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Alternative statistical-mechanical descriptions of decaying two-dimensional turbulence in terms of “patches” and “points”

Z. Yin,^{a)} D. C. Montgomery,^{b)} and H. J. H. Clercx

*Fluid Dynamics Laboratory, Applied Physics Department, Eindhoven University of Technology,
P.O. Box 513, 5600 MB, Eindhoven, The Netherlands*

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Numerical and analytical studies of decaying, two-dimensional Navier–Stokes (NS) turbulence at high Reynolds numbers are reported. The effort is to determine computable distinctions between two different formulations of maximum entropy predictions for the decayed, late-time state. Though these predictions might be thought to apply only to the ideal Euler equations, there have been surprising and imperfectly understood correspondences between the long-time computations of decaying states of NS flows and the results of the maximum entropy analyses. Both formulations define an entropy using a somewhat *ad hoc* discretization of vorticity into “particles.” Point-particle statistical methods are used to define an entropy, before passing to a mean-field approximation. In one case, the particles are delta-function parallel “line” vortices (“points,” in two dimensions), and in the other, they are finite-area, mutually exclusive convected “patches” of vorticity which only in the limit of zero area become “points.” The former are assumed to obey Boltzmann statistics, and the latter, Lynden-Bell statistics. Clearly, there is no unique way to reach a continuous, differentiable vorticity distribution as a mean-field limit by either method. The simplest method of taking equal-strength points and equal-strength, equal-area patches is chosen here, no reason being apparent for attempting anything more complicated. In both cases, a nonlinear partial differential equation results for the stream function of the “most probable,” or maximum entropy, state, compatible with conserved total energy and positive and negative velocity fluxes. These amount to generalizations of the “sinh-Poisson” equation which has become familiar from the “point” formulation. They have many solutions and only one of them maximizes the entropy from which it was derived, globally. These predictions can differ for the point and patch discretizations. The intent here is to use time-dependent, spectral-method direct numerical simulation of the Navier–Stokes equation to see if initial conditions which should relax toward the different late-time states under the two formulations actually do so. © 2003 American Institute of Physics.
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I. INTRODUCTION

It has been known for several years that two-dimensional Navier–Stokes (2-D NS) turbulence with periodic boundary conditions and at high Reynolds numbers (in excess of a very few thousand) will relax to long-lived, quasi-steady states whose topology is preserved and whose energy decays at a rate more or less inversely proportionally to the Reynolds number, computed with respect to a box dimension and a rms initial turbulent velocity. When this energy decay time is large compared to the initial eddy turnover time, the quasi-steady state can be reached at a time when the total energy decay is fractionally small, though the enstrophy decay can be fractionally large. It came as a surprise when, a decade or more ago, a series of such computations^{1–3} revealed that the late-time quasi-steady state for an initially turbulent run at a Reynolds number above 14 000 showed a pointwise hyperbolic sinusoidal dependence between stream function and vorticity.

The prediction of such a dependence, in the context of a mean-field treatment of ideal line vortices (or guiding-center plasma rods), had been given 30 years ago^{4,5} and has since been extended and refined in a series of investigations by several groups:^{6–20} in every case referring to ideal, nonviscous systems. The system is Hamiltonian with a finite phase space, and it is natural to apply Boltzmann statistics to its dynamics, as originally suggested by Onsager²¹ (see also Lin²²). The surprise came in the extent to which the ideal Euler mean-field predictions fit the Navier–Stokes results. At least one attempt was made to define an entropy for the case of finite viscosity²³ but generated new puzzles of its own.

In the late 1980s and early 1990s, an alternative formulation was given by Robert, Sommeria, and Chavanis,^{24–27} by Miller and colleagues,²⁸ and later explored by Brands, Maassen, and Clercx.²⁹ The principal difference was that the vorticity field was discretized not in terms of delta functions, but rather in terms of finite-area, mutually exclusive “patches” of vorticity, to which Lynden-Bell statistics³⁰ could be applied. The choice of parameters in the patch formulation is wider than that of equal-strength point vortices, in that one must decide in advance the size of the patches,

^{a)}Electronic mail: z.yin@tue.nl

^{b)}Also at: Department of Physics & Astronomy, Dartmouth College, Hanover, NH 03755.

and the number of “levels,” or strengths, that the patches carry. There seems to be no deductive mechanism for making such choices, and so we stick with the simplest, taking equal-area patches and never more than three levels, including zero.

The derivations of the “point” and “patch” formulations can be given in terms of straightforward but tedious operations which are frequently treated in an introductory statistical mechanics course. The “point” formulation can be obtained as a limit of the “patch” formulation. In the latter, the vorticity is coarse-grained by replacing it by its average over finite spatial areas that are regarded as nonoverlapping and tile the basic box. For convenience, they can be taken as of equal area. Then box in the x, y plane is ruled off into equal cells each containing one or more of the mutually exclusive patches of vorticity, of different values or “levels.” One of these levels may be zero, i.e., each cell or some fraction of it may be empty of vorticity. The expression adopted for the probability of any configuration (which now amounts to a coarse-grained specification of the vorticity as a function of x and y) is the same one as that used in Lynden-Bell statistics (or Fermi–Dirac statistics, lacking the quantum complications^{28,30}). Its logarithm is maximized subject to the constraints of constant total (discretized) energy and constant total fluxes associated with each vorticity level. The method of Lagrange multipliers is used, and the numbers are considered large enough for Stirling’s approximation to apply. The result is a “most probable” vorticity distribution associated with that energy and set of vorticity fluxes. The “most probable” vorticity states so obtained often lead to only local, rather than absolute, entropy maxima, and further comparisons are required to determine which one is the distribution that actually maximizes the assumed entropy globally. A mean-field approximation is then assumed, in which the sizes of the patches and cells are regarded as negligible, and the patches become more and more numerous. The vorticity distribution is thereafter treated as continuous and differentiable. It is expressed as a function of the stream function, now itself continuous. A final step involves inserting the expression for the entropy-maximizing vorticity field into Poisson’s equation, which must then be solved self-consistently (and numerically) for the associated stream function. One parameter in the procedure remains arbitrary and has no physical basis for its choice: the ratio of the “patch” size to the cell size. The limit in which this ratio is zero gives the original “point” recipe, leading to Eq. (5) in what follows. The “patch” approximation, in which this ratio remains finite, leads to Eq. (6) below. It should be kept in mind that throughout, one has in mind a representation of an ideal, conservative, Euler equation system, with no viscous dissipation of any kind.

Section II will summarize both the numerical method (now well-known in its essentials), due originally to Orszag and Patterson,^{31,32} to which various sets of initial conditions will be subjected. We will also describe the motivation for choosing these initial conditions, in terms of what we expect from them as possible most-probable states. We will concentrate on initial conditions for which it appears that the predicted outcomes using point and patch entropies will be as

different as possible. We will refer the reader to the Appendix for a description of how the sinh-Poisson relation and its patch generalizations are dealt with numerically (a nontrivial task).

Section III contains the main body of the results that we have found. Among other unexpected features, we have found a series of one-dimensional (1-D) solutions to the most-probable state equations, whereas the literature up to now has dealt only with 2-D solutions (“dipoles,” “quadrupoles,” “octupoles,” etc.). There are cases in which the entropies of the 1-D solutions (which we call “bars”) are extremely close to those of the dipole over a considerable range of energy. Which is greater depends on seemingly arbitrary choices, in the patch formulation, such as the areas of the patches used. Classes of initial conditions are found (not essentially turbulent ones) for which relaxation to the “bar” states is observed. It will be easier, in the context of the details of the solutions, to illustrate the variety of behavior we have been able to catalogue.

Section IV will be a summary of what we think we have learned, and of what remains to be learned. One of the more radical, if incidental, conclusions which we believe may be extracted from these runs (to be discussed in more detail later) is that in the context of the initial value problem, all 2-D NS turbulence may result only from its having been there initially. Otherwise, it can come into existence only through random external “stirring.”

Some features of this work have been briefly reported in expanded abstract form previously,^{33,34} with an emphasis on the numerics.

We should caution the reader that the word “turbulence” in this manuscript will be used in a somewhat looser sense than is customary. It will not always mean a state of the fluid in which the kinetic energy is widely shared by many Fourier modes. We have concentrated on initial conditions which seemed to us most relevant to producing effects peculiar to the “patch” description: namely, initial vorticity distributions in which a few flat levels of nearly constant vorticity could be readily identified. We expected, because of instabilities in the shear flows these represent, that turbulence of a typical broad-band modal structure would soon be generated, as it is in three dimensions. We found that instabilities could indeed be activated, but often, broad-band turbulence of the conventional kind was not the result. Instead, energy spectra dominated by only a few low-lying (in Fourier space) modes were nonlinearly dynamically converted to Fourier spectra dominated by only a few (but different) low-lying modes. We have in fact found it nearly impossible to generate genuinely broad-band 2-D NS turbulence through instabilities, rather than through initial conditions or externally imposed stirring. Perhaps surprisingly, the dynamical evolution involved sometimes, but not nearly always, led to late-time quasi-steady states that seemed to correspond to most-probable “patch” predictions: most interestingly, the one-dimensional “bar” state, as will be seen. Broad-band, initially excited turbulence continued to lead to the classical dipolar late-time state, as it has in previous turbulence simulations.

Extension and elaboration of the statistical mechanics of the Euler equations has continued and has led to a variety of

predictions which are not being tested here.^{35–42} Consideration of these mathematically ambitious theories lies outside the scope of this article. Our interest here is focused on what happens as a consequence of Navier–Stokes, not ideal, dynamics, and involves a comparison of the predictions of (historically) the first two Euler entropy maximizations with a Navier–Stokes decay. As far as we are concerned, it is not yet understood why a Navier–Stokes decay, which involves true, well-resolved dissipation by viscosity should lead to anything predicted by any 2-D Euler-equation statistical mechanics; no arguments to date that “coarse-graining” should in any accurate way mimic viscous dissipation seem persuasive to us. In any case, no coarse-graining is involved in the computation, which is well-resolved. There are not likely to be accurate continuum Euler-equation solutions over several hundred eddy turnover times to compare with in the near future. This is not to say that the predictions of the above-mentioned recently presented theories^{35–42} might not eventually be shown to have an even better predictive capacity for viscous turbulent decays than these two. Sharpening up what predictions might be extracted from them that could be compared with a Navier–Stokes decay computation appears as a desirable activity, and actually making the comparisons appears as a demanding one.

II. THE NUMERICAL AGENDA

We address ourselves to the long-time dynamics in 2-D of the Navier–Stokes equation in the usual vorticity representation,

$$\frac{\partial \omega}{\partial t} + \mathbf{v} \cdot \nabla \omega = \nu \nabla^2 \omega, \quad (1)$$

where the fluid velocity $\mathbf{v}$ is to be written in terms of the stream function ψ as $\mathbf{v} = \nabla \times (\hat{\mathbf{e}} \psi)$, and $\hat{\mathbf{e}}$ is a unit vector in the z direction. The velocity $\mathbf{v}$ has only x and y components, which depend only upon x , y , and the time t . In the natural dimensionless units of the problem, the kinematic viscosity ν may be interpreted as the reciprocal of the Reynolds number, which we will specify in more detail presently. The curl of $\mathbf{v}$ is the vorticity $\boldsymbol{\omega} = (0, 0, \omega)$, which has only one component, also in the z direction. The stream function ψ and the non-vanishing component of vorticity ω are related by Poisson’s equation,

$$\nabla^2 \psi = -\omega. \quad (2)$$

We note that dropping the final viscous term in Eq. (1) leaves us with the 2-D equations of an ideal Euler fluid. We note moreover that a time-independent solution of these Euler equations results any time ω is a differentiable function of ψ .

We will work in a periodic box with sides 2π in length, and will use the unit length 1 to define the Reynolds number; thus our basic unit of length is roughly 1/6 of a box dimension. The unit of velocity will typically be a root-mean-square value of the initial velocity, and we will attempt to make this equal to 1 whenever possible. Thus ν in Eq. (1) may reasonably be identified with the reciprocal of an initial Reynolds number, Re . We are interested in values of Re of at

least several thousand, so that the final term in Eq. (1) is formally very small, except in regions of steep vorticity gradients. The “eddy turnover time,” which we shall use as a unit of t , is thus about 1/6 of a box dimension divided by an initial rms velocity. It has been known for many years that under such circumstances, the kinetic energy E , defined by

$$E = \frac{1}{2} \frac{1}{(2\pi)^2} \int \int \mathbf{v}^2 dx dy, \quad (3)$$

decays slowly proportionally to ν . However, the decays of higher-order Euler-equation invariants constructed as integral moments of ω , such as the enstrophy,

$$\Omega = \frac{1}{2} \frac{1}{(2\pi)^2} \int \int \omega^2 dx dy, \quad (4)$$

appear to continue to decay at an $\mathcal{O}(1)$ rate; even a weak dependence of their decay rate upon ν has not been convincingly demonstrated. Note that both ideal invariants are referred to unit volume, in the dimensionless units.

When high Reynolds numbers exist, then, the turbulent decay of energy is seen to be very slow, relative to any other identifiable ideal Euler invariant. When the energy decay time is large enough to be well-separated from the eddy turnover time, it is possible to observe in numerical computations^{1–3} that a quasi-steady state is reached in a time over which the fractional decay of energy is small (a few hundred eddy turnover times). One might expect to find a “selectively decayed” state, in which the enstrophy to energy ratio is minimal,⁴³ and indeed, if one waits long enough, it can be analytically proved that such a state must be approached.⁴⁴ However, it was found³ that long before that time, the quasi-steady, slowly decaying state that is reached has a rather sharp one-to-one pointwise correspondence between ω and ψ . The elucidation and testing of this correspondence is the principal purpose of this paper.

Clearly, even though a slow viscous decay may be superimposed on such a state with ω approximately a function of ψ , the state is also closely approximated by a time-independent solution of the Euler equations. There is no *a priori* reason why this should be so. It is a fact we attempt to incorporate here into a coordinated set of direct numerical simulations of Eq. (1) and a combined analytical and numerical argument, based upon statistical mechanics, in pursuit of what the connection between ω and ψ should be.

The statistical mechanics depends upon a discretization of vorticity in terms of delta-function line vortices (“points”) or in terms of mutually exclusive, finite-area “patches.” Boltzmann statistics are applied to the former, to define an entropy, or logarithm of the probability of a state, and Lynden-Bell statistics are applied to the latter. The limits of zero-area, finite-vorticity patches are points. Thus the patch formulation can be viewed as containing the point version of the theory, and either is or is not a useful generalization of it. In both cases, what results is a “most probable” dependence of vorticity upon stream function. A mean-field limit (infinitely many points or patches, of arbitrarily small

strength) is then taken, to yield continuous and differentiable functions $\omega(\psi)$. It can be shown that for points, the resulting function is

$$\nabla^2 \psi = -\omega = -e^{\alpha_+ - \beta\psi} + e^{\alpha_- + \beta\psi}. \quad (5)$$

For patches, the resulting function is

$$\nabla^2 \psi = -\omega = -\sum_{j=1}^q \frac{M}{\Delta} K_j \frac{e^{\alpha_j - \beta\psi K_j}}{\sum_{l=0}^q e^{\alpha_l - \beta\psi K_l}}. \quad (6)$$

The symbols α , β are Lagrange multipliers, which enter via a maximization of the appropriate entropy subject to the constraints of given energy and positive and negative fluxes of vorticity. They are determined in principle by demanding that the energies and vorticity fluxes calculated on the basis of Eqs. (2) and (5) or (6) match specified values. In Eq. (6), the K_j are the “levels” of vorticity corresponding to the different sized patches, and must be chosen somewhat arbitrarily. Δ/M is a fixed size of a “patch,” and may be chosen arbitrarily, within wide limits.

Referring to the Appendix for a numerical method to solve Eqs. (5) and (6), we proceed in Sec. III to a description of the initial conditions used. Both Eqs. (5) and (6) have infinitely many solutions. The physical ones are interpreted to be those which maximize the appropriate entropy from which they were derived. The others represent local maxima, which, as we shall see, may in some cases represent attainable states in the computations.

The questions before us are:

- (1) How does one determine the entropy-maximizing time-independent solutions to Eqs. (5) and (6)?
- (2) How close may we come to those solutions in a dynamical computation that solves Eqs. (1) and (2) as a time-dependent initial-value problem with large but finite Reynolds number?
- (3) Are there noticeable differences between the predictions of Eqs. (5) and (6) and are there reasons for preferring one to the other as predictors of late-time turbulent decays?

It is to be stressed that all three questions are answerable not in the abstract, but only as a result of somewhat demanding computations. There is no *a priori* reason why Navier–Stokes turbulent decays should be predicted at all by anything having to do with the Euler equations. The latter will never be soluble, for continuous initial conditions, over time intervals long enough to make ideal solutions of much interest, because it is in the nature of Euler codes, having no minimum physically determined length scale, to overrun their own resolution in a very few eddy turnover times. If the predicted states had not shown some empirical relevance to Navier–Stokes solutions, there would be no justification for this activity. Experience with 2-D turbulence computations shows that even though it is known that no singularity occurs for the continuum Euler equations within a finite time, the transfer of excitations to smaller spatial scales proceeds quite rapidly: typically an order of magnitude in wave number per eddy turnover time, or faster. Thus ideal phenomena that

require hundreds of eddy turnover times to see (as these do) will not be feasible to compute with continuum Euler codes within the foreseeable future.

The dynamical code used here is of the now familiar Orszag–Patterson pseudospectral variety.^{31,32} The code is a parallelized MPI FORTRAN 90 version of an earlier FORTRAN 77 code provided by W. H. Matthaeus (private communication). It is fully de-aliased, using the shifted-grid method. It has been run, in the runs reported here, on the SGI Origin 3800 at SARA Supercomputing Centre in Amsterdam. Some of the simulations (resulting in the “bar” final state in section III B 1 b) have also been recomputed with a different pseudospectral Fourier code (provided by Dr. A. H. Nielsen, Risø National Laboratories, Denmark). These computations yielded the same conclusions.

All runs resolve the Kolmogorov dissipation wave number based on enstrophy dissipation. Previously, initial conditions for turbulent decay runs have tended to use randomly loaded Fourier coefficients in the spectra of vorticity fields up to some upper wavenumber. These have typically been chosen to match some cascade wavenumber k spectrum, such as the “–3” direct enstrophy cascade spectrum predicted by Kraichnan.⁴⁵ The phases have been chosen from a random number generator. This has been done to achieve the most disordered (in some sense) initial field compatible with a particular power spectrum. The spectra contain no more useful information for this problem, it should be noted, than the phases of the Fourier coefficients; for this reason, we often give greater emphasis to spatial contour plots of vorticity and stream function, which incorporate the phase information as well as the amplitudes. There is relatively little emphasis on cascade-related considerations, such as power-law wavenumber spectra, since these require external stirring, or continued injection of excitations in k space, and we say nothing about this situation here. Several of the runs to be reported later are therefore presented in the form of contour plots of vorticity and stream function only. The above-described initialization procedure can only yield a vorticity distribution that is analytic in x and y , since only sinusoidal functions are involved, even though the spatial dependence may be wildly fluctuating. Such a vorticity distribution can take on any particular value of ω only over a set of measure zero. For this reason, one might imagine it would tend to de-emphasize features that corresponded to the “levels” in the patch formulation, over which the vorticity is supposed to take on a constant value inside a compact area. Therefore, we have also stressed initial conditions that have large, flat areas of vorticity, separated by thin regions, with as steep spatial gradients between them as the code will resolve. Our original intention was that these would correspond to unstable laminar shear flows whose instabilities would subsequently generate turbulence. Somewhat to our surprise, we have found it easy to produce shear-flow instabilities but not ones that led to turbulence, in the sense of broad k spectra. We have come to suspect that this is a feature inherent to two-dimensional flows, and perusal of the literature has uncovered only three-dimensional turbulence as a consequence of unstable shear flows, but not two-dimensional; but this must be left as a conjecture rather than a demonstrated fact.

FIG. 1. Typical ω - ψ relation from (a) sinh-Poisson, (b) 3-level Poisson equation, and (c) tanh-Poisson equation [special case when $e^{-\alpha} \rightarrow 0$ in Eq. (8)].

In any case, in order to induce a large number of Fourier modes to participate in the subsequent dynamics, we have found it necessary, in the runs evolving from vorticity distributions with flat areas, to add significant amounts of random noise initially, in order to get a subsequent k spectrum that could readily be called “turbulent.” Even then, some of the evolution we find might have its status as turbulence disputed.

Referring to the Appendix for the method of solution of Eqs. (5) and (6), we proceed in Sec. III to a description of initial conditions used in the dynamical runs, the states to which they evolve, and their comparisons with the maximum-entropy, or “most probable” states that Eqs. (5) and (6) lead to.

III. NUMERICAL RESULTS

A. Extracting maximum-entropy (“most probable”) states

We first summarize the results of the most-probable-state solutions, as obtained by the numerical methods described in the Appendix. The goal is not only to find numerical solutions but to determine which among the solutions has the highest entropy for given energy and vorticity fluxes. Generally speaking, the entropies of states with more maxima and minima have been seen to be less than those with fewer, and we can be guided by this. But there are instances where solutions of two different topologies can be found whose entropies lie very close to each other, as found by Pointin and Lundgren,⁸ and we must concentrate on those.

We consider the point relation, Eq. (5), in the absence of any conditions that would suggest asymmetry between posi-

tives and negatives, so that the two Lagrange multipliers $\alpha_+ = \alpha_- = \alpha$. The result is the sinh-Poisson equation [a typical ω - ψ dependence of which is shown in Fig. 1(a)],

$$\nabla^2 \psi = 2e^\alpha \sinh(\beta\psi), \quad \beta < 0. \quad (7)$$

We also consider the patch relation, Eq. (6), specialized to the three-level case of vorticity levels -1 , 0 , and $+1$. Again assuming symmetry between positives and negatives [Fig. 1(b)], we have

$$\nabla^2 \psi = \frac{M}{\Delta} \cdot 1 \cdot \left[\frac{2 \sinh(\beta\psi)}{e^{-\alpha} + 2 \cosh(\beta\psi)} \right], \quad (8)$$

where $\beta < 0$ and Δ/M is the (arbitrary) size of a patch.

Equation (7) can be rewritten as

$$\nabla^2 \Psi = -\lambda^2 \sinh \Psi, \quad (9)$$

where we have defined

$$\Psi = |\beta| \psi \quad (10)$$

and

$$\lambda^2 = 2|\beta|e^\alpha. \quad (11)$$

The form of Eq. (9) makes it most susceptible to numerical solution. Once it is solved, for a given value of λ , we must keep in mind that in order to evaluate the entropy associated with the solution, we must revert to the parameters of Eq. (7), while guaranteeing that α and β satisfy Eq. (11). Since for each fixed value of λ^2 , we have infinite sets of possibilities for α and β , a recipe is needed for choosing them. Since our goal is to plot the entropy versus energy for a fixed value of the vorticity flux,

FIG. 2. Contours of constant stream function ψ for the solutions of Eq. (5). Negative values are shown as broken lines throughout.

FIG. 3. Entropy vs energy for the solutions of Eq. (5) for unit positive and negative vorticity flux, computed for the "point" discretization.

$$\Omega_\psi = \frac{1}{L^2} \int \int e^{\alpha + \beta\psi} dx dy = \frac{1}{L^2} \int \int e^{\alpha - \beta\psi} dx dy$$

for the domain $[0, L] \times [0, L]$ (here we let $L = 2\pi$ and $\Omega_\psi = 1$, for convenience), we must also be assured that the solution we get for ψ satisfies this condition.

Fortunately, these two conditions can be satisfied simultaneously. Combining Eqs. (10), (11) and $\Omega_\psi = 1$ leads to the conclusion that $|\beta| = \Omega_\Psi$:

$$\begin{aligned} \Omega_\Psi &= \frac{1}{L^2} \int \int \frac{\lambda^2}{2} e^{-\Psi} dx dy \\ &= \frac{1}{L^2} \int \int |\beta| e^{\alpha} e^{-|\beta|\psi} dx dy = |\beta|. \end{aligned} \quad (12)$$

Because Eq. (9) is the equation solved, the value of Ω_Ψ is readily obtained. Thus we obtain β and from Eq. (11), α . We have now all the parameters that correspond to fixed

(a)

(b)

(c)

FIG. 4. Entropy vs energy at unit vorticity fluxes for the solutions of Eq. (8): (a) with a relatively large patch size ($M/\Delta = 3.7814$); (b) with a somewhat smaller patch size ($M/\Delta = 25$); (c) with a still smaller patch size ($M/\Delta = 100$). Note that in (c) the dipole solution has become slightly more probable than the bar.

FIG. 5. Equally spaced contours of constant vorticity at three different times (left column) and corresponding modal energies at the lower values of k (right column), during the evolution of the McWilliams–Matthaeus initial conditions. These have no flat patches of vorticity initially, even approximately.

vorticity flux of either sign. From Eqs. (13) and (14),

$$E_\psi = \frac{1}{2L^2} \iint \psi \omega_\psi dx dy = \frac{1}{2L^2 \beta^2} \iint \Psi \omega_\Psi dx dy, \quad (13)$$

$$S_\psi = -2\alpha \Omega_\psi - 2|\beta| E_\psi = -2\alpha - 2|\beta| E_\psi, \quad (14)$$

we can draw plots of the entropy (S_ψ) vs energy (E_ψ) for fixed unit flux of positive and negative vorticity.

It is useful to introduce terms for the solutions of differing topology. For example, for a given energy and vorticity flux, the contours of stream function may look like Fig. 2. Figure 2(a) will be called the “dipole” solution. Figure 2(b) will be called the “quadrupole” solution, and so on through higher “multipoles” that correspond to successively higher (even) numbers of maxima and minima. We have also found one-dimensional solutions as illustrated

in Fig. 2(c). These one-dimensional solutions will be called “bar” solutions, and there are also an infinite sequence of them, with basic periodicities 2π , π , $\pi/2$, ... These states are obtained, as explained in the Appendix, by iterating a trial solution with similar topology until convergence is obtained.

Figure 3 shows a plot of the entropy versus energy, for fixed unit positive and negative vorticity flux, for the quadrupole, bar, and dipole solutions computed for the “point” discretization. Observe the very small difference that makes the entropy of the dipole greater than that of the bar. It will turn out that it is possible to find either solution as a consequence of the development of certain initial conditions, as we shall see later.

Turning to the patch prediction, Eq. (8) can be simplified as

FIG. 6. The ω - ψ scatter plot for the run shown in Fig. 5 [which is close to Fig. 1(a)].

$$\nabla^2 \Psi = -\frac{\lambda^2 \sinh \Psi}{g + \cosh \Psi}, \quad (15)$$

by letting

$$\Psi = |\beta| \psi, \quad (16)$$

$$g = \frac{1}{2} e^{-\alpha}, \quad (17)$$

and

$$\lambda^2 = \frac{M|\beta|}{\Delta}. \quad (18)$$

Solving Eq. (15) is not as easy as solving Eq. (9) because there is an extra parameter, g , which implicitly reflects the arbitrary choice of the patch size. Another strategy is required.

(1) We choose the size of the patch, so that

$$\frac{M}{\Delta} = \frac{\lambda^2}{|\beta|} = \text{const.} \quad (19)$$

(2) Then a trial value of g is chosen before solving Eq. (15), again iterating about a trial solution of desired topology. Using Eq. (19), we have the value of β , and α from Eq. (17), E_ψ and Ω_ψ are given by

$$E_\psi = \frac{1}{2L^2\beta^2} \iint \omega_\psi \Psi dx dy, \quad (20)$$

$$\Omega_\psi = \frac{M}{L^2|\beta|} \iint \frac{e^\Psi}{2g + 2 \cosh \Psi} dx dy. \quad (21)$$

(3) However, the parameters obtained in this way are not usable since the condition of unit positive and negative vorticity flux is not in general satisfied. We must return to the second step and change g until the desired accuracy is obtained from Eq. (21) with $\Omega_\psi = 1$. Once this has been done, we are then in a position to evaluate the entropy from the algebraic expression:

FIG. 7. The initial vorticity field as a function of x and y for a run intended to exhibit patch characteristics. (a) The initial vorticity field without random noise, (b) the initial vorticity field with substantial random noise added to it.

$$S_\psi = -2\alpha\Omega_\psi - 2|\beta|E_\psi + \frac{\lambda^2}{|\beta|L^2} \iint \ln(1 + e^{\alpha - \Psi} + e^{\alpha + \Psi}) dx dy. \quad (22)$$

We may plot the entropy of Eq. (22) in Fig. 4(a), which is obtained by the choice of $M/\Delta = 3.7814$, a relatively large “patch” size. It will be seen that for this large patch size, the entropy of the bar solution is greater than that of the dipole, a different conclusion than that of the point calculation. However, if we reduce the size of the patch, we find that the result approaches that of the point formulation. In Fig. 4(b), we see the result when $M/\Delta = 25$, and in Fig. 4(c) the result when $M/\Delta = 100$, where the maximum entropy state is again the dipole one. Considering the very small entropy differences between the dipole and the bar solutions shown in Figs. 3 and 4, it might not be thought surprising if the fluid had difficulty making up its mind which state to relax into. It may appear to the reader that the shift of the bar state to its status as “most probable,” solely as a consequence of enlarging the patch size seems artificial (Fig. 4), since there seems to be no unique physical determination for the patch size. It seems somewhat artificial to us also, but the fact that the dynamical code seeks out the bar solution for some quite dissimilar initial conditions, as will be seen in what follows, convinces us that the bar solution has some reality.

Before turning to the results of the dynamical computations, we offer a few observations on the relations of the patch versions of the theory to the point version, and to each other. It is clear from the derivation that in the limit that the patch sizes become smaller and smaller, at fixed and finite separation, the point version of the theory is recovered. Depending upon the number of levels chosen for the patch formulation, there are many versions of the most-probable-state patch equation; the three-level version of Eq. (8) is not the most general by any means. Each will have different solutions. There is, however, no apparent unique or practical prescription for how many levels, or what size patches, should be used to represent a particular initial analytic vorticity distribution with high accuracy. It is not clear that it can be done without requiring the patch size to shrink to zero, at which

FIG. 8. Contours of constant vorticity (left column) and constant stream function (right column) at three different times for the run originating from the initial vorticity distribution (with noise) shown in Fig. 7(b).

point it becomes indistinguishable from a point representation. In fairness, we should also say that there is another aspect to the point formulation that is also ambiguous: namely, there is no reason to choose the point vortices of the mean-field theory to be of equal strength. If some are of different strength, Eq. (7) will also change. Keeping these in mind, we turn now to the effort to see some of the solutions as consequences of direct numerical solution of the Navier–Stokes equation.

B. Dynamical solutions and comparisons

We now present the results of dynamical, pseudo-spectral-method solutions of the 2-D NS equation, using a resolution of 512^2 . The time step is 0.0005. The initial en-

ergy, using the normalization of Eq. (3), is 0.5. There is no hyperviscosity or small-scale smoothing of any kind.

1. “Dipole” and “bar” final states

a. Random, broad-band initial conditions. The first run we report here is essentially a reproduction of the run of Matthaeus *et al.*,^{1–3} but with a lower Reynolds number ($1/\nu=5000$), corresponding to an initial Taylor-scale Reynolds number

$$R_\lambda = \sqrt{\frac{10}{3}} \frac{E}{\nu \sqrt{\Omega}} \approx 558.$$

R_λ has increased to 3143 by the end of the run.

FIG. 9. The ω - ψ scatter plot of the late-time state achieved in the run shown in Fig. 8 [which is close to Fig. 1(c)].

This run served as a benchmark for our parallelized code in the beginning, and the McWilliams initial conditions⁴⁶ that were used in the computations by Matthaeus *et al.* can also be used as noise, with variable amplitude, to be added to later simulations to break unwanted symmetries and accelerate the dynamical development. The run also provides an opportunity to introduce the types of data displays that will be used throughout the rest of this section. The Fourier modal energy spectrum $E(\mathbf{k}) \equiv (1/2)|\mathbf{v}(\mathbf{k})|^2$ is initialized according to

$$E(k) = \frac{C}{1 + \left(\frac{k}{6}\right)^4}$$

for $1 \leq k \leq 120$ (here, and for this purpose only, the wave-number k is binned in integer values by a standard FORTRAN routine, and the partition of energies among modes with the same integer k is decided by a random number generator),

FIG. 10. Low- k modal energy spectra (left column) and vorticity histogram (right column) at three different times for the run shown in Figs. 7(b) and 8. The vorticity histogram shows the number of times a particular value of ω is recorded as one cycles through the computational grid. Notice that there is substantial variation from one time to another. (An Euler equation solution should preserve this histogram in time.)

FIG. 11. Time evolution of a 64-pole initial condition, triggered only by round-off error, into a bar state [the scatter plot of final state is close to Fig. 1(c)]. No noise beyond round-off error has been added to the initial condition.

and zero otherwise. The phases of the Fourier coefficients are chosen from a Gaussian random number generator. C is a constant to be adjusted to make the total initial energy equal to 0.5.

The results were quite similar in all respects to those of Matthaeus *et al.*,¹⁻³ so only a few figures (Figs. 5 and 6) are shown here. The contour plots of vorticity and stream function relax to the familiar dipole final states. The left column of Fig. 5 shows the vorticity contours at three separate times. The right column of Fig. 5 shows the modal energy spectra, for the lower part of k space (the maximum k is 241) at those same times, exhibited as three-dimensional perspective plots. They are initially broad-band, but evolve as expected to be concentrated in the lowest values of k . In Fig. 6, a scatter plot of ω vs ψ shows a good agreement with the hyperbolic sinusoidal dependence of vorticity on stream function, as predicted in the point formulation (Fig. 3).

b. "Bar." A different, and interesting, evolution results ($1/\nu=5000$) if we initialize using the quadrupole solution from Eq. (15). In this run, R_λ increases from 2391 initially to 4920 at the end. This is not predicted to be the maximum entropy state from Eq. (22), or any other equation, patch or point; and so should evolve when initialized with some random noise. The quadrupole solution was found to have a symmetry which persists in time, and the noise, it is hoped, will permit that symmetry to be broken. It was found that the symmetry persisted until the noise was raised to quite substantial levels. Figure 7 presents perspective plots of the initial vorticity as a function of x and y , with and without the

noise added. The noise level in this case is about twice the quadrupole vorticity, and its randomness should break any symmetry that might be present.

Figure 8 shows the evolution of the vorticity and stream function contours that result from the initial conditions shown in Fig. 7(b). The intermediate panel ($t=15$) shows no obvious connection to the symmetry of Fig. 7(a) or to that of the final panels ($t=1250$) of Fig. 8 (the first is odd about the center lines of the basic square, but clearly, the $t=15$ state has no such symmetry). The relaxation to the one-dimensional "bar" state at $t=1250$ has been quite robust in several such runs. Figure 9 is an ω - ψ scatter plot for the bar state. There has been considerable decay of vorticity, as evidenced by the disappearance in Fig. 9 of the high and low values visible in Fig. 7(b). By $t=1250$, the energy (E) has decreased to 58.4% of the initial value.

When we consider the modal energy spectra in Fig. 10, for this evolution, we see that the initial and final states are dominated by four and two Fourier modes, respectively. In fact, the evolution after $t=40$ has this character. The energy spectrum at $t=15$ is somewhat more broad-band than either, but whether it should properly be called "turbulent" may be debated; it never achieves the fully broad-band character of the evolution shown in Fig. 5, for example. The final state achieved is consistent with the maximum-entropy prediction of the "patch" formulation for suitably large-sized patches [Fig. 4(a)], but not with the same formulation for smaller-sized ones [Fig. 4(c)]. The right column of Fig. 10 shows a histogram of the vorticity at three different times in this evo-

FIG. 12. Time evolution of the vorticity contours, starting from the same initial condition as in Fig. 11, but with a healthy addition of random noise. (In this run, R_λ increases from 2036 initially to 11 400 at the end.) The last panel is the late-time ω - ψ scatter plot for the resulting dipole [which is close to Fig. 1(a)].

FIG. 13. Evolution of the one-dimensional “8-bar” initial condition, with random noise added initially. (In this run, R_λ increases from 3228 initially to 11 500 at the end.) The evolution is toward a dipole, now, as seen in the bottom panel [which is close to Fig. 1(a)].

lution; it is in effect a frequency distribution for the appearance of various values of the vorticity over the plane, one per computational cell in the 512^2 array. It will be seen that this distribution changes enormously over the course of the run, which it cannot do, of course, in any ideal Euler picture.

Figures 7(b)–10, show, then, a reproducible evolution of an initial condition consisting of a patch quadrupole plus large amounts of random noise into a one-dimensional “bar” most-probable state. This state is more probable than the dipole, according to a “patch” analysis if the patch size is big enough, though it is not for a smaller but finite patch size.

A similar initial condition is used by Segre and Kida.⁴⁷ However, their computation made use of hyperviscosity and appears not to have run long enough for the proper late-time state to evolve.

2. Local maximum states

a. Evolution from a 64-pole initial condition. The contrasting evolutions of the two initial states in Sec. III B 1 naturally arouse curiosity about whether there is a limit in which one behavior goes over into the other. Here, we attempt an answer to this by considering initial conditions which originate in high-order multipole solutions of the patch formulation, not maximum entropy states in either formulation, but intuitively closer to the random initial conditions that led to the dipole solution before, in the first run reported.

There are four numerical solutions in this group, with $1/\nu = 10\,000$ for all four runs. They all originate in the 64-pole solution of the patch formulation, using the 3-level equation, Eq. (15). The same conclusions that we reach can

FIG. 14. Evolution from initial conditions corresponding to a sinh-Poisson quadrupole, plus random noise. Note the differences in evolution from those shown in Figs. 7(b)–10, whose initial conditions are superficially similar. The low- k part of the Fourier space contains most of the energy throughout, so the evolution again is not classically “turbulent.”

FIG. 15. Scatter plot of the pointwise dependence of ω upon ψ at two different times, for the evolution shown in Fig. 14 [both of them are close to Fig. 1(a)].

also be reached using 16-pole initial states, but we will not display those results here.

The motivation was to see if the high-order multipole initializations would lead to “bar” final states, the way the quadrupole does. Intuitively, they would seem to be closer to our picture of what true turbulence might look like. The bar states are no longer found, but there are “local maximum” entropy states that can be achieved in the limit of low noise in the initial conditions.

First consider what happens to the 64-pole initial conditions without any noise, as shown in Fig. 11. A straightforward laminar evolution occurs, with the end product ($t=250$) being a one-dimensional bar state with a total of eight maxima and minima. This state is essentially dominated by one Fourier mode, as indicated by the essentially linear pointwise dependence of vorticity on stream function. In Fig. 12, we show the evolution of the same initial conditions with only a low level of noise: 1/2000 of the amplitude of the 64-pole solution, not visible on the $t=0$ contour plot. In this case, fully developed turbulence does seem to result, with a dipole final state as the end result.

The third run in the group concerns the result of taking the final bar state as shown in Fig. 11. We raise the initial energy: $E(t=0)=0.5$, without putting any noise into it, and allow it to run. This appears to be a time-independent state, stable in the presence of round-off error for the duration of the run, but not a maximum-entropy state and not stable in the presence of noise of greater amplitude. This is clear from Fig. 13, which shows the development of the unstable evolution induced by putting on the same level of random noise as in Fig. 12, with a fully developed turbulence and a dipole final state as the result.

FIG. 16. Evolution of the global quantities as functions of time for the run shown in Figs. 14 through 15. The merger of vortices around $t=175$ (from quadrupole to dipole) leads to the “spike” in (d). This is a relatively short process (from $t \approx 170$ to $t \approx 190$). Palinstrophy, or mean square vorticity gradient, is proportional to the enstrophy dissipation rate, and it is known that large vortex merger is an abrupt event, resulting in a rather dramatic loss of total enstrophy (Ref. 2).

FIG. 17. Evolution of a 4-bar initial condition, plus random noise, that ceases to evolve (perhaps because of loss of nonlinearity due to decay) before either a dipole or a bar final state is achieved.

There are thus some subtleties revealed by these four runs. Bar final states, predicted by the patch theory with sufficiently large patch size, do result from the evolution of a patch quadrupole plus sufficient noise. However, higher order multipoles from the patch formulation seem to be unstable at low levels of random noise, and evolve into dipoles, consistently. Still less easy to fit into the picture is the laminar evolution of the “local maximum” 64-pole state without noise into the bar state shown in Fig. 11. Neither state is in this case an absolute maximum entropy one, though both are in some sense local maxima.

The initial conditions illustrated in the first panel of Fig. 11 are not those of an approximately steady state. What Figs. 11–13 apparently show is that such a checkerboard initial condition can evolve to a metastable quasi-stationary state of less than absolute maximum entropy which is nonetheless unstable if it is excited by greater random noise than pseudospectral round-off error provides. The high degree of symmetry apparent in the initial condition may prejudice the evolution, also. The differences in the evolutions in Figs. 11 and 12 support this speculation.

b. Evolution starting from a sinh-Poisson quadrupole. We have started several runs from initial conditions that originate with a sinh-Poisson quadrupole state, with $1/\nu = 10\,000$. The first is the quadrupole state without any added random noise, and the others are the same state plus smaller or larger amounts of random noise.

The zero-random noise initial conditions seem to be stable and to remain in the same shape in the presence of only round-off error for the duration of the run, as reported previously.⁴⁸ Even after putting in enough random noise to manifestly break the symmetry (Fig. 14), the quadrupolar shape is maintained for quite a long time before breaking down into a dipole by $t = 300$. The ω - ψ scatter plots are shown in Fig. 15. Evolution of the global quantities for this run is shown in Fig. 16, showing abrupt changes in enstrophy and palinstrophy ($P \equiv \sum_{\mathbf{k}} (1/2) k^2 |\omega(\mathbf{k})|^2$) when vortex merger occurs, but otherwise not displaying characteristics of turbulent behavior. Note that energy is well conserved for this case. In this run, the quadrupole persists for a long time because the energy is concentrated in the four well-separated vortices, before breaking down into a dipole. This is notice-

ably different behavior than when we started with the patch quadrupole.

3. Oddities

(a) A run ($1/\nu = 20\,000$) that leads to an “unclassifiable” final state is shown in Fig. 17. It begins with a fourfold bar state, plus a considerable amount of random noise, and ends at a state that shows features of both dipoles and bars. A tentative interpretation of this evolution is that the nonlinearity is simply exhausted by the viscous decay before either evolution can be completed. Once the amplitudes fall below amounts or in Fourier configurations at which the activity is effectively nonlinear, the pattern in place is “frozen” in its topology, and can only slowly decay. The scatter plot has features of both the point and patch predictions.

(b) Several other runs demonstrated odd features. Metastable states were found that would persist for a long time as a consequence of evolution that, though disordered, might be thought to be less than totally turbulent. An example appears in Fig. 18. In this run ($1/\nu = 12\,500$), we began with a 16-pole solution of the 3-level patch equation, removed some of the squares of vorticity, and used the rest as an initial condition. Most of these went to the dipolar states, but one of the untypical ones reached a vaguely quadrupolar one, in the last panel of Fig. 18. Figure 19 is a ω - ψ scatter plot, showing a bilinear form, as if two locally “most probable” states had formed into a quasi-equilibrium that was slow to break up and decay toward anything globally “most probable.”

Several runs ($1/\nu = 10\,000$) were carried out using this metastable quasi-quadrupole as the basis for initial conditions, plus a significant amount of added random noise. The evolution then was typically that the system, after having lingered awhile, evolved into the dipole configuration.

IV. CONCLUSIONS AND DISCUSSION

We have set out to test the relevance of the predictions of Eqs. (5) and (6) and their respective vorticity discretizations to the numerically determined, long-time states of a 2-D NS fluid with Reynolds numbers of several thousand subject to doubly periodic boundary conditions. Equation (5) has been

FIG. 18. A second run, whose late-time state is outside the patch/point classifications, that seems to have reached some metastable late-time state. Here, a large area of zero vorticity was created in the initial conditions by removing, asymmetrically, four pieces of a 16-pole patch solution.

derived by modeling the vorticity distribution as a mean-field limit of equal delta-function point vortices, while Eq. (6) models the vorticity distribution as made up of flat mutually exclusive patches of an area whose choice is arbitrary. Both equations have an infinite number of solutions, characterized by different topologies, but for fixed vorticity fluxes and total energy, all but one are only local maxima and there seems to be always one uniquely defined maximum-entropy prediction. As seen in Figs. 3 and 4, it is sometimes a matter of painstaking analysis to determine which state this is, however. We did computational runs of two basic types: runs with broad-band, totally disordered initial conditions of the type previously investigated, and runs originating from vorticity distributions that consisted of large areas of nearly flat vorticity patches plus random noise to break possible unwanted symmetries. In the former case, there were no surprises: the previously found dipolar late-time states inevitably resulted. This was not the case for the second kind of initial conditions, however. They did not generate broad-band turbulence in which energy was shared widely among many Fourier modes, but did exhibit considerable nonlinear activity in which the energy remained primarily in the lower parts of Fourier space. It is perhaps a semantic quibble as to whether the evolution should be called fully turbulent. This second class of initial conditions, in any case, exhibited a more diverse range of possible behavior than the broad-band initializations did, and often came close to a late-time state that could be identified as one of the solutions of Eq. (6). The most interesting of those was the one-dimensional “bar” state exemplified by Figs. 8 and 11; in the former case, it is attained with the help of added initial random noise, and in the latter, without it. As Figs. 3 through 4 illustrate, it may or may not be considered to be the most-probable patch state, depending upon the choice of the patch size Δ/M , which seems to be arbitrary. In any case, it is not the most probable point state predicted from Eq. (5). In the finite-size patch theory leading to Eq. (6), the arguments for it proceed from “coarse graining” the Euler equation behavior and presuppose a minimum observational scale below which fine spatial structure cannot be resolved. In a well-resolved NS computation (presumably including all of the ones reported here), there is no such scale, and observation of all scales participating is feasible. Thus that the patch formulation of Eq. (6), with a large enough value of Δ , can predict a radically different

topology than it does with a smaller Δ , and then substantiate that prediction in a dissipative NS computation, is another of the accumulating puzzles which remain to be deciphered. In our opinion, there can be said to be an interesting 2-D NS regime to be explored that has opened up in these investigations in which nonlinear evolution, perhaps not fully turbulent in the conventional sense, nevertheless leads from one state that can be identified to another laminar, late-time state that is quantitatively more probable than its initial conditions: an essentially thermodynamic behavior. The conditions for assigning this probability remain incompletely defined, and seem to us worthy of further numerical investigation.

The general question of whether there is a sharp sense in which Euler equation dynamics can satisfactorily mimic turbulent Navier–Stokes decays must, in our opinion, remain open. It seems beyond question, however, that entropies defined in terms of ideal fluid models have some predictive power for high Reynolds number decays that go beyond coincidence. It remains to be determined what precisely the limitations, for this purpose, of the two entropies considered here may be.

FIG. 19. The ω - ψ scatter plot of the late-time state achieved in Fig. 18, suggesting two independent coexisting sinh-Poisson states which have developed asymmetrically. This is for the evolution shown in Fig. 18.

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APPENDIX: SOLVING EQS. (5) AND (6)

The sinh-Poisson equation [Eq. (9)] has been solved by several researchers numerically and analytically (see Ref. 20, and references therein). Probably the most direct way to do it is to put the equation into spectral space and do the iteration there:

$$(\widehat{\Psi}_{n+1})_{\mathbf{k}} = \frac{\lambda^2}{|\mathbf{k}|^2} (\widehat{\sinh \Psi_n})_{\mathbf{k}}. \quad (\text{A1})$$

We show three solutions in Fig. 2. In addition to the solutions exhibited, there are an infinite number more, characterized by more and more maxima and minima. They are obtained by starting with a trial function on the right-hand side of Eq. (A1) that has the desired number of final maxima and minima, as noted by several authors: McDonald,⁶ Book *et al.*,⁷ Lundgren and Pointin.⁸ Typically, the solutions with many maxima and minima are lower entropy solutions, for fixed energy and vorticity fluxes, and so we do not discuss them in any detail.

FIG. 20. A periodic quadrupolar solution may be used to generate a dipolar one, depending upon the spatial subvolume chosen, as shown. The “dipole” solution which is cut from the “quadrupole” can be zoomed [by a factor of $\sqrt{2}$ in (x,y) space] to fit the $[0,2\pi] \times [0,2\pi]$ domain. But it is not the solution of Eq. (15) anymore. It is a solution of $\nabla^2 \Psi = -(\lambda^2 \sinh \Psi)/(2g + 2 \cosh \Psi)$ instead.

The spectral method loses its advantage when we try to use it to solving Eq. (15), because we might have some step function solutions. Here we adopt the numerical method used by McDonald.⁶

The first step is to get the nontrivial solution for Eq. (15) with $\Psi = 0$ on a square boundary of $[0, \pi] \times [0, \pi]$. Starting from a trial solution $W(x,y)$, we get the solution of $v(x,y)$ from

$$\nabla^2 v + v f'(W) = R(W, W), \quad (\text{A2})$$

subject to $v = 0$ on the boundary. Here

$$(A, B) = \int_0^\pi \int_0^\pi A(x, y) B(x, y) dx dy, \quad (\text{A3})$$

$$f(\Psi) = \frac{\lambda^2 \sinh \Psi}{g + \cosh \Psi}, \quad (\text{A4})$$

$$R(x, y) = \frac{\nabla^2 W + f(W)}{(W, W)}. \quad (\text{A5})$$

Then we can correct the trial solution:

$$W \rightarrow W + \frac{v(x, y)(W, W)}{2(v, W) - (W, W)}, \quad (\text{A6})$$

until we get a sufficiently accurate solution.

Perhaps the most important change we made to McDonald's methods⁶ is using the quadruple precision instead of double precision in the calculation (and this is not a luxury on modern computers). The accuracy problem mentioned by McDonald in this method is overcome. As a result, if we define the root-mean-square error (rms):

$$\epsilon = \left(\frac{\sum_{i,j} f^2}{\sum_{i,j} g^2} \right)^{1/2},$$

where

$$f = \Delta \Psi(i, j) + \lambda^2 \sinh(\Psi(i, j))$$

and

$$g = |\Delta \Psi(i, j)| + |\lambda^2 \sinh(\Psi(i, j))|.$$

Our solutions can at least reach an accuracy of 10^{-7} .

We may construct higher-order multipole solutions by putting monopole solutions side by side with alternating signs, and achieve doubly periodic boundary conditions in this way, in a “checkerboard” solution, so that dipole (Fig. 20) and higher-multipole solutions can be constructed. There is some loss of accuracy in this procedure, and we have made sure that the accuracy of all solutions is of the order of 10^{-6} (rms) in our calculations.

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