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SEPARATE AND JOINT GEVREY VECTORS FOR REPRESENTATIONS OF LIE GROUPS

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Separate and joint Gevrey vectors for representations of Lie groups

by

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Abstract

In the present paper we prove that for every Lie group there exists a basis in its Lie algebra such that for each $\lambda \geq 1$ and each of its unitary representations, a vector in the Hilbert space which is a Gevrey vector of order $\lambda$ in each direction of the basis elements is in fact a (joint) Gevrey vector of order $\lambda$. As a corollary we obtain that a vector is a Gevrey vector if and only if the corresponding positive definite function is a Gevrey function.

AMS 1980 Subject Classification: 22E45.
1 Introduction and notations

Let $G$ be a $d$-dimensional (real) Lie group, let $Y_1, \ldots, Y_d$ be analytic vector fields on $G$ which are linearly independent at each point of $G$, let $H$ be a Hilbert space and let $\lambda \geq 1$. A function $f$ from $G$ into $H$ is called a Gevrey function of order $\lambda$ if and only if $f$ is infinitely differentiable and for each compact subset $K$ of $G$ there exist $c, t > 0$ such that for all $n \in \mathbb{N}_0$

$$\sup_i \sup_{g \in K} \| (Y_1 \circ \ldots \circ Y_n f)(g) \| \leq ct^n n!^\lambda.$$ 

It follows from [Nel, Theorem 2] and [GW, Theorem 1.1] that this definition does not depend on the choice of $Y_1, \ldots, Y_d$. The Gevrey functions of order 1 are just the real analytic functions from $G$ into $H$. Now let $\pi$ be a representation of $G$ into $H$. For $u \in H$ define $\bar{u} : G \to H$ by

$$\bar{u}(g) := \pi g u \quad (g \in G).$$

A vector $u \in H$ is said to be infinitely differentiable, analytic respectively a Gevrey vector of order $\lambda$ for $\pi$ if and only if the map $\bar{u}$ is infinitely differentiable, (real) analytic respectively a Gevrey function of order $\lambda$ from $G$ into $H$. (Cf. [Går], [Nel] and [GW] respectively.) Let $H^\infty(\pi), H^\omega(\pi)$ and $H_\lambda(\pi)$ denote the space of all infinitely differentiable vectors, of all analytic vectors and of all Gevrey vectors of order $\lambda$ for $\pi$, respectively. Note that $H^\omega(\pi) = H_1(\pi)$. We only consider continuous unitary representations. For each $X$ in the Lie algebra $\mathfrak{g}$ of $G$ let $d\pi(X)$ be the infinitesimal generator of the one-parameter unitary group $t \mapsto \pi_{\exp tX}$. Let $\partial \pi(X)$ be the restriction of $d\pi(X)$ to $H^\infty(\pi)$. The map $X \mapsto \partial \pi(X)$ extends uniquely to an associative algebra homomorphism, denoted by $\partial \pi$ also, from the complex universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ into the set of all linear operators from $H^\infty(\pi)$ into $H^\infty(\pi)$.

There exist infinitesimal characterizations for the spaces $H^\infty(\pi), H^\omega(\pi)$ and $H_\lambda(\pi)$, i.e. characterizations in terms of the infinitesimal generators. To this end we introduce the concept of multi-index. Let $V$ be a non-empty finite set. We define the set $M(V)$ of multi-indices over $V$ by

$$M(V) := \bigcup_{n=0}^{\infty} \mathbb{V}^n.$$

Here $\mathbb{V}$ is the set with one element, called the empty sequence, which is denoted by ( ). For $\alpha \in M(V)$ define the length of $\alpha$ by $||\alpha|| = n$, where $n \in \mathbb{N}_0$ is the unique number such that $\alpha \in \mathbb{V}^n$ and the reverse $\alpha^r$ of $\alpha$ by

$$(\ )^r := (\ )$$

$$(j_1, \ldots, j_n)^r := (j_n, \ldots, j_1) \quad (n \in \mathbb{N}, j_1, \ldots, j_n \in V).$$

Now let $V := \{1, \ldots, d\}$ and let $A_1, \ldots, A_d$ be operators in $H$. For $\alpha \in M(V)$ define the operator $A_{\alpha}$ by

$$A(\ ) := \mathbb{I},$$

$$A_{(j_1, \ldots, j_n)} := A_{j_1} \circ \ldots \circ A_{j_n} \quad (n \in \mathbb{N}, j_1, \ldots, j_n \in V).$$

Define the joint $C^\infty$-domain $D^\infty(A_1, \ldots, A_d)$ of the operators $A_1, \ldots, A_d$ by

$$D^\infty(A_1, \ldots, A_d) := \bigcap_{\alpha \in M(V)} D(A_{\alpha}).$$
Here $D(T)$ denotes the domain of the operator $T$. Let $\lambda \geq 1$. Define the Gevrey space $S_\lambda(A_1, \ldots, A_d)$ of order $\lambda$ relative to $\{A_1, \ldots, A_d\}$ by

$$S_\lambda(A_1, \ldots, A_d) := \{ u \in D^\infty(A_1, \ldots, A_d) : \exists c, t > 0 \forall \alpha \in \mathbb{N} \{ ||A_\alpha u|| \leq c t^{||\alpha||} ||\alpha||^{\lambda} \} \}.$$

(Cf. [GW, Section 1.]) We have the following infinitesimal description of the spaces $H^\infty(\pi)$ and $H_\lambda(\pi)$.

**Theorem 1** Let $\pi$ be a representation of a Lie group $G$ in a Hilbert space $H$. Let $X_1, \ldots, X_d$ be any basis in the Lie algebra $\mathfrak{g}$ of $G$. Let $\lambda \geq 1$. Then

$$H^\infty(\pi) = D^\infty(\pi(X_1), \ldots, \pi(X_d)) = \bigcap_{k=1}^d D^\infty(\pi(X_k))$$

and

$$H_\lambda(\pi) = S_\lambda(\pi(X_1), \ldots, \pi(X_d)).$$

**Proof.** See [Goo2, Proposition 1.1] and [Goo1, Theorem 1.1] for the space $H^\infty(\pi)$ and [GW, Proposition 1.5] for the space $H_\lambda(\pi)$. $\square$

So a vector in $H$ is infinitely differentiable for $\pi$ if and only if it is infinitely differentiable for each $\pi|_{G_k}$ separately, where $G_k$ is the one dimensional group $\{ \exp t X_k : t \in \mathbb{R} \}$.

Let $\lambda \geq 1$. Then clearly

$$S_\lambda(\pi(X_1), \ldots, \pi(X_d)) \subset \bigcap_{k=1}^d S_\lambda(\pi(X_k)) \quad (1)$$

for any basis $X_1, \ldots, X_d$ in $\mathfrak{g}$. The space on the right hand side of (1) is much easier to describe than the space on the left hand side. In the present paper we shall show that there exists a basis $X_1, \ldots, X_d$ in $\mathfrak{g}$ such that

$$H_\lambda(\pi) = S_\lambda(\pi(X_1), \ldots, \pi(X_d)) = \bigcap_{k=1}^d S_\lambda(\pi(X_k)). \quad (2)$$

As a corollary of (2) we obtain that for every $\lambda \geq 1$ a vector $u \in H$ is a Gevrey vector of order $\lambda$ for $\pi$ if and only if the corresponding positive definite function $g \mapsto (\pi_g u, u)$ from $G$ into $\mathbb{C}$ is a Gevrey function of order $\lambda$.

In fact, (2) generalizes a theorem of Flato and Simon ([FS, Theorem 3]), which states that there exists a basis $X_1, \ldots, X_d$ in $\mathfrak{g}$ such that $H^\omega(\pi) = \bigcap_{k=1}^d S_1(\pi(X_k))$. Taking $\lambda = 1$, this paper yields an easier proof for the theorem of Flato and Simon.

## 2 Gevrey vectors for direct sums

In this section we prove that in case the operators $A_1, \ldots, A_d$ span a Lie algebra, we can always write the corresponding Gevrey spaces as an intersection of two Gevrey spaces relative to a reduced number of operators.
Theorem 2 Let $\mathfrak{g}$ be a Lie algebra of skew-Hermitian operators in a Hilbert space defined on a common invariant domain. Let $d \in \mathbb{N}$, $d \geq 2$ and let $A_1, \ldots, A_d \in \mathfrak{g}$. Suppose $\mathfrak{g} = \text{span}(\{A_1, \ldots, A_d\})$.

Let $\lambda \geq 1$ and let $d_1 \in \{1, \ldots, d - 1\}$. Then

$$S_\lambda(A_1, \ldots, A_d) = S_\lambda(A_1, \ldots, A_{d-1}) \cap S_\lambda(A_{d-1} + 1, \ldots, A_d).$$

**Proof.** Let $V_1 := \{1, \ldots, d_1\}$, $V_2 := \{d_1 + 1, \ldots, d\}$ and $V := \{1, \ldots, d\}$. By assumption, for all $i, j \in V$ there exist $c_{i,j}^1, \ldots, c_{i,j}^d \in \mathbb{R}$ such that $[A_i, A_j] = \sum_{k=1}^d c_{i,j}^k A_k$. Let $M := 1 + \max\{|c_{i,j}^k| : i, j, k \in V\}$.

Let $u \in S_\lambda(A_1, \ldots, A_{d-1}) \cap S_\lambda(A_{d-1} + 1, \ldots, A_d)$. Then there exist $c > 0$ and $t \geq Md$ such that $\|A_\alpha u\| \leq ct\|\alpha\|^\lambda$ and $\|A_\beta u\| \leq ct\|\beta\|^\lambda$ for all $\alpha \in M(V_1)$ and $\beta \in M(V_2)$. For $N \in \mathbb{N}$ the hypothesis $P(N)$ states:

$$|(A_\gamma A_\alpha u, u)| \leq c^2 t \|\alpha\| \|\gamma\| \lambda \sum_{\alpha} \|\alpha\| \|\gamma\| \lambda$$

Clearly hypothesis $P(0)$ holds. Let $N \in \mathbb{N}$ and suppose hypothesis $P(N-1)$ holds. Let $\gamma \in V^N$ and let $\alpha \in M(V_1)$. Let $j_1, \ldots, j_N \in V$ such that $\gamma = (j_1, \ldots, j_N)$. We consider two cases.

**Case I.** Suppose $j_i \in V_2$ for all $i \in \{1, \ldots, N\}$.

Then $\gamma \in M(V_2)$, so by Schwarz' inequality we obtain that

$$|(A_\gamma A_\alpha u, u)| = |(A_\alpha u, A_{\gamma^R} u)| \leq \|A_\alpha u\| \|A_{\gamma^R} u\| \leq ct\|\alpha\| \|\gamma^R\| \lambda \leq c^2 t \|\alpha\| \|\gamma^R\| \lambda.$$

**Case II.** Suppose there exists $i \in \{1, \ldots, N\}$ with $j_i \in V_1$.

Let $k := j_i$. Pushing the operator $A_k$ in $A_\gamma$ to the ultimate right hand side and taking into account all commutators, we obtain that there exist $\delta \in V^{N-1}$, $c_1, \ldots, c_{(N-1)d} \in \mathbb{R}$ and $\theta_1, \ldots, \theta_{(N-1)d} \in V^{N-1}$ such that

$$A_\gamma = A_\delta A_k + \sum_{p=1}^{(N-1)d} c_p A_{\delta_p}$$

and $|c_p| \leq M$ for all $p$. Now the induction hypothesis $P(N-1)$ and the inequality $dM \leq t$ yield the estimates

$$|(A_\gamma A_\alpha u, u)| \leq c^2 t \|\alpha\| \|\gamma\| \lambda \sum_{\alpha} \|\alpha\| \|\gamma\| \lambda + (\|\gamma\| - 1) dM c^2 t \|\alpha\| \|\gamma\| \lambda \leq c^2 t \|\alpha\| \|\gamma\| \lambda \left(1 + \frac{\|\gamma\| - 1}{\|\alpha\| \|\gamma\| \lambda}\right) \leq c^2 t \|\alpha\| \|\gamma\| \lambda \left(1 + \frac{1}{\|\alpha\| \|\gamma\| \lambda}\right).$$

This proves hypothesis $P(N)$.

By induction, for all $\gamma \in M(V)$ and $\alpha \in M(V_1)$ we obtain that
Now let \( \gamma \in M(V) \). Then
\[
\|A_{\gamma}u\|^2 = \|A_{\gamma}A_{\gamma}u, u\| \leq c^2(2t)^{\|H\|\|t\|}(\|\alpha\| + \|\gamma\|)^{1/2}
\]
and the theorem follows

For other theorems in this direction, we refer to the thesis [1E].

The third equality in the following corollary has been firstly proved by Flato and Simon. (See [FS, Theorem 2].)

**Corollary 3** Let \( G \) be a real Lie group with Lie algebra \( \mathfrak{g} \). Let \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) be subalgebras of \( \mathfrak{g} \) such that \( \mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2 \). (Not necessarily a direct sum.) Let \( G_1 \) and \( G_2 \) be subgroups of \( G \) which have Lie algebras \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) respectively. Let \( \pi \) be a representation of \( G \) in a Hilbert space \( H \), and let \( \pi_1 \) and \( \pi_2 \) be the restrictions of \( \pi \) to \( G_1 \) and \( G_2 \) respectively. Let \( Z_1, \ldots, Z_d \) be a basis in \( \mathfrak{g} \), let \( X_1, \ldots, X_d_1 \) be a basis in \( \mathfrak{g}_1 \) and let \( Y_1, \ldots, Y_d_2 \) be a basis in \( \mathfrak{g}_2 \). Let \( \lambda \geq 1 \). Then
\[
S\lambda(d\pi(Z_1), \ldots, d\pi(Z_d)) = S\lambda(d\pi_1(X_1), \ldots, d\pi_1(X_d_1)) \cap S\lambda(d\pi_2(Y_1), \ldots, d\pi_2(Y_d_2))
\]
In particular,
\[
H\lambda(\pi) = H\lambda(\pi_1) \cap H\lambda(\pi_2)
\]
and
\[
H^\omega(\pi) = H^\omega(\pi_1) \cap H^\omega(\pi_2).
\]

**Proof.** By the previous theorem we obtain that \( S\lambda(\partial\pi(X_1), \ldots, \partial\pi(X_d_1), \partial\pi(Y_1), \ldots, \partial\pi(Y_d_2)) = S\lambda(\partial\pi(X_1), \ldots, \partial\pi(X_d_1)) \cap S\lambda(\partial\pi(Y_1), \ldots, \partial\pi(Y_d_2)) \). By an elementary counting argument, it follows that \( S\lambda(\partial\pi(X_1), \ldots, \partial\pi(X_d_1), \partial\pi(Y_1), \ldots, \partial\pi(Y_d_2)) = S\lambda(\partial\pi(Z_1), \ldots, \partial\pi(Z_d)) \). Since \( H^\omega(\pi) = H^\omega(\pi_1) \cap H^\omega(\pi_2) \) by Theorem 1, the corollary follows.

**Corollary 4** Let \( \pi \) be a representation of a solvable real Lie group \( G \). Let \( X_1, \ldots, X_d \) be a basis in the Lie algebra \( \mathfrak{g} \) of \( G \) such that \( L_i := \text{span}\{X_1, \ldots, X_i\} \) is a subalgebra of \( \mathfrak{g} \) and \( L_i \) is an ideal in \( L_{i+1} \) for all \( i \in \{1, \ldots, d\} \). Let \( \lambda \geq 1 \). Then
\[
S\lambda(d\pi(X_1), \ldots, d\pi(X_d)) = \bigcap_{k=1}^d S\lambda(d\pi(X_k)).
\]

### 3 Separate and joint Gevrey vectors

The following lemma in case \( \lambda = 1 \) can be found in [FS, Lemma 1], but there the proof is based on different arguments.

**Lemma 5** Let \( A \) be a Hermitian or skew-Hermitian operator in a Hilbert space which has an invariant domain \( D \). Let \( \lambda \geq 1 \). Let \( c, t > 0 \). Then
Proof. Let $s := 4't$. Let $u \in D$ and suppose that $\|A^{2m}u\| \leq ct^{2m}(2^m)!^\lambda$ for all $m \in \mathbb{N}_0$. For $n \in \mathbb{N}$ hypothesis $P(n)$ states

$$\|A^ku\| \leq cs^k k!^\lambda \quad \text{for all } k \in \{1, \ldots, n\}.$$  

Clearly hypotheses $P(1)$ and $P(2)$ are valid. Let $n \in \mathbb{N}$, $n \geq 2$ and suppose $P(n-1)$ is valid. If $2 \log n \in \mathbb{N}$ then hypothesis $P(n)$ holds. Suppose $2 \log n \not\in \mathbb{N}$. There exist unique $m, k \in \mathbb{N}_0$ such that $n = 2^m + k$ and $1 \leq k < 2^m$. Then $2k < 2^m + k = n$, hence $2k \leq n - 1$. So by assumption and hypothesis $P(n-1)$ we obtain:

$$\|A^n u\|^2 = |(A^{2^{m+1}}u, A^{2k}u)| \leq \|A^{2^{m+1}}u\| \cdot \|A^{2k}u\| \leq c^2 t^{2^{m+1}} s^2 (2^{m+1})!^\lambda (2k)!^\lambda$$

$$\leq c^2 t^{2^{m+1}} s^2 2^m 2^{m+1} (2^m)!^\lambda (2k)!^\lambda \leq c^2 t^{2^{m+1}} s^2 (2^m)^{(2^m+k)} (2^m+k)!^\lambda$$

$$\leq [cs^n n!^\lambda]^2.$$  

By induction, the lemma follows.  

For the case $\lambda = 1$ the following result has been stated in [FS, Theorem 1].

**Theorem 6** Let $G$ be a compact Lie group with Lie algebra $\mathfrak{g}$. Let $\lambda \geq 1$ and let $\pi$ be a representation of $G$ in a Hilbert space $H$. Let $X_1, \ldots, X_d$ be any basis in $\mathfrak{g}$. Then

$$S_\lambda(d\pi(X_1), \ldots, d\pi(X_d)) = \bigcap_{k=1}^d S_\lambda(d\pi(X_k)).$$

**Proof.** The compactness of $G$ insures that there exists a positive definite invariant real symmetric bilinear form $\beta$ on $\mathfrak{g} \times \mathfrak{g}$. (See [Hoc, Theorem XIII.1.1].) Let $Y_1, \ldots, Y_d$ be a basis in $\mathfrak{g}$ such that $\beta(Y_i, Y_j) = \delta_{ij}$ for all $i, j \in \{1, \ldots, d\}$. By [Nel, Lemma 6.1] there exists a constant $M_1 \geq 1$ such that for all $i, j \in \{1, \ldots, d\}$ and all $u \in H^\infty(\pi)$ we have

$$\|\partial \pi(\beta_{ij}) u\| \leq M_1 \|\partial \pi(I - \Delta_{ij}) u\|,$$

where $\Delta_{ij} := \sum_{k=1}^d X_k^2 \in U(\mathfrak{g})$. So there exists a constant $M \geq 1$ such that for all $u \in H^\infty(\pi)$:

$$\|\partial \pi(I - \Delta_{ij}) u\| \leq \frac{M}{d+1} \|\partial \pi(I - \Delta_{ij}) u\|,$$

where $\Delta_{ij} := \sum_{k=1}^d X_k^2 \in U(\mathfrak{g})$.

Now let $u \in \bigcap_{k=1}^d S_\lambda(d\pi(X_k))$. Then $u \in \bigcap_{k=1}^d D^\infty(d\pi(X_k)) = H^\infty(\pi)$. (See Theorem 1.) Since $G$ is compact, there exists an index set $I$ and for all $\alpha \in I$ there exists a $\pi$-invariant subspace $H_\alpha$ of $H$ such that $\pi_\alpha := \pi|_{H_\alpha}$ is irreducible and $H = \bigoplus_{\alpha \in I} H_\alpha$. Let $u_\alpha \in H_\alpha$ be the projection of $u$ on $H_\alpha$. Note that $H_\alpha \subset H^\infty(\pi)$.

Let $\alpha \in I$. By [Bou, Chapter I §3.7 Proposition 11], $\Delta_{ij}$ belongs to the center of $U(\mathfrak{g})$. Since $\pi_\alpha$ is irreducible, it follows by Schur's lemma that there exists $\delta_\alpha \in \mathbb{C}$ such that $\partial \pi_\alpha(\Delta_{ij}) = -\delta_\alpha I$. Because the operator $\partial \pi(\Delta_{ij})$ is negative, we obtain that $\delta_\alpha \geq 0$.

Then $(1 + \delta_\alpha)\|u_\alpha\| = \|\partial \pi_\alpha(I - \Delta_{ij}) u_\alpha\| = \|\partial \pi(I - \Delta_{ij}) u_\alpha\| \leq \frac{M}{d+1} \|\partial \pi(I - \Delta_{ij}) u_\alpha\| \leq \frac{M}{d+1} \sum_{k=0}^d \|\partial \pi(X_k)^2 u_\alpha\|$, where $X_0 := I \in \mathbb{C} \subset U(\mathfrak{g})$.

For all $m \in \mathbb{N}_0$ let hypothesis $P(m)$ state:
We have already proved hypothesis $P(0)$. Let $m \in \mathbb{N}_0$ and suppose $P(m)$ holds. Then by Hölders inequality:

\[(1 + \delta_{\alpha})^{2m+1} \| u_\alpha \| \| u_\alpha \| = \left[ (1 + \delta_{\alpha})^{2m} \| u_\alpha \| \right]^2 \leq \frac{1}{(d+1)^2} M^{2m+1} \left[ \sum_{k=0}^{d} \| \partial \pi(X_k)^{2m+1} u_\alpha \| \right]^2 \leq \frac{1}{d+1} M^{2m+1} \sum_{k=0}^{d} \| \partial \pi(X_k)^{2m+1} u_\alpha \|^2 \]

\[= \frac{1}{d+1} M^{2m+1} \sum_{k=0}^{d} \| \partial \pi(X_k) \|^{2m+2} u_\alpha u_\alpha \| \leq \frac{1}{d+1} M^{2m+1} \sum_{k=0}^{d} \| \partial \pi(X_k)^{2m+2} u_\alpha \| \| u_\alpha \| \cdot \]

So $P(m)$ is valid for all $m \in \mathbb{N}_0$.

Since $u \in \cap_{k=1}^{d} S_\lambda^1(d\pi(X_k))$, there exist $c, t > 0$ such that for all $k \in \{1, \ldots, d\}$ and all $n \in \mathbb{N}_0$: $\| d\pi(X_k)^n u \| \leq c t^n n!$. Let $m \in \mathbb{N}_0$. Then

\[\| \partial \pi(I - \Delta_Y)^{2m} u \|^2 = \sum_{\alpha \in I} \| \partial \pi(I - \Delta_Y)^{2m} u_\alpha \|^2 = \sum_{\alpha \in I} \left[ (1 + \delta_{\alpha})^{2m} \| u_\alpha \| \right]^2 \leq \sum_{\alpha \in I} \left[ \frac{1}{d+1} M^{2m} \sum_{k=0}^{d} \| \partial \pi(X_k)^{2m+1} u_\alpha \| \right]^2 \leq \frac{1}{d+1} \sum_{\alpha \in I} \sum_{k=0}^{d} \left[ M^{2m} \| \partial \pi(X_k)^{2m+1} u_\alpha \| \right]^2 \leq \frac{1}{d+1} \sum_{k=0}^{d} \left[ M^{2m} \| \partial \pi(X_k)^{2m+1} u \| \right]^2 \leq \left[ c \left( 2^{2\lambda} M t^2 \right)^{2m} \left( 2^{2m} \right)^{2\lambda} \right].\]

So

\[\| \partial \pi(I - \Delta_Y)^{2m} u \| \leq c \left( 2^{2\lambda} M t^2 \right)^{2m} \left( 2^{2m} \right)^{2\lambda}\]

for all $m \in \mathbb{N}_0$. Hence $u \in S_{2\lambda}(\partial \pi(I - \Delta_Y))$ by Lemma 5. Since $\lambda \geq 1$ we can deduce that $u \in H_\lambda(\pi)$ from [GW] Example following Theorem 1.7.

We arrive at the main theorem of this section.

**Theorem 7** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Then there exists a basis $X_1, \ldots, X_d$ in $\mathfrak{g}$ such that for all $\lambda \geq 1$ and all representations $\pi$ of $G$ we have

\[S_\lambda(d\pi(X_1), \ldots, d\pi(X_d)) = \bigcap_{k=1}^{d} S_\lambda(d\pi(X_k)).\]
Proof. We prove the theorem by induction to \( \dim \mathfrak{g} \). If \( \dim \mathfrak{g} = 1 \) then nothing has to be proved. For \( d \in \mathbb{N} \) let hypothesis \( P(d) \) state:

For any Lie group \( G \) with Lie algebra \( \mathfrak{g} \) and \( d_1 := \dim \mathfrak{g} \leq d \) there exists a basis \( X_1, \ldots, X_{d_1} \) in \( \mathfrak{g} \) such that for all \( \lambda \geq 1 \) and all representations \( \pi \) of \( G \) we have

\[
S_{\lambda}(d\pi(X_1), \ldots, d\pi(X_{d_1})) = \bigcap_{k=1}^{d_1} S_{\lambda}(d\pi(X_k)).
\]

Let \( d \in \mathbb{N}, d \geq 2 \) and suppose hypothesis \( P(d - 1) \) is valid. Let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \). Suppose \( \dim \mathfrak{g} = d \). We shall prove:

Assertion 1: There exists a basis \( X_1, \ldots, X_d \) in \( \mathfrak{g} \) such that for all \( \lambda \geq 1 \) and all representations \( \pi \) of \( G \) we have

\[
S_{\lambda}(d\pi(X_1), \ldots, d\pi(X_d)) = \bigcap_{k=1}^{d} S_{\lambda}(d\pi(X_k)).
\]

First we prove the following assertion:

Assertion 2: Let \( \mathfrak{g}_1, \mathfrak{g}_2 \) be subalgebras of \( \mathfrak{g} \) such that \( \mathfrak{g} \) is the direct sum of \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \).

Suppose \( \dim \mathfrak{g}_1 \geq 1 \) and \( \dim \mathfrak{g}_2 \geq 1 \). Then Assertion 1 holds.

Proof of Assertion 2. Let \( G_1 \) and \( G_2 \) be subgroups of \( G \) which have Lie algebras \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) respectively. By induction hypothesis \( P(d - 1) \) there exist a basis \( X_1, \ldots, X_{d_1} \) in \( \mathfrak{g}_1 \) and a basis \( Y_1, \ldots, Y_{d_2} \) in \( \mathfrak{g}_2 \) such that for every representation \( \pi_1 \) of \( G_1 \) and for every representation \( \pi_2 \) of \( G_2 \) and all \( \lambda \geq 1 \) we have

\[
S_{\lambda}(d\pi_1(X_1), \ldots, d\pi_1(X_{d_1})) = \bigcap_{k=1}^{d_1} S_{\lambda}(d\pi_1(X_k))
\]

and

\[
S_{\lambda}(d\pi_2(Y_1), \ldots, d\pi_2(Y_{d_2})) = \bigcap_{k=1}^{d_2} S_{\lambda}(d\pi_2(Y_k)).
\]

Then \( X_1, \ldots, X_{d_1}, Y_1, \ldots, Y_{d_2} \) is a basis in \( \mathfrak{g} \).

Now let \( \pi \) be a representation of \( G \) in a Hilbert space \( H \) and let \( \lambda \geq 1 \). Let \( \pi_1 \) and \( \pi_2 \) be the restrictions of \( \pi \) to \( G_1 \) and \( G_2 \) respectively. Then \( d\pi(X_k) = d\pi_1(X_k) \) for all \( k \in \{1, \ldots, d_1\} \) and \( d\pi(Y_k) = d\pi_2(Y_k) \) for all \( k \in \{1, \ldots, d_2\} \). So by Corollary 3 we obtain that

\[
S_{\lambda}(d\pi(X_1), \ldots, d\pi(X_d)) = \bigcap_{k=1}^{d_1} S_{\lambda}(d\pi_1(X_k)) \cap \bigcap_{k=1}^{d_2} S_{\lambda}(d\pi_2(Y_k)).
\]
This proves Assertion 2.

Now we prove Assertion 1. If $\mathfrak{g}$ is solvable, then Assertion 1 follows by Corollary 4. So we may assume that $\mathfrak{g}$ is not solvable. Let $\mathfrak{q}$ be the radical of $\mathfrak{g}$. By [Var1, Theorem 3.14.1] there exists a semisimple subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g}$ is the direct sum of $\mathfrak{q}$ and $\mathfrak{m}$. (This is a Levi decomposition of $\mathfrak{g}$.) If $\dim \mathfrak{q} \geq 1$, then Assertion 1 follows by Assertion 2.

So we may assume that $\dim \mathfrak{q} = 0$. Then $\mathfrak{g} = \mathfrak{m}$ is semisimple. Let $\mathfrak{g} = \mathfrak{t} + \mathfrak{a} + \mathfrak{n}$ be an Iwasawa decomposition of $\mathfrak{g}$. (See [Hel, Theorem VI.3.4].) Let $\mathfrak{s} := \mathfrak{a} + \mathfrak{n}$. Then $\mathfrak{t}$ and $\mathfrak{s}$ are subalgebras of $\mathfrak{g}$, $\mathfrak{s}$ is solvable and $\mathfrak{g}$ is the direct sum of $\mathfrak{t}$ and $\mathfrak{s}$. Since $\mathfrak{g}$ is semisimple, always $\dim \mathfrak{t} \geq 1$. In case $\dim \mathfrak{s} \geq 1$, Assertion 1 follows again by Assertion 2.

So we may assume that $\dim \mathfrak{s} = 0$. Then $\mathfrak{g} = \mathfrak{t}$. So the Lie algebra $\mathfrak{g}$ is compact.

But the Lie group $G$ need not be compact and we cannot immediately apply Theorem 6. Corresponding to the Lie algebra $\mathfrak{g}$ there exists a connected simply connected Lie group $G_1$ with Lie algebra $\mathfrak{g}$. (See [Var1, Theorem 3.15.1].) Then $G_1$ is compact by [Wal, Theorem 3.6.6]. Let $X_1, \ldots, X_d$ be any basis in $\mathfrak{g}$. Let $\lambda \geq 1$ and let $\pi$ be a representation of $G$ in a Hilbert space $H$. Now $X \mapsto \partial \pi(X)$ is a representation of the Lie algebra $\mathfrak{g}$ by skew-symmetric operators in $H$ and the operator $\partial \pi(X_1)^2 + \ldots + \partial \pi(X_d)^2$ is essentially self-adjoint. (See [Hel, Theorem 6].) So by [Hel, Corollary 9.1] there exists a representation $\sigma$ of $G_1$ such that $d\sigma(X) = d\pi(X)$ for all $X \in \mathfrak{g}$. Therefore we obtain by Theorem 6 that

$$S_\lambda(d\pi(X_1), \ldots, d\pi(X_d)) = S_\lambda(d\sigma(X_1), \ldots, d\sigma(X_d))$$

$$= \bigcap_{k=1}^d S_\lambda(d\sigma(X_k)) = \bigcap_{k=1}^d S_\lambda(d\pi(X_k)).$$

This proves the theorem.

The following corollary is a kind of Hartogs’ theorem.

**Corollary 8** Let $\pi$ be a representation of a Lie group $G$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and let $\lambda \geq 1$. Then

$$H_\lambda(\pi) = \bigcap_{G_1 \text{ subgroup of } G \atop \dim G_1 = 1} H_\lambda(\pi|_{G_1}) = \bigcap_{X \in \mathfrak{g}} S_\lambda(d\pi(X)).$$

4 Gevrey vectors and positive definite functions

Let $\pi$ be a representation of a Lie group $G$ in a Hilbert space $H$. For $u \in H$ define $p_u : G \to \mathbb{C}$ by $p_u(g) := (\pi_g u, u)$ ($g \in G$). Now we present a description of the infinitely differentiable vectors for $\pi$ and the Gevrey vectors of order $\lambda$ for $\pi$ in terms of the positive definite functions $p_u$, $u \in H$. More precisely, we prove that

$$H^\infty(\pi) = \{ u \in H : \text{the function } p_u \text{ is infinitely differentiable on } G \}$$

and

$$H_\lambda(\pi) = \{ u \in H : \text{the function } p_u \text{ is a Gevrey function of order } \lambda \text{ on } G \}$$
for all \( \lambda \geq 1 \). We need a well-known lemma.

**Lemma 9** Let \( A \) be a self-adjoint operator in a Hilbert space \( H \). Let \( u \in H \). Define \( F: IR \rightarrow C \) by

\[
F(t) := (e^{itA}u, u) \quad (t \in IR).
\]

Let \( V \) be an open neighborhood of \( 0 \). Suppose the restriction \( F|_V \) is infinitely differentiable. Then \( u \in D^\infty(A) \). Moreover, for all \( n \in IN_0 \) we have \( F^{(2n)}(0) = (-1)^n \|A^n u\|^2 \).

**Proof.** We may assume that \( A \) is the multiplication operator by the function \( h \) in the Hilbert space \( H = L^2(Y, m) \) for some measure space \( (Y, B, m) \). Let \( f : IR \rightarrow IR \) be defined by \( f(t) := \text{Re} F(t) = \int \cos(th)|u|^2 dm, t \in IR \). Since \( f \) is an even function and infinitely differentiable on \( V \), we obtain that \( f'(0) = 0 \). Then by Fatou's lemma:

\[
\int h^2|u|^2 dm = 2 \int \liminf_{n \rightarrow \infty} n^2 \left( 1 - \cos(\frac{1}{n}h) \right) |u|^2 dm \\
\leq 2 \liminf_{n \rightarrow \infty} \int n^2 \left( 1 - \cos(\frac{1}{n}h) \right) |u|^2 dm \\
= -2 \liminf_{n \rightarrow \infty} n^2 \left( f(\frac{1}{n}) - f(0) - \frac{1}{n}f'(0) \right) \\
= -f''(0) < \infty.
\]

So \( u \in D(A) \) and \( f''(0) = -\|Au\|^2 \). Hence by Lebesque's theorem on dominated convergence we obtain that \( F \) is twice differentiable on \( V \) and \( F''(t) = -(e^{itA}Au, Au) \) for all \( t \in V \). By induction, the lemma follows. \( \square \)

**Theorem 10** Let \( \pi \) be a representation of a Lie group \( G \) in a Hilbert space \( H \). Let \( \lambda \geq 1 \). Then

\[
H^\infty(\pi) = \{ u \in H : p_u \in C^\infty(G) \}
\]

and

\[
H_\lambda(\pi) = \{ u \in H : \text{the function } p_u \text{ is a Gevrey function of order } \lambda \text{ on } G \}
\]

In particular,

\[
H^\omega(\pi) = \{ u \in H : \text{the function } p_u \text{ from } G \text{ into } C \text{ is real analytic} \}.
\]

**Proof.** Let \( u \in H \). Suppose \( p_u \in C^\infty(G) \). Let \( X_1, \ldots, X_d \) be a basis in the Lie algebra \( g \) of \( G \). Let \( k \in \{1, \ldots, d\} \). Then the function \( t \mapsto (e^{it\pi(X_k)}u, u) = p_u(\exp tX_k) \) from \( IR \) into \( C \) is infinitely differentiable, so by Lemma 9 we obtain that \( u \in D^\infty(d\pi(X_k)) \). Therefore, \( u \in \bigcap_{k=1}^d D^\infty(d\pi(X_k)) = H^\infty(\pi) \) by Theorem 1.

Now let \( \lambda \geq 1 \), let \( u \in H \) and suppose the function \( p_u \) is a Gevrey function of order \( \lambda \) on \( G \). Then \( p_u \in C^\infty(G) \), so by the previous part we obtain that \( u \in H^\infty(\pi) \). Let \( K \) be a compact neighborhood of the identity \( e \) in \( G \). Let \( X \in g \). Then by definition there exist \( c, t > 0 \) such that for all \( n \in IN_0 \) and all \( g \in K \) we have \( \|[\tilde{X}^n p_u](g)\| \leq C t^n n^\lambda \), where \( \tilde{X} \) denotes the
corresponding left invariant vector field on \( G \). Define \( F : \mathbb{R} \to \mathbb{C} \) by \( F(t) := (e^{td\pi(X)}u, u) \), \( t \in \mathbb{R} \). Then by Lemma 9 we obtain for all \( n \in \mathbb{N}_0 \):

\[
||[d\pi(X)]^n u||^2 = |F^{(2n)}(0)| = \left| \left( \frac{d}{dt} \right)^{2n} p_u(\exp tX) \right|_{t=0} \\
= ||\tilde{X}^{2n} p_u(e)|| \leq Ct^{2n}(2n)!^{2\lambda} \leq C(2t)^{2n}n!^{2\lambda}.
\]

Hence \( u \in S_{\lambda}(d\pi(X)) \). Therefore \( u \in \bigcap_{X \in G} S_{\lambda}(d\pi(X)) = H_{\lambda}(\pi) \) by Corollary 8.

A vector \( u \in H \) is called weakly analytic if for all \( v \in H \) the function \( g \mapsto (\pi_g u, v) \) is an analytic function from \( G \) into \( \mathbb{C} \).

**Corollary 11** Let \( \pi \) be a representation of Lie group in a Hilbert space \( H \). Let \( u \in H \). Then \( u \) is weakly analytic if and only if \( u \) is analytic.

This corollary has been proved before in [Var2, page 303], by using the Baire category theorem.

**References**


