus the new one(s) correspond to as many active processes. Again at least one of those edges can be chosen differently, such that one endpoint \(\not\in C_1 \cup C_2\) (Lemma 5). This process can be repeated. In the end there must be some edge which can be replaced by one with one endpoint in an other component. If this edge is contained in some cycle, which can always be (re)established, the number of connected components is then reduced by one.

An illustration of the concept "derivated graph" is given in fig. 8. The numbers in the nodes say to which transportation tree this node belongs. The active processes are given by A, B and C; only the chosen representative arcs of these processes are drawn. The structure of the derivated graph is given below the basis graph.

---

**fig. 8**, Part of a basis graph with derivated graph.
5. Conclusion

The lemma's in the previous chapter prove:

**Theorem:** A basis graph in a processing network is a spanning tree consisting of transportation arcs and representative arcs of active processes plus \((m_i - 1)\) additional processing arcs for each set \(PA(i), i \in PN\).

In matrix terms the basis structure can be summarized as in the following picture:
References


