On the distribution of θ modulo 1

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ON THE DISTRIBUTION OF $n\theta$ MODULO 1

R. L. GRAHAM AND J. H. VAN LINT

Introduction. In recent work of E. Arthurs and L. A. Shepp on a problem of H. Dym concerning the existence of an ergodic stationary stochastic process with zero entropy (cf. **1**), the function $d_\theta(n)$ was introduced as follows:

For an irrational number θ , let

$$0 = a_0 < a_1 < a_2 < \dots < a_n < a_{n+1} = 1$$

be the sequence of points $\{l\theta\}$, $1 \leq l \leq n$, (where $\{x\}$ denotes $x - [x]$, the fractional part of x) and define*

$$d_\theta(n) = \max(a_i - a_{i-1}), \quad 1 \leq i \leq n+1.$$

It is our purpose in this paper to establish several asymptotic results for $d_\theta(n)$. In particular, we prove that

$$\sup_{\theta} \liminf_{n \rightarrow \infty} n d_\theta(n) = \frac{1 + \sqrt{2}}{2}$$

and

$$\inf_{\theta} \limsup_{n \rightarrow \infty} n d_\theta(n) = 1 + \frac{2\sqrt{5}}{5}$$

(cf. Theorems 1 and 2).

Notation. We consider an irrational number θ . $[b_0, b_1, b_2, \dots]$ is the simple continued fraction expansion of θ , i.e.,

$$\theta = b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \dots}}$$

The convergents h_n/k_n of θ satisfy (cf. **3**)

$$h_{-1} = 1, \quad h_0 = b_0, \quad h_i = b_i h_{i-1} + h_{i-2}, \quad i \geq 1,$$

$$k_{-1} = 0, \quad k_0 = 1, \quad k_i = b_i k_{i-1} + k_{i-2}, \quad i \geq 1.$$

We define θ_n by

$$\theta_0 = \theta, \quad \theta_{i+1} = 1/(\theta_i - [\theta_i]), \quad i \geq 0.$$

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*It should be noted that the related function $d'_\theta(n) = \min_{1 \leq i \leq n+1} (a_i - a_{i-1})$ has been extensively studied by Sós, Halton, and others (cf. **2**; **4**; **5**; **6**; **7**; **8**; and **9**).

We have (cf. 3)

$$b_n = [\theta_n]$$

and

$$\theta - \frac{h_n}{k_n} = \frac{(-1)^n}{k_n(k_n\theta_{n+1} + k_{n-1})}.$$

Finally, we define x_n and y_n by

$$x_n = k_{n+1}/k_n, \quad y_n = 1/\theta_{n+2}.$$

We then have

$$x_{n+1} = [1/y_n] + 1/x_n, \quad y_{n+1} = -[1/y_n] + 1/y_n, \quad n \geq 0.$$

It follows easily from the definitions that

$$(1) \quad x_n = [b_{n+1}, b_n, \dots, b_1]$$

and

$$(2) \quad y_n = 1/[b_{n+2}, b_{n+3}, b_{n+4}, \dots].$$

We shall use the following basic lemma.

LEMMA 1.

$$d_\theta(m) = |k_n\theta - h_n + \alpha(k_{n+1}\theta - h_{n+1})|$$

if

$$k_n + (\alpha + 1)k_{n+1} - 1 \leq m \leq k_n + (\alpha + 2)k_{n+1} - 2$$

and $0 \leq \alpha \leq b_{n+2} - 1$.

The proof of this result appears implicitly in (5) and (7) and will not be given here. It depends upon the somewhat surprising and apparently little-known fact that the set of numbers $\{a_{i+1} - a_i: 0 \leq i \leq n\}$ (using the notation in § 1) always consists of at most three elements.

We are now prepared to prove the statements given in the introduction.

The main results.

THEOREM 1.

$$\sup_{\theta} \liminf_{n \rightarrow \infty} nd_\theta(n) = \frac{1 + \sqrt{2}}{2}.$$

Proof. We observe that, for $n \rightarrow \infty$,

$$\begin{aligned} \liminf nd_\theta(n) &\leq \liminf (k_n + k_{n+1})|k_n\theta - h_n| \\ &= \liminf (1 + x_n)(y_n + x_n)^{-1}. \end{aligned}$$

We first show that

$$(3) \quad \liminf (1 + x_n)(y_n + x_n)^{-1} \leq \frac{1}{2}(1 + \sqrt{2}), \quad n \rightarrow \infty.$$

Equivalently, we must show that

$$(4) \quad \limsup (y_n + x_n)(1 + x_n)^{-1} \geq 2(1 + \sqrt{2})^{-1} = 2(\sqrt{2} - 1), \quad n \rightarrow \infty.$$

We prove (4) by establishing

LEMMA 2. Let $\theta = [b_0, b_1, b_2, \dots]$, where $b_n \leq M$ for all n . Then

$$\limsup x_n y_n \geq 1, \quad n \rightarrow \infty,$$

with equality if and only if the b_n are eventually constant.

Proof. By (1) and (2) we have

$$\begin{aligned} x_n y_n &= [b_{n+1}, b_n, \dots, b_1] / [b_{n+2}, b_{n+3}, \dots] \\ &> (b_{n+1} + (M + 1)^{-1})(b_{n+2} + 1)^{-1}. \end{aligned}$$

(i) If $b_{n+1} > b_{n+2}$ infinitely often, then $b_{n+1} \geq 1 + b_{n+2}$ infinitely often and hence, for $n \rightarrow \infty$,

$$\begin{aligned} \limsup x_n y_n &\geq \limsup (b_{n+1} + (M + 1)^{-1})(b_{n+2} + 1)^{-1} \\ &\geq \limsup (b_{n+2} + 1 + (M + 1)^{-1})(b_{n+2} + 1)^{-1} \\ &\geq 1 + (M + 1)^{-2} > 1. \end{aligned}$$

(ii) If $b_{n+1} > b_{n+2}$ for just a finite number of values of n , then there is an N such that $b_m = N$ for all sufficiently large m . Hence, if $\alpha = [N, N, N, \dots]$ then

$$\lim x_n y_n = (\lim x_n)(\lim y_n) = \alpha \cdot (1/\alpha) = 1, \quad n \rightarrow \infty.$$

This proves Lemma 2.

It follows that, for any $\epsilon > 0$, infinitely many of the pairs (x_n, y_n) lie in the hyperbolic region given by $x \geq 0$ and $xy \geq 1 - \epsilon$. We observe that this region is contained in that defined by $x \geq 0$ and $(y + x)/(1 + x) \geq 2(\sqrt{2} - 1) - \epsilon$, since the last boundary line passes below the hyperbola, for all sufficiently small ϵ ; and (4) now follows. Thus (4) holds in case the b_n are bounded. On the other hand, if the b_n are unbounded, then the x_n are unbounded and

$$\limsup (y_n + x_n)(1 + x_n)^{-1} \geq \limsup x_n(1 + x_n)^{-1} = 1 > 2(\sqrt{2} - 1), \quad n \rightarrow \infty.$$

This proves (4).

Finally, suppose that $\theta = 1 + \sqrt{2}$. Then $b_n = 2$ for $n = 0, 1, 2, \dots$. The relations for h_n and k_n can be solved to give

$$\begin{aligned} h_n &= (2\sqrt{2})^{-1}[(1 + \sqrt{2})^{n+2} - (1 - \sqrt{2})^{n+2}], \\ k_n &= (2\sqrt{2})^{-1}[(1 + \sqrt{2})^{n+1} - (1 - \sqrt{2})^{n+1}], \end{aligned}$$

and all $\theta_n = 1 + \sqrt{2}$. Hence, we have

$$k_n \theta - h_n = (-1)^n (\sqrt{2} - 1)^{n+1}.$$

By Lemma 1,

$$d_\theta(m) = (\sqrt{2} - 1)^{n+1} [1 - \alpha(\sqrt{2} - 1)]$$

if $m = k_n + (\alpha + 1)k_{n+1} + c$, where $0 \leq \alpha \leq b_{n+2} - 1$ and $-1 \leq c \leq k_{n+1} - 2$; that is, if, for large n ,

$$m \sim (2\sqrt{2})^{-1}(1 + \sqrt{2})^{n+1}[1 + (\alpha + 1)(1 + \sqrt{2})],$$

where $\alpha = 0$ or 1 and $-1 \leq c < (2\sqrt{2})^{-1}(1 + \sqrt{2})^{n+1} - 1$. When $m \rightarrow \infty$, $n \rightarrow \infty$; therefore

$$\begin{aligned} \liminf_{m \rightarrow \infty} md_\theta(m) &= \inf_{\alpha} (2\sqrt{2})^{-1}[1 - \alpha(\sqrt{2} - 1)][1 + (\alpha + 1)(1 + \sqrt{2})] \\ &= \inf_{\alpha} (2\sqrt{2})^{-1}[2 + \sqrt{2} + \alpha - \alpha^2] = \frac{1 + \sqrt{2}}{2}. \end{aligned}$$

This completes the proof of the theorem.

THEOREM 2.

$$\inf_{\theta} \limsup_{n \rightarrow \infty} nd_\theta(n) = 1 + \frac{2\sqrt{5}}{5}.$$

Proof. By Lemma 1, it is sufficient to prove

$$(5) \quad \begin{aligned} \limsup (k_n + 2k_{n+1})|k_n\theta - h_n| &= \limsup (1 + 2x_n)(y_n + x_n)^{-1} \\ &\geq 1 + 2\sqrt{5}/5, \quad n \rightarrow \infty, \end{aligned}$$

in order to show that

$$\limsup nd_\theta(n) \geq 1 + 2\sqrt{5}/5, \quad n \rightarrow \infty.$$

If $y_n \leq \frac{1}{2}$ infinitely often, then

$$(1 + 2x_n)(y_n + x_n)^{-1} \geq 2$$

infinitely often and we have

$$\limsup (1 + 2x_n)(y_n + x_n)^{-1} \geq 2, \quad n \rightarrow \infty.$$

If $y_n > \frac{1}{2}$ for all sufficiently large n , then $b_n = 1$ for all sufficiently large n . Hence, as $n \rightarrow \infty$,

$$\lim x_n = [1, 1, 1, \dots] = (1 + \sqrt{5})/2, \quad \lim y_n = (-1 + \sqrt{5})/2$$

and

$$\lim (1 + 2x_n)(y_n + x_n)^{-1} = 1 + 2\sqrt{5}/5.$$

This proves (5). An easy calculation shows that

$$\lim nd_\theta(n) = 1 + 2\sqrt{5}/5, \quad n \rightarrow \infty,$$

for $\theta = (1 + \sqrt{5})/2$, and Theorem 2 is proved.

We note that if

$$k_n + k_{n+1} - 1 \leq m \leq k_n + (\alpha + 2)k_{n+1} - 2,$$

where $\alpha = b_{n+2} - 1$, we have

$$\max_m md_\theta(m) = \max_{0 \leq \mu \leq b_{n+2}-1} (1 + (\mu + 2)x_n)(1 - \mu y_n)(x_n + y_n)^{-1}.$$

We conclude with

THEOREM 3.

$$\limsup nd_{\theta}(n) = \infty \Leftrightarrow \limsup b_n = \infty, \quad n \rightarrow \infty.$$

Proof. (i) If $\limsup_{n \rightarrow \infty} b_n = \infty$, then $\liminf_{n \rightarrow \infty} y_n = 0$. If y_n is sufficiently small, then we can take $\mu = [1/2y_n] - 1$ (since this is less than $b_{n+2} - 1$) and we find

$$(1 + (\mu + 2)x_n)(1 - \mu y_n)(x_n + y_n)^{-1} \geq x_n(2y_n)^{-1}(\frac{1}{2})(x_n + 1)^{-1} \rightarrow \infty$$

for a subsequence of y_n which tends to 0.

(ii) If $\limsup_{n \rightarrow \infty} nd_{\theta}(n) = \infty$, then certainly

$$\limsup(1 + (\mu^* + 2)x_n)(1 - \mu^* y_n)(x_n + y_n)^{-1} = \infty, \quad n \rightarrow \infty,$$

where

$$\mu^* = (2y_n)^{-1} - (2x_n)^{-1} - 1$$

(the expression considered is a quadratic form in μ with a maximum for $\mu = \mu^*$). Hence, as $n \rightarrow \infty$,

$$\limsup 2^{-1}[(x_n + y_n)(2x_n y_n)^{-1} + 1][1 + 2x_n y_n(x_n + y_n)^{-1}] = \infty$$

and this implies $\liminf_{n \rightarrow \infty} y_n = 0$, i.e., $\limsup_{n \rightarrow \infty} b_n = \infty$ and the theorem is proved.

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