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Structure development during chaotic mixing in the journal bearing flow

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Laminar mixing in the two-dimensional time-periodic Stokes flows between eccentric cylinders [journal bearing flow (JBF)] is studied using the extended mapping method [Galaktionov *et al.*, *Int. J. Multiphase Flows* **28**, 497 (2002)] with the emphasis on the material stretching, e.g., the interface generation abilities, of the flow. With this flexible and *computational advantageous* method both, the macroscopic material transport and the evolution of the microstructure can be described. It enables a convenient way for studying the material stretching in the flow and moreover, it provides spatial distribution of locally averaged stretching values instead of pointwise statistics, which was typical for previous studies [Liu *et al.*, *AIChE J.* **40**, 1273 (1994); Muzzio *et al.*, *Phys. Fluids A* **3**, 822 (1991)]. The results clearly indicate how the total amount of stretching generated by the flow depends on the parameters of the flow protocol, and that this is not just proportional to the work done on the system, as was suggested earlier in Muzzio *et al.*, *Phys. Fluids A* **3**, 822 (1991). It was found that when self-similar patterns are established, distinctive zones in the flow, which we call “microstructural demixing zones,” are observed, where interfaces are contracted during a typical period of the mixing process. Spatial nonuniformity of stretching in chaotic flows calls for additional mixing measures that reflect the nonuniformity of self-similar stretching patterns, created by time-periodic mixing flows. © 2002 American Institute of Physics. [DOI: 10.1063/1.1494810]

I. INTRODUCTION

Laminar mixing of viscous fluids is extensively studied due to widespread occurrence of such processes in nature and their importance in industrial applications. Although important advances in the understanding of the basic mechanisms of laminar mixing (see, for example Refs. 1 and 2) were achieved, the work on a general framework that would allow for quantifying (the dynamics of) mixture quality and mixing abilities of different flows is still not completed and continues. For example, the coexistence of chaotic and non-chaotic regions in mixing flows present significant difficulties for analysis, as earlier observed by Liu *et al.*³ Chaotic trajectories of fluid particles restrict the capability of explicit tracking of deforming fluid volumes (see, for example, Ref. 4 and references therein), while purely statistical approaches are not very suited for studying structure development in systems that have regular (nonchaotic) zones³ co-existing with chaotic regions.

Well-defined time-periodic prototype flows present useful models to study the mechanisms and typical features of laminar mixing. For this type of flows many of these problems are overcome by using the mapping method. This versatile method, similar to the approach proposed by Welander⁵ and Spencer and Wiley,⁶ was first used for studying concentration distributions, Kruijt *et al.*⁷ and Galaktionov *et al.*,⁸ and recently extended to stretching distributions in mixing flows by Galaktionov *et al.*,⁹ enabling the analysis of the dynamics of the microstructure and, for the

first time, optimizing mixing procedures. The flow domain is divided into (a large number of) subdomains and for a given quantity (for example, concentration) the transport from one subdomain to others after a characteristic period of time is calculated. This information is stored in a matrix which is then repetitively used to simulate a mixing process.

Muzzio *et al.*¹⁰ investigated the stretching statistics in the journal bearing flow (JBF). It was observed that the amount of stretching experienced by the fluid elements is highly non-uniform and that the distribution of stretching values quickly *becomes self-similar*. The stretching statistics was computed in a pointwise manner. A large ensemble of material points was distributed in the flow domain and the evolution of the deformation tensor $\mathbf{F}(t)$, associated with every material point advected in the flow, was determined numerically. Next, the stretching was obtained as an average length increase over the randomly oriented initial vectors. The stretching statistics show typical features of laminar mixing in periodic flows, i.e., highly irregular stretching distributions, self-similar behavior, a fast onset of asymptotic orientation patterns, etc. The stretching statistics computed in a pointwise manner provides important information about the stretching abilities of the chaotic flow, but this information may be incomplete. The stretching analysis can be extended if locally averaged values of stretching are considered, thus, representing the whole flow domain instead of large but finite number of material points (see the discussion in the Sec. V).

Muzzio *et al.*¹¹ pointed to the close relation of the total amount of stretching experienced by a material point and the amount of intermaterial area (interface) created if an affinely

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FIG. 1. (a) Schematic drawing of the flow domain (see text for comments). (b) The streamline pattern, created by the rotation of inner cylinder, while the outer one is fixed. (c) The streamline pattern, created by the rotation of outer cylinder.

stretching microscopic blob is placed at this location. The stretching distribution is used to characterize the rate of generation of the material interface and, consequently, the development of the mixture microstructure. Thus, the stretching analysis describes the microstructure development in an indirect way. The work of Muzzio *et al.*¹² demonstrates an essentially reverse approach: the initial stage of the development of microstructure is studied by explicitly tracking the intermaterial interfaces. These data are then translated into a coarse grain density of the interface. This coarse grain “intermaterial area density”¹² is then used to investigate the stretching statistics.

The main goal of this paper is to study the stretching and structure development in the JBF. The *extended mapping technique*⁹ is applied. It enables the direct simulation of the microstructure development as well as macroscopic transport in the flow in a computationally advantageous way. Within this method the microstructure is described statistically using the coarse grain quantity called “area tensor”¹³ that contains information on the volumetric density of the interfaces and their orientation. This provides the opportunity to study intermaterial area growth, taking as initial condition the description of the mixture containing, for example, uniformly distributed microscopic drops (isotropic orientation of interfaces). Then, the volume-averaged stretching statistics can be extracted from these results.

II. THE JOURNAL BEARING FLOW

We consider a time-periodic Stokes flow ($Re \ll 1$) of a viscous incompressible Newtonian fluid in the gap between two eccentric cylinders. The flow domain is shown in Fig. 1(a). This is a well-known, experimentally realizable, chaotic prototype flow extensively used for studying laminar mixing mechanisms (see, for example, Refs. 1, 11, and 10). The flow between eccentric cylinders was already considered in classical works on lubrication theory,^{14,15} because it describes the idealized flow of a lubricating fluid between a rotating journal and its cylindrical support [journal bearing flow (JBF)]. In the early works approximate solutions for the velocity field were used. Later, the availability of the closed analytical expressions for the velocity field¹⁶ enabled accurate simulations of fluid motion. A chaotic flow can be generated easily

by rotating the cylinders in a time-periodic fashion, which makes the JBF a convenient prototype flow for studying mixing phenomena.

In the current work we use the same family of periodic flow protocols as was studied by Muzzio *et al.*¹⁰ The results on stretching statistics provided by different approaches are compared and extended. The geometry of the flow domain is characterized by two dimensionless parameters: the ratio of the radii of the inner and outer cylinder, r_{in}/r_{out} , and the dimensionless eccentricity $e = d/r_{out}$, where d is the distance between the centers of the two cylinders [see Fig. 1(a)]. The periodic flow is induced by a discontinuous two-step protocol: during the first half-period the outer cylinder is rotated, while the inner remains stationary; during the second half-period the outer cylinder is fixed and the inner one is rotating. The streamline patterns created by the rotation of the inner and outer cylinder are shown in Figs. 1(b) and 1(c), respectively. To clearly show the structure of the flow, two different families of equidistant isolines of the stream function are used, including the separatrix that delimits the vortex and the main flow. In case of a Stokes flow the result of the fluid motion is completely defined by the rotation angles of the cylinders. Thus, the flow protocol can be described by two dimensionless parameters: rotation angle θ of the outer cylinder and the ratio Ω of the rotation angles of both cylinders. Following Muzzio *et al.*¹⁰ we fix the geometric parameters to $r_{in}/r_{out} = 1/3$, $e = 0.3$ and the ratio of the rotation angles is kept $\Omega = 3.0$. The angle θ takes a sequence of fixed values that includes those considered in Ref. 10, but we expand to a twice larger maximum rotation angle: $\theta = 8\pi$.

As it was shown in Ref. 10, depending on the rotation angle θ , this protocol can generate completely different chaotic flows with or without large islands (zones of bad mixing). Figure 2 shows Poincaré maps for some typical cases ($\theta = 0.5\pi, \pi, 2\pi$). The Poincaré maps are obtained by tracking a small number of markers in the flow and marking their position after each period of the flow. This dynamical systems technique reveals zones of regular and chaotic fluid motion. Using the algorithm that exploits the symmetry properties of the particle trajectories during each half-period¹⁷ some first-order periodic points were located and classified. Unstable (hyperbolic) first-order periodic points are plotted on the appropriate Poincaré maps as filled

FIG. 2. The examples of Poincaré maps for the flows with different values of θ : (a) $\theta=0.5\pi$; (b) $\theta=1.0\pi$; (c) $\theta=2.0\pi$. The location of some first-order hyperbolic periodic points is shown by black dots. Numbers indicate the corresponding maximum stretching values.

circular markers. The numbers, shown for each point, are the largest absolute values of the eigenvalues of the deformation tensor associated with that periodic point. They describe the largest possible stretching after one period of the flow of an infinitesimal vector located at that point. These eigenvalues were estimated in an attempt to see a correlation between the maximum stretching in the vicinity of hyperbolic points and average stretching in the whole domain, assuming that the chaotic mixing is controlled by unstable manifolds of these points. Such a correlation was not observed: apparently, the high stretching values in the vicinity of a hyperbolic point do not necessarily mean a high material stretching efficiency of the flow as a whole.

The flow with $\theta=0.5\pi$ [Fig. 2(a)] contains a large number of islands of different order and a wide ring of regular flow along the outer boundary. No first-order periodic points were found. The flow with a twice larger rotation angle $\theta=\pi$ [Fig. 2(b)] can be regarded as more chaotic. It contains only two relatively large (period-2) islands in the bulk of the flow domain. The protocol with $\theta=2\pi$ [Fig. 2(c)] results in an almost globally chaotic flow: there are no noticeable islands. It seems, however, that all chaotic flows belonging to this family of protocols have thin regular layers adjacent to both cylinders. In case of $\theta=2\pi$ a stable (elliptic) periodic point was detected near the inner cylinder—its position is shown by nonfilled circular marker. Note that the islands that are not single-connected (in this flow they enclose the inner cylinder) do not necessarily have to contain a periodic point of appropriate order inside the island itself.

III. THE EXTENDED MAPPING TECHNIQUE

A. Mathematical formulation

We apply the extended mapping method⁹ to analyze the JBF. The flow domain Ω is divided into N nonoverlapping subdomains Ω_i with boundaries $\partial\Omega_i$ and the mixture is described with the coarse-grain concentration C_i within each cell. The column vector containing all the cell values at time t_k is $\{C\}^k$. The mapping method advances these values over relatively large, discrete time steps using the matrix multiplication:

$$C_i^{k+1} = \sum_{j=1}^N \Psi_{ij} C_j^k. \quad (1)$$

The *mapping matrix* Ψ is constructed using the velocity field¹⁶ in the flow domain and an accurate adaptive interface tracking algorithm.⁴ This matrix can be very large (in the examples of this paper it has 3.6×10^9 elements), but is normally essentially *sparse*. Usage of appropriate algorithms makes it affordable to store and manipulate such matrices. The mapping matrices are computed for the set of different time steps and for different boundary conditions. Then, mapping protocols in the system under study are approximated by the appropriate sequences of the steps for which the mapping matrices were computed. Details and validation of the technique can be found in Ref. 8, while some properties of the mapping matrices and their eigenvalues and eigenvectors are addressed in Ref. 18.

The original mapping method⁷ describes the mixture only on macrolevel: the smallest unit is a cell and only structures larger than the typical cell size are resolved. The extended mapping technique, introduced in Ref. 9, uses a multiscale approach. The microstructure within each cell is described using the second order *area tensor*,¹³ which is defined for the cell with volume V as follows:

$$\mathbf{A} = \frac{1}{V} \int_{\Gamma} \mathbf{n} \mathbf{n} dS, \quad (2)$$

where $\mathbf{n}$ denotes the unit normal vector to the increment of the interfacial area dS . The integral is computed over all intermaterial interfaces contained in the averaging volume V . The area tensor $\mathbf{A}$ defined in this way describes the amount and orientation of the interface. Figure 3 illustrates the definition of the area tensor and gives some typical examples of the lamellar and isotropic (spherical droplets) microstructures and corresponding area tensors. The trace of the area tensor, $\text{tr} \mathbf{A}$, has a simple physical meaning: it gives the total interfacial area per unit volume (denoted as S_v in Fig. 3).

The algorithm to determine the transformation of the area tensor under finite deformation was described in Ref. 9, while passive advection is described by the mapping matrix itself, because the area tensor, defined by (2) is additive. To

FIG. 3. Definition of the area tensor components (a) and examples of particular mixture morphology and corresponding area tensors: (b) lamellar and (c) isotropic (circular drops).

determine the evolution of the area tensor due to local deformation, the average value of the deformation gradient should be known for each intersection of the deformed and undeformed grid cells. For each nonzero mapping coefficient the components of the corresponding averaged deformation gradient are computed and stored. In previous work⁹ the deformation was estimated at the centroid of the cell intersection. In this paper we use a somewhat different approach. The boundary of the cell intersection is described by a closed polygon, for which the coordinates of its vertices are computed before and after the deformation. This is used to obtain the better estimate of the volume-averaged value of deformation gradient, using the numerical technique developed by Peters (Ref. 19, Sec. 3.3).

The evolution of the area tensor under finite strain can not be expressed explicitly in terms of the original area tensor and the local deformation gradient. Wetzel and Tucker¹³ suggested a good approximation based on the use of the so-called *droplet shape tensor*. For any area tensor and concentration there corresponds a unique set of identical ellipsoidal droplets, whose shape, size and distribution is described by droplet shape tensor.^{20,21} Transformation of the droplet shape tensor can be expressed explicitly using the deformation gradient. The evolution of the area tensor is computed in three steps. First, the area tensor is converted to its equivalent droplet shape tensor. Next, the droplet shape tensor is transformed using the deformation gradient tensor. And, finally, the droplet shape tensor is transformed back into area tensor.

The extended mapping method updates the area tensor at each time step according to

$$\mathbf{A}_i^{k+1} = \sum_{j=1}^N \Psi_{ij} (\mathbf{A}_j^k \otimes \mathbf{F}_{ij}^{-1}), \quad (3)$$

where $\mathbf{F}_{ij}^{-1}$ is the inverse deformation gradient tensor that describes the deformation of the material transported from

subdomain j to subdomain i . That is, the area tensor $\mathbf{A}_i^{k+1}$ in cell i at time $k+1$ is the sum of contributions from all donor cells, after the donor tensors from time k have been transformed by the appropriate deformation gradients. This operation is denoted by the operator $\mathbf{A} \otimes \mathbf{F}^{-1}$. Note, that this is merely a convenient notation: the actual transformation of $\mathbf{A}_j^k$ under the deformation, described by $\mathbf{F}_{ij}^{-1}$ is done indirectly, using the above described technique involving the droplet shape tensor. Although this approximation works well, no simple formula relating the trace of area tensor (used later to describe the amount of interface generated) before and after the deformation results from this transformation. Note that, although in the illustrations only the trace of area tensor is used, which gives the amount of interfacial area per unit volume, a complete tensor must be used in computations, since it contains essential information about the interface orientation.

The extended mapping technique is a *multiscale* method in the sense that it treats simultaneously the macroscopic transport, using coarse grain concentration, and describes statistically the intermaterial interface patterns finer than the grid cell size using the coarse grain *area tensor*. The term “multiscale” is used in a sense that variations (i.e., large striation in a mixture) larger than the grid cell size are resolved due to concentration differences—typical coarse grain density approach, while, at the same time, the structure on a subgrid level is described using the area tensor. These features allow the technique to work for both initial and advanced stages of mixing process, when macroscopic concentration variations disappear. Muzzio *et al.*¹² observed that “for time-periodic chaotic flows, the mixing process is controlled by a stationary multiplicative operator that generates structures that are self-similar with respect to time,” although this operator was not obtained. The extended mapping technique actually provides this operator.

Figure 4 illustrates the type of cell grids used to imple-

FIG. 4. Example of the undeformed and deformed grids (coarse grids containing only 150 cells are shown): (a) initial grid; (b) deformed grid after rotation of inner cylinder by $\pi/2$; (c) deformed grid after rotation of outer cylinder by $\pi/2$.

FIG. 5. Comparison of the mapping (b), (e), extended mapping (c), (f) and front tracking results (a), (d). Top row (a), (b), (c) shows initial distributions, bottom (d), (e), (f) after two periods of the flow with $\theta = \pi$ and $\Omega = 3$. In subplot (d) the result of the deformation of the initial contour (a) determined using the adaptive front tracking technique is shown. Gray filling of the contour in (a) and (d) is used only to indicate the interior of the blob, while leaving the sharp contour visible. The subplots (b), (e) show in gray scale the concentration distribution before and after deformation. Similar, the subplots (c), (f) show the distribution of the trace of area tensor.

ment the mapping technique for the journal bearing flow (actual grids used in computations are much finer and contain 6×10^4 cells). Figure 4(b) shows the grid deformed by a counter-clockwise rotation of the inner cylinder by $\pi/2$ and Fig. 4(c) for a rotation of the outer cylinder. The mapping matrices and deformation gradient tensor components are computed for the separate rotations of $\pi/2$, π and 2π of each cylinder (i.e., six different mappings). All protocols studied in this paper are represented as a sequence of these precomputed steps.

Before proceeding with the stretching distribution analysis we show that the extended mapping technique provides an appropriate, quantitative description of the interface stretching.

B. Mapping versus explicit front tracking and accuracy

To verify the ability of the mapping technique to predict quantitatively the amount of interface stretching, a comparison was made between results from extended mapping and explicit adaptive tracking of the interface.⁴ For this purpose a test blob is used, see Fig. 5(a). The concentration distribution describing this blob [Fig. 5(b)] has slightly blurred edges due to the discretization. Figure 5(c) shows how the boundary of the initial blob is described with the area tensor (trA is plot-

ted). The boundary description is as sharp as possible with the given grid: only a single layer of cells contains nonzero area tensor. The bottom row of images in the same figure show how the blob is deformed by two periods of the flow with $\theta = \pi$ and $\Omega = 3$. Comparison to the interface tracking shows that the mapping approach properly represents the deforming blob.

The deformation of the blob, depicted in Fig. 5 was computed for three different flows with $\theta = 0.5\pi$, $\theta = \pi$, and $\theta = 2\pi$ (in all three cases $\Omega = 3$) by using both front tracking and extended mapping approach. Total length of the interface was recovered from the mapping results as a sum over all subdomains (cells):

$$L(t) = \sum_{i=1}^N S_i \text{tr} \mathbf{A}_i, \quad (4)$$

where $\mathbf{A}_i$ is the area tensor in subdomain i that has an area equal to S_i .

Figure 6 shows the evolution of the relative length $L(t)/L_0$ of the blob boundary, where L_0 is the initial length at $t=0$. Computations were performed for five periods with $\theta = 0.5\pi$, $\theta = \pi$, and $\theta = 2\pi$. Notice that the number of periods is limited by the tracking method only as the number of points needed to describe the contour is increasing exponen-

FIG. 6. Interface stretching obtained using front tracking and extended mapping technique.

tially (computational efforts for tracking are typically in the order of hours, while mapping requires seconds). The lines show the evolution of the contour length, computed using the adaptive front tracking technique while markers represent the mapping results. In all three cases the agreement is rather good. The discrepancy between the contour length, computed using two approaches did not exceed 4%, while an interface stretching of more than 10^4 was obtained. Thus, despite the “numerical diffusion” caused by using coarse grain density values, i.e., by averaging within subdomains, the extended mapping technique is able to correctly predict the interface stretching.

To justify the quantitative results on average stretching we examine the influence of the size of mapping steps. The evolution of the average stretching Λ (see next section) was computed for the same flow using different sizes of mapping steps. For example, for the flow with $\theta = 2\pi$ the stretching of the initially isotropic pattern was computed for 10 periods. Mapping matrices were computed for the rotation angles $\pi/2$, π and 2π . Three test computations were performed: one with the largest mapping steps possible, next one with rotation angle $\pi/2$ and π , and, finally, one with only the smallest steps of $\pi/2$. The predicted logarithms of average stretching were respectively $\log \Lambda = 7.640$, $\log \Lambda = 7.651$, and $\log \Lambda = 7.646$. This demonstrates the insensitivity of the average stretching to the size of the mapping steps and, thus, the extended mapping technique can predict stretching in the journal bearing flow reliably. Since the stretching predictions are not sensitive to the size of the mapping steps, the largest possible steps are used in the following computations, reducing the computational expenses.

IV. RESULTS

The extended mapping technique gives both the macroscopic material transport and the subcell microstructure evolution. In this paper, however, we focus our attention on the microstructure development. Our primary interest is the stretching ability of the flow. Similar to the approach of Muzzio *et al.*¹⁰ we want to obtain the distribution of stretch-

ing values, averaged over all possible orientations. But unlike their approach we intend to obtain the *locally volume averaged* rather than *point wise* stretching values. The extended mapping approach allows for such computations in a very efficient way and is applied here to study the kinematics of the chaotic mixing processes.

A. Stretching distribution

Volume averaging is an intrinsic feature of the mapping technique, since the mixture is described using coarse grain variables. To evaluate the stretching averaged over all possible initial interface orientations, we prescribe the initial distribution that has an isotropic area tensor with unit trace in every cell: $\mathbf{A}_i = 0.5\mathbf{I}$, where $\mathbf{I}$ is the unit tensor. This is equivalent to a mixture containing small spherical droplets, where all orientations of the interface have equal probability. The composing Newtonian fluids are supposed to have identical viscosities and zero surface tension, so the droplets are deforming affinely with the flow. The average stretching λ of the material in a cell is estimated as the ratio of the trace of area tensor (which equals the total length of interface, divided by the cell area) with its initial value (which, in this case, equals one):

$$\lambda_i(t) = \text{tr } \mathbf{A}_i(t) / \text{tr } \mathbf{A}_i(0) = \text{tr } \mathbf{A}_i(t). \quad (5)$$

We also define the arithmetic mean stretching in the whole domain as

$$\Lambda = \sum_i^N \lambda_i S_i / \sum_i^N S_i, \quad (6)$$

where S_i is the area of the cell with number i and N is the total number of cells. The definition of Λ is closely related to the *arithmetic mean stretching* used by Muzzio *et al.*¹⁰ (discussed later in Sec. IV B).

Figure 7 shows the logarithm of the stretching distribution after one period with the parameter θ equal to $\pi/2$, π and 2π , respectively. We use the logarithm of the stretching value because the growth of Λ with time is exponential. Even after a single period of the flow, the stretching distribution is highly nonuniform with zones of high stretching closely interleaved with zones of weak stretching. In all three plots the same logarithmic scale is used to reveal better the whole range of stretching values (dark corresponds to higher stretching). Notice that the results, presented in Fig. 7, shows the stretching during a single period starting from isotropic initial distribution. These patterns do *not* represent the long-term stretching in the corresponding flows.

Stretching of isotropically oriented interfaces during one period of flow does not fully characterize the stretching performance of the flow. After just a few periods the distribution of the stretching values evolves into a self-similar pattern and asymptotic orientation of interfaces is established and persists. This was shown by Muzzio *et al.*,¹⁰ who named this phenomenon “asymptotic directionality.” This particular orientation created by the flow itself strongly influences the stretching rates. To evaluate the asymptotic stretching behavior of the flow, we computed the average stretching in each cell during the preceding period of the flow. To do this, the

FIG. 7. Stretching of an initially isotropically oriented interfaces during one period of the flow: (a) $\theta = \pi/2$, (b) $\theta = \pi$, (c) $\theta = 2\pi$. The gray level represents the decimal logarithm $\log_{10} \lambda$ of the local stretching values.

area tensor distribution after at least 10 periods of flow was taken as an initial condition. This ensures that the interfaces acquired a stable orientation and that the distribution of $\text{tr} \mathbf{A}$ becomes self-similar. This distribution was then mapped for one more period after which we compared the resulting distribution of $\text{tr} \mathbf{A}$ with the values which would result if the interfaces would just have been passively convected (to do this, the scalar value of $\text{tr} \mathbf{A}$ was mapped in the same way as concentration). The ratio of the two values of $\text{tr} \mathbf{A}$ gives an estimate of the average stretching experienced by interfaces in each cell during this last period of the flow.

Figure 8 demonstrates the stretching λ^* of asymptotically oriented interfaces for the same flows as in Fig. 7. The gray level of each cell corresponds to the logarithmic average stretch experienced during the last period of the flow. The plots in Fig. 8 show some remarkable features. First, the asymptotic stretching rates are generally lower than those on an initial isotropic pattern (see Fig. 7). Moreover, the stretching fields now have a more complex structure.

Another important feature is that stretching ratios less than one are observed ($\log \lambda^* < 0$). This means that the material in these cells underwent contraction during the last period and we call these regions zones of *local microstructural demixing*. The zones of the interface contraction are shown separately in Fig. 9. In this figure the contour lines define the zones where contracted material resides. Inside

these zones the cells are shaded to show the contraction ratio: a darker gray level corresponds to a higher contraction ratio. Interfaces are observed to contract as strong as six to nine times. Notice that the stretching (or contraction) computed in this way is the average value over the interface area. So, it takes into account the interface distribution and properly describes the interface generation by the flow when typical self-similar distributions are established.

The presence of the local microstructural demixing zones does not mean that the amount of interface at some locations decreases with time. Quite contrary, it grows exponentially. The material that is being contracted was previously highly stretched in other parts of the flow. These local demixing zones, however, quite closely match the parts of the flow domain where the trace of area tensor grows slower (and, thus, the striation thickness is larger) and where homogenization during the initial stage of the mixing is slower.⁹ From our observations, also based on the previous work dealing with the flow in a lid-driven rectangular cavity,⁹ location of these demixing zones is not directly associated with stable islands. The islands do typically contain the regions, where the interfaces contract alongside with the regions where material experiences stretching. However, the largest contraction ratios are observed in chaotic regions, mainly in the zones, where sharp folds of material layers are formed.

FIG. 8. One-period stretching of asymptotically oriented microstructure: (a) $\theta = \pi/2$, (b) $\theta = \pi$, (c) $\theta = 2\pi$. Gray levels represent the logarithm $\log_{10} \lambda^*$ of the stretching values over one period.

FIG. 9. Microstructural demixing (interface contraction) factor: (a) $\theta = \pi/2$, (b) $\theta = \pi$, (c) $\theta = 2\pi$. Contours delimit the demixing zones. Darker color indicates a stronger contraction.

B. Arithmetic mean stretching

Since the extended mapping approach gives quantitatively good results on stretching, as was demonstrated in Sec. III B, we compare the rate of interface generation in the JBF predicted by the mapping method to that computed by Muzzio *et al.*¹⁰ They obtained the stretching statistics pointwise by computing the stretching values associated with a set of markers with zero size. The stretching averaged over the volume of the fluid may provide a better estimation of the ability of the mixing flow to stretch the material and generate intermaterial area. In this case the result is not dependent on the distribution of sample points (markers) and all possible orientations of initial interfaces are taken into account because of the statistical description using the area tensor.

Since the arithmetic mean stretching Λ_a is growing exponentially with number of the periods n as

$$\Lambda_a \sim \exp(\beta_a n), \quad (7)$$

Muzzio *et al.*¹⁰ computed the rate of exponential stretching β_a for the flows with $\Omega = 3$ and θ ranging from $\pi/2$ to 4π . They concluded that in the examined range β_a increases linearly with the angle θ and total stretching is thus determined only by the total rotation angle of the cylinders. Such a conclusion seems not very convincing even with the data they

presented. To clarify this issue we estimated the value of β_a based on the current model. To make sure that the trend is captured properly, mapping computations cover a wider range of rotation angle, up to $\theta = 8\pi$. The results are presented in Fig. 10(a), where the data from Ref. 10 are also plotted.

For some values of θ our results coincide with those from Muzzio *et al.*,¹⁰ for some parameters the mapping approach predicts higher stretching rates. This discrepancy can possibly be attributed to the difference in the way the stretching is computed. The mapping approach, however, is in very good agreement with the explicit tracking results in this range. Both approaches show the same trend in the range $0.5\pi < \beta < 4\pi$. We do not regard this as a linear proportionality between β_a and θ , since the line bends downwards; the mapping data is best fitted with a power function $\beta_a \sim \theta^{0.7816}$. When the data computed for the values of $4\pi < \theta < 8\pi$ are taken into account, nonlinear dependency of the stretching efficiency on the rotation angle θ becomes even more obvious, see Fig. 10(a). This can be clarified by rescaling the same data and plotting β_a/θ versus θ , see Fig. 10(b). A remarkable result is that both, our results and the analysis performed by Muzzio *et al.*¹⁰ show, although it is not indicated in Ref. 10, that the higher stretching for the same total

FIG. 10. The exponential stretching coefficients computed using extended mapping technique for the flows with $\Omega = 3$ and different values of θ , compared with the data from Muzzio *et al.* [Phys. Fluids A 3, 822 (1991)].

angular displacement (and, consequently, the same energy) is achieved by the flow with $\theta = \pi$.

V. CONCLUSIONS AND DISCUSSION

In this paper the stretching and microstructure development in the time-periodic Stokes flows between eccentric cylinders was studied using the extended mapping method. The extended mapping approach uses coarse grain values to describe both the concentration distribution and the area tensor. The latter gives the intermaterial area patterns on the subgrid scale. The ability of the mapping approach to correctly predict stretching rates for this type of the flow was verified by comparing the deformation of the boundary of a blob of passive tracer as predicted by the mapping technique with the result obtained from explicit tracking. In the test examples the discrepancy for the predicted total interface length did not exceed 4% while the average stretching of the interface exceeded a factor of 10^4 .

The extended mapping method fills the gap between techniques that rely on pointwise characteristics of deformation (for example, Ref. 10) and on explicit description of interfaces. The latter approach was taken in Ref. 12 (and the references therein), where the interfaces were explicitly tracked and then the coarse grain values were calculated. Explicit front tracking is limited to the early stages of mixing, since the total length of the interface grows exponentially and computational resources limit the amount of the interface that can be explicitly described. Mapping exploits the interface tracking to build the mapping matrices, but the statistical way of the interface description in terms of area tensor¹³ allows to extend the computations to the later stages of mixing.

The arithmetic mean stretching, predicted by the mapping approach was in some cases nearly the same as computed in Ref. 10 and in some cases higher. The *computational efficiency* of the mapping approach allowed to extend the range of the parameters of the flow protocols being considered. It was found that the conclusion, drawn with some reservations in Ref. 10, that the total amount of stretching is proportional to the total rotation angle (and, hence, the work done on the system), was not right. The maximum stretching efficiency for the studied mixing protocols is achieved within the range, examined by Muzzio *et al.*¹⁰ but doubling of the investigated range of the rotation angles made the trend in the dependency of stretching on the protocol parameter more obvious.

Typical features of time-periodic chaotic flows phenomena, such as a strongly nonuniform interfacial area distribution and a quick onset of the dominant interface orientation—the phenomenon referred in Ref. 12 as “asymptotic directionality” (AD), were found using the mapping method. It was revealed that when self-similar patterns are established, there exist distinctive zones in the flow, which we called “microstructural demixing zones,” where

interfaces can be strongly contracted. The reason for such behavior is that the established orientation of interfaces does not favor stretching there. The total amount of interface, however, grows due to advection of the material to and from zones where it experiences high stretching rates.

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