Let \( D \) be the dual code of \( C \). By Corollary 2.1, \( D \) is an \([39,11,15]\)-code, which contains the all-one vector \( I \) (since \( C \) is even-weight). The dual code of \( D \) is just \( C \) and so has minimum weight \( 6 \). Let \( A_i \) be the number of codewords of weight \( i \) in \( D \). Then \( A_i = A_{39-i} \), for each \( i \) (since \( I \in D \)) and \( A_0 = A_6 = 0 \) (by Lemma 2.4, the residual of \( D \) with respect to a codeword of weight 21 is an \([18,10,5]\)-code, which does not exist by Table I).

The MacWilliams’ identities (2.1) with \( \tau = 0, 2, 4, \) and 6 now give

\[
A_{18} + A_{16} + A_{17} + A_{19} = 1023,
\]

\[
21A_{10} + 5A_{14} - 7A_{17} - 19A_{19} = -741,
\]

\[
-309A_{11} - 181A_{13} - 29A_{15} + 171A_{17} = -82251,
\]

\[
1519A_{12} + 1407A_{14} + 595A_{16} - 969A_{18} = -326263 + 1024B_6,
\]

which lead to

\[
\begin{align*}
A_{18} &= (3588 - 9A_{19})/14, \\
A_{16} &= (-726 + 5A_{19})/2, \\
A_{17} &= (7008 - 20A_{19})/7, \\
A_{19} &= 30720 - 8B_6.
\end{align*}
\]

From a), d), b) and c), we get respectively

\[
A_{10} = 6 \mod 7, \quad A_{12} = 0 \mod 8, \quad A_{14} = 146, \quad \text{and} \quad A_{16} = 350,
\]

which imply that \( A_{19} = 1 \) of 160, 216, 272, or 328. There are just the four possible weight distributions for \( D \).

\[
\begin{array}{cccccccc}
A_0 & A_2 & A_4 & A_6 & A_8 & A_{10} & A_{12} & A_{14} & A_{16} \\
W_1: & 1 & 282 & 37 & 544 & 160 & 160 & 544 & 37 & 282 & 1 \\
W_2: & 1 & 246 & 177 & 216 & 216 & 177 & 246 & 1 \\
W_3: & 1 & 210 & 317 & 224 & 272 & 224 & 317 & 210 & 1 \\
W_4: & 1 & 174 & 457 & 64 & 328 & 328 & 64 & 457 & 174 & 1
\end{array}
\]

For each of the four cases, the \( B_i \)’s were calculated (with the aid of a computer program) from the MacWilliams’ identities (2.1) in order to check whether they were all integer-valued. Indeed they were, but in each case exactly one \( B_i \) was negative. It is easily confirmed by hand calculation that

- for \( W_1 \), \( B_{18} = -5 \),
- for \( W_2 \), \( B_{18} = -3 \),
- for \( W_3 \), \( B_{18} = -1 \),
- for \( W_4 \), \( B_{18} = -6 \).

So we have a contradiction in each case. \( \Box \)

**Corollary 3.15:** \( d(39 + i, 26 + i) \leq 5 \) and \( d(38 + i, 26 + i) \leq 4 \), for \( 0 \leq i \leq 4 \).

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**References**


**On the Minimum Distance of Combinatorial Codes**

L. TOLHUIZEN AND J. H. VAN LINT

Abstract — A conjecture of Da Rocha concerning the minimum distance of a class of combinatorial codes is proven.

**I. Introduction**

The generator matrix of the first-order Reed–Muller code \( (1,m) \) of length \( n = 2^m \) consists of all possible column-vectors from \( (F_2)^m \). The combinatorial code \( C(m,s) \) has as generator matrix the matrix \( A(m,s) \) of length \( (\binom{m}{s}) \), that has all possible column-vectors of weight \( s \) as columns.

These codes were introduced by V. C. Da Rocha [2]. It is an easy exercise to show that the weight of the sum of any \( j \) rows of \( A(m,s) \) only depends on \( j \), \( m \), and \( s \). If we denote this weight by \( F(m,j,s) \), then we have for \( 1 \leq j \leq m \)

\[
F(m,j,s) = \frac{1}{2} \left( \binom{m}{s} - P(j;m) \right)
\]

where \( P(j;m) \) is a Kravchouk polynomial (cf. [1], p. 130, [2], Th. 2). Note that \( F(m,1,s) = \binom{m-1}{s-1} \).

In [2], Da Rocha conjectures that the minimum weight of \( C(m,s) \) is \( \binom{m-1}{s-1} \) for \( s < m/2 \). We shall prove this conjecture and, in fact, we shall prove the following theorem.

**Theorem 1:** For \( m \geq 1, 2s < m \) and \( 1 \leq j \leq m - 1 \) we have

\[
\binom{m-1}{s-1} \leq F(m,j,s) = \binom{m-1}{s-1} + 1.
\]

**II. Relations for** \( F(m,j,s) \)

By adding all the rows of \( A(m,s) \), or by replacing all 0’s by 1’s and vice versa, one obtains the following two trivial relations ([2], Theorems 3, 4)

\[
F(m,m-j,s) = \begin{cases} F(m,j,s), & \text{if } s \text{ is even,} \\ \left( \binom{m}{s} - F(m,j,s) \right), & \text{if } s \text{ is odd.} \end{cases}
\]

\[
F(m,j,m-s) = \begin{cases} F(m,j,s), & \text{if } j \text{ is even,} \\ \left( \binom{m}{s} - F(m,j,s) \right), & \text{if } j \text{ is odd.} \end{cases}
\]
From these we obtain

\[ F(2s, j, s) = \begin{cases} \frac{1}{2} \binom{2s}{s} = \binom{2s-1}{s-1}, & \text{if } j \text{ is odd.} \end{cases} \]

(2.3)

Note that by a permutation of columns, we can give \( A(m+1, s+1) \) the form

\[ A(m+1, s+1) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ A(m, s) & A(m, s+1) \end{pmatrix}. \]

From this we immediately find two more relations:

\[
\begin{align*}
F(m+1, j, s+1) &= F(m, j, s) + F(m, j-1, s), \\
F(m+1, j, s+1) &= \binom{m+1}{j}.
\end{align*}
\]

(2.4)

\[
\begin{align*}
F(2s+1, j, s+1) &= F(2s+1, j, s) + F(2s, j-1, s-1), \\
F(2s+1, j, s+1) &= \binom{2s+1}{j} - F(2s+1, j-1, s) + F(2s+1, j-1, s). \\
\end{align*}
\]

(2.5)

III. PROOF OF THEOREM 1

We prove the theorem by induction on \( m \). For small values of \( m \) the theorem is easily checked by hand. Assume the theorem is true for \( m \leq k \). Let \( 2s < k+1, 1 \leq j \leq k \). We distinguish three cases.

Case a) \( j = k \). We have by (2.2)

\[
F(k+1, k, s) = \begin{cases} 1 & \text{if } s \text{ is even}, \\
\binom{k+1}{s} - F(k+1, s) = \binom{k+1}{s} - \binom{k}{s-1} = \binom{k}{s}, & \text{if } s \text{ is odd.}
\end{cases}
\]

Case b) \( 1 \leq j < k-1 \) and \( 2s < k \). Now we use (2.4)

\[
F(k+1, j, s) = F(k, j, s-1) + F(k, j, s),
\]

so by the induction hypothesis

\[
F(k+1, j, s) \leq \binom{k+1}{s} - F(k, j, s) = \binom{k+1}{s} - \binom{k}{s-1} = \binom{k}{s},
\]

and

\[
F(k+1, j, s) \geq \binom{k-1}{s-2} + \binom{k-1}{s-1} = \binom{k}{s}.
\]

Case c) \( 1 \leq j < k-1 \) and \( 2s = k \). We must now distinguish between odd and even values of \( j \). Let \( j \) be odd. By (2.4) and (2.3) we have

\[
F(2s+1, j, s) = F(2s+1, j, s) + F(2s, j, s-1) - \binom{2s-1}{s-1} + F(2s, j, s-1),
\]

and then the induction hypothesis yields

\[
\begin{align*}
F(2s+1, j, s) \leq \binom{2s-1}{s} + \binom{2s-1}{s-1} = \binom{2s}{s}.
\end{align*}
\]

(2.6)

\[
\begin{align*}
F(2s+1, j, s) \geq \binom{2s-1}{s-1} + \binom{2s-1}{s-2} = \binom{2s}{s-1}.
\end{align*}
\]

Let \( j \) be even. By (2.5) and (2.3) we have

\[
\begin{align*}
F(2s+1, j, s) &= \binom{2s}{s} - F(2s+1, j-1, s-1) + F(2s+1, j-1, s) \\
&= \binom{2s}{s-1} + \binom{2s-1}{s-1} - F(2s+1, j-1, s-1),
\end{align*}
\]

and now the induction hypothesis yields

\[
\begin{align*}
F(2s+1, j, s) &\leq \binom{2s}{s-1} + \binom{2s-1}{s-1} - \binom{2s}{s-2} \\
&= \binom{2s-1}{s-1}.
\end{align*}
\]

(2.6)

\[
\begin{align*}
F(2s+1, j, s) &\geq \binom{2s}{s-1} + \binom{2s-1}{s-1} - \binom{2s-1}{s-2} = \binom{2s-1}{s-1}.
\end{align*}
\]

Cases a), b), c) show that the theorem is also true for \( m = k + 1 \) and the proof is complete.

Note that the theorem has some combinatorial interest. It is nice to know that codewords cannot have weight less than the rows of the generator, but one should also realize that these codes are not good. Also as anticodes they do not seem to be very promising.

For the sake of completeness we mention the following facts concerning \( C(m, s) \), (cf. [2]):

\[
C(m, s) \text{ has dimension } \begin{cases} m & \text{if } s \text{ is odd}, \\
 m-1 & \text{if } s \text{ is even}.
\end{cases}
\]

By adding the all-one vector of the code \( C(m, s) \) if \( s \) is even, a code with dimension \( m \) is obtained with minimum weight \( d(m, s) \) where

\[
d(m, s) = \begin{cases} m-1 & \text{if } 2s < m, \\
m & \text{if } 2s > m, \\
2 & \text{if } 2s = m.
\end{cases}
\]

For \( 2s > m \), the assertion about the minimum distance is a consequence of the following obvious extension of Theorem 1.

Theorem 1':

a) For \( m \geq 1, 2s > m \) and \( 1 \leq j \leq m-1 \) we have

\[
\begin{align*}
\binom{m-1}{s-1} &\leq F(m, j, s) \leq \binom{m-1}{s-1}.
\end{align*}
\]

b) For \( s \geq 1 \) and \( 1 \leq j \leq 2s-1 \) we have

\[
\begin{align*}
F(2s, j, s) &= \binom{2s-1}{s-1}, & \text{odd}, \\
2 \binom{2s-2}{s-2} &\leq F(2s, j, s) \leq 2 \binom{2s-2}{s-1} & \text{even}.
\end{align*}
\]

Proof:

a) Combination of Theorem 1 and (2.2).

b) Combination of Theorem 1, (2.3), (2.4) and a.

References
