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NON INTERACTING CONTROL
BY MEASUREMENT FEEDBACK

by

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ABSTRACT

In this paper we shall solve the problem of non interacting control by measurement feedback for systems that in addition to a control input and a measurement output have two exogenous inputs and two exogenous outputs. That is, we shall derive necessary and sufficient conditions that can actually be verified for the existence of a measurement feedback compensator such that the transfer matrix of the resulting closed loop system, when partitioned according to exogenous inputs and outputs, has off-diagonal blocks equal to zero.

Keywords & Phrases
Non interacting control, measurement feedback, \((A,B)\)-compatibility, \((C,A)\)-compatibility, linear matrix equations.
1. Introduction

Consider a system in state space form that, in addition to a control input and a measurement output, has two exogenous inputs and two exogenous outputs. Controlling such a system by means of a measurement feedback compensator results in a closed loop system that has two exogenous inputs and two exogenous outputs. Therefore, the transfer matrix of the closed loop system can be partitioned according to the dimensions of the exogenous inputs and outputs as a two by two block matrix. The problem that will be addressed in this paper can then be formulated as follows.

Given a system as described above, does there exist a measurement feedback compensator such that the transfer matrix of the closed loop system has off-diagonal blocks equal to zero? And if so, how can this compensator be computed?

The above problem formulation falls within the framework of non interacting control. The approach towards non interacting control as described in this paper was initiated in Willems [5] and was developed in Trentelman & Van der Woude [4]. In the latter paper also the distinction was made clear between the present approach and the point of view towards non interacting control as exposed in Morse & Wonham [2] and Hautus & Heyman [1]. The main contribution of this paper is that unlike Willems [5] and Trentelman & Van der Woude [4], where (dynamic) state feedback was required in the solution of the problem formulated above, in this paper we allow the problem to be solved by (dynamic) measurement feedback. In this context we also refer to Van der Woude [7] where also measurement feedback is used to solve the almost version of the problem formulated above.

The outline of the paper is as follows.

In Section 2 we shall give a mathematical formulation of the main problem of this paper. Furthermore we shall recall some well-known results coming from the geometric approach towards control theory. In Section 3 we shall derive some preliminary results. In fact, the main result of Section 3 consists of sufficient conditions for the solvability of our main problem. Necessary and sufficient conditions for the solvability of our main problem will be derived in Section 4. In Section 5 we shall state some remarks and conclusions. Furthermore, in Section 5 we shall give a conceptual algorithm, that, if it exists, provides a compensator that achieves non interaction.
2. Problem Formulation

Consider the finite-dimensional linear time-invariant system \( \Sigma \) given by

\[
\begin{align*}
\dot{x}(t) &= A x(t) + B u(t) + G_1 v_1(t) + G_2 v_2(t), \\
y(t) &= C x(t) , \\
z_1(t) &= H_1 x(t) , \\
z_2(t) &= H_2 x(t) .
\end{align*}
\]

Here \( x(t) \in \mathbb{R}^n \) denotes the state of the system, \( u(t) \in \mathbb{R}^m \) the control input, \( v_1(t) \in \mathbb{R}^{q_1} \), \( v_2(t) \in \mathbb{R}^{q_2} \) the two exogenous inputs, \( y(t) \in \mathbb{R}^p \) the measurement output and \( z_1(t) \in \mathbb{R}^{r_1} \), \( z_2(t) \in \mathbb{R}^{r_2} \) the two exogenous outputs. \( A, B, C, G_1, G_2, H_1 \) and \( H_2 \) are real matrices of appropriate dimensions.

Assume that the system \( \Sigma \) is controlled by means of a measurement feedback compensator \( \Sigma_c \) described by

\[
\begin{align*}
\dot{w}(t) &= K w(t) + L y(t) , \\
u(t) &= M w(t) + N y(t) ,
\end{align*}
\]

with \( w(t) \in \mathbb{R}^k \) the state of the compensator and \( K, L, M \) and \( N \) real matrices of appropriate dimensions.

Interconnection of the system \( \Sigma \) with the compensator \( \Sigma_c \) results in a closed loop system \( \Sigma_{cl} \) with two exogenous inputs \( v_1(t), v_2(t) \) and two exogenous outputs \( z_1(t), z_2(t) \). The closed loop system \( \Sigma_{cl} \) is described by

\[
\begin{align*}
\dot{x}_c(t) &= A_c x_c(t) + G_{1,c} v_1(t) + G_{2,c} v_2(t) , \\
z_1(t) &= H_{1,c} x_c(t) , \\
z_2(t) &= H_{2,c} x_c(t) ,
\end{align*}
\]

where we have denoted

\[
x_c(t) = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} , \quad A_c = \begin{bmatrix} A + BNC & BM \\ LC & K \end{bmatrix} , \quad G_{i,c} = \begin{bmatrix} G_i \\ 0 \end{bmatrix} \quad (i = 1, 2) ,
\]

\[
H_{i,c} = [H_i \ 0] \quad (i = 1, 2) .
\]

Let \( T(s) \) be the transfer matrix of the closed loop system \( \Sigma_{cl} \). Then \( T(s) \) can be partitioned as

\[
T(s) = \begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{bmatrix}
\]

where \( T_{ij}(s) \) denotes the \( r_i \times q_j \) transfer matrix between the \( j \)-th exogenous input and the \( i \)-th exogenous output. It is clear that

\[
T_{ij}(s) = H_{i,c} (sI - A_c)^{-1} G_{j,c} .
\]

We are now able to give the following problem formulation.
Definition 2.1.
Let $\Sigma$ be given. The non interacting control problem by measurement feedback (NICPM) consists of finding a measurement feedback compensator $\Sigma_c$ such that in the closed loop system $T_{12}(s) = 0$ and $T_{21}(s) = 0$.

If a measurement feedback compensator $\Sigma_c$ is such that it solves (NICPM), then it is said that $\Sigma_c$ achieves non interaction.

In Section 4 we shall derive necessary and sufficient conditions for the solvability of (NICPM). The conditions obtained will be stated in geometric terms. To that extent we recall some well-known concepts originating from the geometric approach towards control theory (cf. Wonham [6], Schumacher [3]).

Consider the dynamical system described by
\[ \dot{x}(t) = A x(t) + B u(t), \quad y(t) = C x(t) \]
with state space $\mathbb{R}^n$, control input space $\mathbb{R}^m$, measurement output space $\mathbb{R}^p$ and matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$. With respect to this system we now introduce the following.

A linear subspace $V$ in $\mathbb{R}^n$ is called an $(A, B)$-invariant subspace if $AV \subseteq V + \text{im} B$. It is well known, that this subspace inclusion is equivalent to the existence of a matrix $F \in \mathbb{R}^{m \times n}$ such that $(A + BF)V \subseteq V$.

Following Wonham [6] we call two $(A, B)$-invariant subspaces $V_1$ and $V_2$ compatible with respect to the pair $(A, B)$ (or simply: $(A, B)$-compatible) if there exists a matrix $F \in \mathbb{R}^{m \times n}$ such that both $(A + BF)V_1 \subseteq V_1$ and $(A + BF)V_2 \subseteq V_2$. It can be proved (see Wonham [6], Ex. 9.1) that two $(A, B)$-invariant subspaces $V_1$ and $V_2$ are $(A, B)$-compatible if and only if their intersection $V_1 \cap V_2$ is an $(A, B)$-invariant subspace. If $K$ is a linear subspace in $\mathbb{R}^n$, then $V^*(K)$ will denote the largest $(A, B)$-invariant subspace contained in $K$. $V^*(K)$ can be calculated by means of the algorithm given in Wonham [6], Chapter 4.

Dualizing the concepts introduced above we obtain the following (cf. Schumacher [3]).

A linear subspace $S$ in $\mathbb{R}^n$ is called a $(C, A)$-invariant subspace if $A(S \cap \text{ker} C) \subseteq S$. This subspace inclusion is equivalent to the existence of a matrix $J \in \mathbb{R}^{n \times p}$ such that $(A + JC)S \subseteq S$. Furthermore, two $(C, A)$-invariant subspaces $S_1$ and $S_2$ are said to be compatible with respect to the pair $(C, A)$ (or simply $(C, A)$-compatible) if there exists a matrix $J \in \mathbb{R}^{n \times p}$ such that both $(A + JC)S_1 \subseteq S_1$ and $(A + JC)S_2 \subseteq S_2$. By the previous it is clear
that two \((C,A)\)-invariant subspaces \(S_1\) and \(S_2\) are \((C,A)\)-compatible if and only if their sum \(S_1 + S_2\) is a \((C,A)\)-invariant subspace. If \(L\) is a linear subspace in \(\mathbb{R}^n\), then \(S^*(L)\) will denote the smallest \((C,A)\)-invariant subspace containing \(L\). An algorithm to calculate \(S^*(L)\) can be found in Schumacher [3]. The latter algorithm is in fact the dual of the algorithm mentioned previously for the determination of \(V^*(K)\) with respect to a given linear subspace \(K\).
3. Sufficient Conditions

In this section we shall derive a preliminary result that we shall need in the proof of our main result. The result provides sufficient conditions for the solvability of (NICPM). In order to establish these sufficient conditions, we shall make use of the following two results. The first result that we need is very general and is concerned with the existence of a common solution to a pair of linear matrix equations.

**Theorem 3.1.**
Let \( A_i \in \mathbb{R}^{s_i \times v_i}, B_i \in \mathbb{R}^{w_i \times v_i}, C_i \in \mathbb{R}^{s_i \times w_i} \) (\( i = 1, 2 \)) be given matrices. There exists a matrix \( X \in \mathbb{R}^{v \times w} \) such that \( A_1 X B_1 = C_1 \) and \( A_2 X B_2 = C_2 \) if and only if \( \text{im}\, A_i \supseteq \text{im}\, C_i \) (\( i = 1, 2 \)), \( \ker B_i \subseteq \ker C_i \) (\( i = 1, 2 \)) and
\[
\begin{bmatrix}
C_1 & 0 \\
0 & -C_2
\end{bmatrix} \ker [B_1, B_2] \subseteq \text{im} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.
\]

**Proof.** See Van der Woude [7].

For the second result that we need here, we refer to the linear system \( \dot{x}(t) = A x(t) + B u(t) \), \( y(t) = C x(t) \) as described in the previous section.

**Theorem 3.2.**
Let \( S_1, S_2 \) be \((C, A)\)-invariant subspaces and \( V_1, V_2 \) be \((A, B)\)-invariant subspaces in \( \mathbb{R}^n \) such that \( S_1 \subseteq V_1 \) and \( S_2 \subseteq V_2 \). Then there exists a matrix \( N \in \mathbb{R}^{m \times p} \) such that \((A + BNC)S_i \subseteq V_i \) and \((A + BNC)S_2 \subseteq V_2 \) if and only if
\[
\begin{bmatrix}
A & 0 \\
0 & -A
\end{bmatrix} ((S_1 \oplus S_2) \cap \ker [C, C]) \subseteq (V_1 \oplus V_2) + \text{im} \begin{bmatrix} B \\ B \end{bmatrix}.
\]

Here we have adopted the following notation. If \( L_1 \) is a linear subspace in \( \mathbb{R}^{l_1} \) and \( L_2 \) is a linear subspace in \( \mathbb{R}^{l_2} \), then \( L_1 \oplus L_2 \) denotes the linear subspace in \( \mathbb{R}^{l_1 + l_2} \) defined as
\[
L_1 \oplus L_2 = \left\{ \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \in L_1 \oplus L_2 \right\}.
\]

**Proof.** Let \( X_1, X_2, T_1 \) and \( T_2 \) be matrices such that \( \text{im}\, T_i = S_i \) (\( i = 1, 2 \)) and \( \ker T_i = V_i \) (\( i = 1, 2 \)). Then there exists a matrix \( N \in \mathbb{R}^{m \times p} \) such that \((A + BNC)S_i \subseteq V_i \) (\( i = 1, 2 \)) if and only if there exists a matrix \( N \in \mathbb{R}^{m \times p} \) such that \( T_i A X_i + T_i BNC X_i = 0 \) (\( i = 1, 2 \)).

By Theorem 3.1 the latter is equivalent to \( \text{im}\, T_i B \supseteq \text{im}\, T_i A X_i \) (\( i = 1, 2 \)), \( \ker C X_i \subseteq \ker T_i A X_i \) (\( i = 1, 2 \)) and
\[
\begin{bmatrix}
T_1 A X_1 & 0 \\
0 & -T_2 A X_2
\end{bmatrix} \ker [C X_1, C X_2] \subseteq \text{im} \begin{bmatrix} T_1 B \\ T_2 B \end{bmatrix}.
\]

In its turn this is equivalent to \( A S_i \subseteq V_i + \text{im}\, B \) (\( i = 1, 2 \)), \( A (S_i \cap \ker C) \subseteq V_i \) (\( i = 1, 2 \)).
and
\[
\begin{bmatrix}
A & 0 \\
0 & -A
\end{bmatrix}
((S_1 \oplus S_2) \cap \ker [C, C]) \subseteq (V_1 \oplus V_2) + \text{im} \begin{bmatrix} B \\ B \end{bmatrix}.
\]

The proof can now be completed using the observation that the conditions \( A S_i \subseteq V_i + \text{im} B \) \((i = 1, 2)\) and \( A(S_i \cap \ker C_i) \subseteq V_i \) \((i = 1, 2)\) are fulfilled trivially since \( S_1, S_2 \) are \((C, A)\)-invariant subspaces, \( V_1, V_2 \) are \((A, B)\)-invariant subspaces and \( S_i \subseteq V_i, S_j \subseteq V_j \).

Now the following theorem is the main result of this section.

**Theorem 3.3.**

Let the system \( \Sigma \) be given. Let \( S_1, S_2 \) be \((C, A)\)-invariant subspaces and let \( V_1, V_2 \) be \((A, B)\)-invariant subspaces such that the following conditions are satisfied.

(a) \( \text{im} G_1 \subseteq S_1 \subseteq V_1 \subseteq \ker H_2, \) \( \text{im} G_2 \subseteq S_2 \subseteq V_2 \subseteq \ker H_1, \)

(b) \( V_1 \cap V_2 \) is an \((A, B)\)-invariant subspace,

(c) \( S_1 + S_2 \) is an \((A, B)\)-invariant subspace and

(d) \( A = \begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix}, (S_1 \oplus S_2) \cap \ker [C, C]) \subseteq (V_1 \oplus V_2) + \text{im} \begin{bmatrix} B \\ B \end{bmatrix}.
\]

Then there exists a measurement feedback compensator \( \Sigma_c \) such that in the closed loop system \( T_{12}(s) = 0 \) and \( T_{21}(s) = 0 \).

**Proof.** Because of (a), (d) and Theorem 3.2 there exists a matrix \( N \in \mathbb{R}^{m \times p} \) such that \( (A + BNC)S_i \subseteq V_i \) \((i = 1, 2)\). By (b) and (c) it follows that there exist matrices \( F \in \mathbb{R}^{m \times n} \) and \( J \in \mathbb{R}^{n \times p} \) such that \( (A + BF)V_i \subseteq V_i \) \((i = 1, 2)\) and \( (A + JC)S_i \subseteq S_i \) \((i = 1, 2)\). Let \( W_{1,e} \) and \( W_{2,e} \) be linear subspaces in \( \mathbb{R}^{2n} \) defined by

\[
W_{i,e} = \left\{ \begin{bmatrix} s \\ v \end{bmatrix} \mid s \in S_i, v \in V_i \right\} \quad (i = 1, 2),
\]

and let \( A_e \in \mathbb{R}^{2n \times 2n} \) be a matrix defined as

\[
A_e = \begin{bmatrix}
A + BNC & BF - BNC \\
BNC - JC & A + BF + JC - BNC
\end{bmatrix}.
\]

The matrix \( A_e \) can considered to be obtained by the interconnection of the system \( \Sigma \) and the compensator \( \Sigma_c \) where \( K = A + BF + JC - BNC, L = BN - J \) and \( M = F - NC \).

For every \( x_e \in W_{1,e} \) there exist vectors \( s \in S_1 \) and \( v \in V_1 \) such that

\[
A_e x_e = A_e \left( \begin{bmatrix} s \\ 0 \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix} \right) = \begin{bmatrix} (A + JC)s \\ 0 \end{bmatrix} - \begin{bmatrix} (A + JC)s \\ (A + JC)s \end{bmatrix} + \begin{bmatrix} (A + BNC)s \\ (A + BNC)s \end{bmatrix} + \begin{bmatrix} (A + BF)v \\ (A + BF)v \end{bmatrix}.
\]

Since \( (A + B)V_1 \subseteq V_1 \), \( (A + JC)S_1 \subseteq S_1 \) and \( (A + BNC)S_1 \subseteq V_1 \) it is immediate that \( A_e x_e \in W_{1,e} \) for every \( x_e \in W_{1,e} \). Hence, \( A_e W_{1,e} \subseteq W_{1,e} \). Analogously, it can be shown that
A_\varepsilon W_{2,\varepsilon} \subseteq W_{2,\varepsilon}.

By (a) it is clear that im \text{G}_{1,\varepsilon} \subseteq W_{1,\varepsilon} \subseteq \ker H_{2,\varepsilon} \text{ and } im \text{G}_{2,\varepsilon} \subseteq W_{2,\varepsilon} \subseteq \ker H_{1,\varepsilon}.

Now there follows that \text{H}_{1,\varepsilon} A_\varepsilon^k \text{G}_{2,\varepsilon} = 0 \text{ and } \text{H}_{2,\varepsilon} A_\varepsilon^k \text{G}_{1,\varepsilon} = 0 \text{ for all } k \geq 0 \text{ from which it is clear that } T_{12}(s) = H_{1,\varepsilon}(sl - A_\varepsilon)^{-1} G_{2,\varepsilon} = 0 \text{ and } T_{21}(s) = H_{2,\varepsilon}(sl - A_\varepsilon)^{-1} G_{1,\varepsilon} = 0. \quad \square
4. Main result

In this section we shall derive the main result of the present paper. The result establishes necessary and sufficient conditions for the solvability of (NICPM) in state space terms that can actually be verified. However, before stating this result we have to introduce the following.

If \( Z_1, Z_2 \) and \( Z \) are linear subspaces in \( \mathbb{R}^n \) such that \( Z_1 + Z_2 \subseteq Z \) then we define the set \( \Phi(Z_1, Z_2, Z) \) as follows.

\[
\Phi(Z_1, Z_2, Z) = \{(M_1, M_2) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \mid (M_1-I)Z_1 = \{0\}, (M_2-I)Z_2 = \{0\} \text{ and } (M_1+M_2-I)Z = Z_1 \cap Z_2\}.
\]

Note that the set \( \Phi(Z_1, Z_2, Z) \) is not empty.

Indeed, let \([L_0, L_1, L_2, L_3, L_4] \in \mathbb{R}^{n \times n}\) be a square invertible matrix such that \( \text{im} L_0 = Z_1 \cap Z_2, \text{im} [L_0, L_1] = Z_1, \text{im} [L_0, L_2] = Z_2 \) and \( \text{im} [L_0, L_1, L_2, L_3] = Z \). Then any pair of matrices \( (M_1, M_2) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \) such that \( M_1 [L_0, L_1, L_2, L_3, L_4] = [L_0, L_1, 0, M_{13}, M_{14}] \) and \( M_2 [L_0, L_1, L_2, L_3, L_4] = [L_0, 0, L_2, L_3 - M_{13}, M_{24}] \) is an element of \( \Phi(Z_1, Z_2, Z) \). In the latter \( M_{13}, M_{14} \) and \( M_{24} \) are arbitrary matrices of appropriate dimensions.

Given the system \( \Sigma \) we shall adopt the following notation

\[
S_1^* = S^*(\text{im} G_1), \quad S_2^* = S^*(\text{im} G_2), \quad S^* = S^*(\text{im} [G_1, G_2]),
\]

\[
V_1^* = V^*(\text{ker} H_2), \quad V_2^* = V^*(\text{ker} H_1), \quad V^* = V^*(\text{ker} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}).
\]

Since \( S_1^* \subseteq S^*, S_2^* \subseteq S^*, V^* \subseteq V_1^* \) and \( V^* \subseteq V_2^* \) there holds

\[
S_1^* + S_2^* \subseteq S^* \quad \text{and} \quad V^* \subseteq V_1^* \cap V_2^*.
\]

Now the main result of the present paper reads as follows.

**Theorem 4.1.**

Let the system \( \Sigma \) be given.

Then \( \text{(NICPM)} \) is solvable if and only if there exist pairs of matrices \( (D_1, D_2) \in \Phi(S_1^*, S_2^*, S^*) \) and \( (E_1^*, E_2^*) \in \Phi(V_1^*, V_2^*, V^*) \) such that \( D_1 S^* \subseteq V_1^*, D_2 S^* \subseteq V_2^*, E_1 S_1^* \subseteq V^*, E_2 S_2^* \subseteq V^*, (D_1 + E_1 - I)S^* \subseteq V^* \) and \( (AD_1 + E_1 A - A)(S^* \cap \ker C) \subseteq V^* + \text{im} B \).

In the above \( ^\top \) stands for matrix transposition and \( ^\perp \) denotes the orthogonal complement with respect to the euclidean innerproduct.

Before giving the proof of Theorem 4.1, we have to make some remarks.

**Remark 4.2.**

In the proof of Theorem 4.1 we shall frequently make use of specific non unique representations of the linear subspaces \( S_1^*, S_2^*, S^*, V_1^*, V_2^* \) and \( V^* \). More concretely, we let
$[X_0,X_1,X_2,X_3]$ be an injective (full column rank) matrix such that \( \text{im} X_0 = S_i^* \cap S_2^* \), \[
\begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3 
\end{bmatrix}
\]
be a surjective (full row rank) matrix such that \( \ker T_0 = V_1^* + V_2^* \), \( \ker \begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \end{bmatrix} = V^* \).

In the sequel we denote \( X = [X_0,X_1,X_2,X_3] \), \( \hat{X} = [0,X_1,X_2,X_3] \), \( T = \begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \end{bmatrix} \) and \( \hat{T} = \begin{bmatrix} 0 \\ T_1 \\ T_2 \\ T_3 \end{bmatrix} \)

and we let \( Q_X \) and \( Q_T \) be square matrices such that \( \hat{X} = XQ_X \) and \( \hat{T} = Q_T T \).

Then using these specific representations, the subspace inclusions in the "unknown" matrices \( D_1, D_2, E_1 \) and \( E_2 \) appearing in Theorem 4.1 can be reformulated as linear matrix equations in the "unknown" matrices \( D_1, D_2, E_1 \) and \( E_2 \). Next, the linear matrix equations obtained can, by means of Kronecker products (see Lancaster [8]), be transformed into linear equations whose solvability can be checked using standard techniques.

Hence, the conditions of Theorem 4.1 can actually be verified.

**Proof of Theorem 4.1.**

**(if)**

Let \( S_i^*, S_2^*, S^*, V_1^*, V_2^* \) and \( V^* \) have a representation as indicated in Remark 4.2.

Because \( (D_1, D_2) \in \Phi(S_i^*, S_2^*, S^*) \) there follows that with respect to the chosen representation \( D_1 [X_0,X_1,X_2,X_3] = [X_0,X_1,0,D_{13}] \) and \( D_2 [X_0,X_1,X_2,X_3] = [X_0,0,X_2,D_{23}] \) with \( D_{13} + D_{23} = X_3 \).

Indeed, because \( (D_1 - I)S_i^* = \{0\} \) and \( (D_2 - I)S_2^* = \{0\} \) it is clear that \( D_1 [X_0,X_1,X_2,X_3] = [X_0,X_1,D_{12},D_{13}] \) and \( D_2 [X_0,X_1,X_2,X_3] = [X_0,D_{21},X_2,D_{23}] \). and because \( (D_1 + D_2 - I)S^* = S_i^* \cap S_2^* \) there follows that \( D_{12} = 0, D_{21} = 0 \) and \( D_{13} + D_{23} = X_3 \).

Dually, because \( (E_1^T, E_2^T) \in \Phi(V_1^*, V_2^*, \hat{S}_i^* \cap \hat{S}_2^*) \) there follows that with respect to the chosen representation
To To To To

\[
\begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3
\end{bmatrix}
E_1 =
\begin{bmatrix}
T_0 \\
T_1 \\
0 \\
E_{31}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3
\end{bmatrix}
E_2 =
\begin{bmatrix}
T_0 \\
0 \\
T_2 \\
E_{32}
\end{bmatrix}
\]
with \(E_{31} + E_{32} = T_3\).

With \(X\) and \(T\) as introduced in Remark 4.2 the subspace inclusions of Theorem 4.1 imply that

\[
\begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3
\end{bmatrix}
D_1X = 0, \quad \begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3
\end{bmatrix}
D_2X = 0, \quad TE_1[X_0, X_1] = 0, \quad TE_2[X_0, X_2] = 0,
\]

\[
T(D_1 + E_1 - I)X = 0 \quad \text{and} \quad T(AD_1 + E_1 A - A)X \ker CX \subseteq \im TB.
\]

Decompose \(-E_{32}D_{23} = U_3Y_3\) with \(U_3\) a surjective matrix and \(Y_3\) an injective matrix. (It is always possible to find such a decomposition; for instance, \(U_3 = \begin{bmatrix} [E_{32}, 0] \\
Y_3 \end{bmatrix}\) and \(Y_3 = \begin{bmatrix} D_{23} \\
0 \\
1 \end{bmatrix}\).

Let \(g\) denote the number of rows of \(Y_3\) and define matrices \(R, P^T \in \mathbb{R}^{g \times n}\) as

\[
R [X_0, X_1, X_2, X_3] = \begin{bmatrix} 0, 0, 0, Y_3 \end{bmatrix}, \quad P = \begin{bmatrix} 0 \\
0 \\
0 \\
U_3 \end{bmatrix}.
\]

Denote

\[
A_g = \begin{bmatrix} A & 0 & 0 \\
0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{(n+g) \times (n+g)}, \quad B_g = \begin{bmatrix} B & 0 & 0 \\
0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{(n+g) \times (n+g)},
\]

\[
G_i, g = \begin{bmatrix} G & 0 \\
0 & 0 \end{bmatrix} \in \mathbb{R}^{(n+g) \times q_i}, \quad (i = 1, 2), \quad C_g = \begin{bmatrix} C & 0 \\
0 & 0 \end{bmatrix} \in \mathbb{R}^{(p+g) \times (n+g)} \quad \text{and}
\]

\[
H_i, g = \begin{bmatrix} H_i & 0 \\
0 & 0 \end{bmatrix} \in \mathbb{R}^{(r_i+g) \times (n+g)} \quad (i = 1, 2).
\]

Now let \(S_{1,g}, S_{2,g}, V_{1,g}\) and \(V_{2,g}\) be linear subspaces in \(\mathbb{R}^{n+2}\) defined as

\[
S_{1,g} = \im \begin{bmatrix} D_1X \\
RX \end{bmatrix}, \quad S_{2,g} = \im \begin{bmatrix} D_2X \\
-RX \end{bmatrix}, \quad V_{1,g} = \ker [TE_1, TP] \quad \text{and} \quad V_{2,g} = \ker [TE_2, TP].
\]

In order to complete the proof of the (if)-part of Theorem 4.1 it suffices to show that the system characterized by the matrices \(A_g, B_g, C_g, G_{1,g}, G_{2,g}, H_{1,g}\) and \(H_{2,g}\) together with the linear subspaces \(S_{1,g}, S_{2,g}, V_{1,g}\) and \(V_{2,g}\) satisfy the conditions of Theorem 3.3. Indeed, if that is the case, then by Theorem 3.3 there exist matrices \(K_g, L_g, M_g\) and \(N_g\) such that for \((i, j) = (1,2),(2,1)\)
where we have decomposed $L = [L_1, L_2]$, $M = [M_1, M_2]$ and $N = [N_{11}, N_{12}]$. From the latter we can conclude that the compensator given by

$$
\begin{bmatrix}
  w_1(t) \\
  w_2(t)
\end{bmatrix} =
\begin{bmatrix}
  N_{22} & N_2 \\
  L_2 & K_g
\end{bmatrix}
\begin{bmatrix}
  w_1(t) \\
  w_2(t)
\end{bmatrix} +
\begin{bmatrix}
  N_{21} \\
  L_1
\end{bmatrix} y(t),
$$

achieves non interaction and therefore solves (NICPM).

So, it remains to show that the system described by the matrices $A_g$, $B_g$, $C_g$, $G_{1,g}$, $G_{2,g}$, $H_{1,g}$ and $H_{2,g}$ together with the linear subspaces $S_{1,g}$, $S_{2,g}$, $V_{1,g}$ and $V_{2,g}$ satisfy the conditions of Theorem 3.3.

To that extent note that $\text{im} G_{1,g} = \text{im} G_1 \oplus \{0\} \subseteq S_1^* \oplus \{0\} = \text{im} \begin{bmatrix} D_1 \\ R \end{bmatrix} [X_0, X_1] \subseteq S_{1,g}$.

Analogously it can be shown that $\text{im} G_{2,g} \subseteq S_{2,g}$, $V_{1,g} \subseteq \ker H_{2,g}$ and $V_{2,g} \subseteq \ker H_{1,g}$.

Next we claim that $S_{1,g} \subseteq V_{1,g}$ and $S_{2,g} \subseteq V_{2,g}$.

It is clear that proving this claim is equivalent to proving that $T(E_1 D_1 + PR)X = 0$ and $T(E_2 D_2 + PR)X = 0$. Therefore, observe that

$$T(E_2 D_2 + PR)X =
\begin{bmatrix}
  T_0 X_0 & 0 & T_0 X_2 & T_0 D_{23} \\
  0 & 0 & 0 & 0 \\
  T_2 X_0 & 0 & T_2 X_2 & T_2 D_{23} \\
  E_{32} X_0 & E_{32} X_2 & E_{32} D_{23} & U_3 Y_3
\end{bmatrix}.
$$

Since $\begin{bmatrix} T_0 \\ T_2 \end{bmatrix} D_2 X = 0$, $T E_2 [X_0, X_2] = 0$ and $E_{32} D_{23} + U_3 Y_3 = 0$ there follows that $T(E_2 D_2 + PR)X = 0$. The latter implies also that $Q_T T(E_2 D_2 + PR)X Q_X \overset{\circ}{=} T(E_2 D_2 + PR)X$ where we used $T$, $X$, $Q_T$ and $Q_X$ as introduced in Remark 4.2.

Note that $RX = RX$, $TP = TP$, $D_1 X + D_2 X \overset{\circ}{=} D_1 X + D_2 X = X$ and $TE_1 + TE_2 = TE_1 + TE_2 = T$. Then there follows $0 = T(E_2 D_2 + PR)X = T(I - D_1 - E_1)X + T(E_1 D_1 + PR)X$. Since $T(D_1 + E_1 - I)X = 0$ we obtain that also $T(E_1 D_1 + PR)X = 0$. So $S_{1,g} \subseteq V_{1,g}$ and $S_{2,g} \subseteq V_{2,g}$.

Next we claim that the subspaces $S_{1,g}$, $S_{2,g}$ and $S_{1,g} + S_{2,g}$ are $(C_g, A_g)$-invariant subspaces.
in $\mathbb{R}^{n+g}$.

Indeed, because the matrix $Y_3$ is injective there holds

$$A_g(S_{1,g} \cap \ker C_g) = A_g \left( \begin{bmatrix} X_0 & X_1 & D_{13} \\ 0 & 0 & Y_3 \end{bmatrix} \cap (\ker C \oplus \{0\}) \right) =$$

$$= A_g \left( (\im [X_0, X_1] \cap \ker C) \oplus \{0\} \right) = \left( A(S_{1}^* \cap \ker C) \right) \oplus \{0\} \subseteq S_{1,g}^* \oplus \{0\} \subseteq S_{1,g}$$

and

$$A_g((S_{1,g} + S_{2,g}) \cap \ker C_g) = A_g \left( \begin{bmatrix} X_0 & X_1 & X_2 & X_3 & D_{13} \\ 0 & 0 & 0 & 0 & Y_3 \end{bmatrix} \cap (\ker C \oplus \{0\}) \right) =$$

$$= A_g \left( \im [X_0, X_1, X_2, X_3] \cap \ker C \right) \oplus \{0\} =$$

$$= (A(S^* \cap \ker C)) \oplus \{0\} \subseteq S^* \oplus \{0\} \subseteq \im \begin{bmatrix} X_0 & X_1 & X_2 & X_3 & D_{13} \\ 0 & 0 & 0 & 0 & Y_3 \end{bmatrix} = S_{1,g} + S_{2,g}.$$  

Hence, $S_{1,g}, S_{2,g}$ and $S_{1,g} + S_{2,g}$ are $(C_g, A_g)$-invariant subspaces in $\mathbb{R}^{n+g}$. Dually we can prove that $V_{1,g}, V_{2,g}$ and $V_{1,g} \cap V_{2,g}$ are $(A_g, B_g)$-invariant subspaces in $\mathbb{R}^{n+g}$.

Finally we have to prove that

$$\begin{bmatrix} A_g & 0 \\ 0 & -A_g \end{bmatrix} ((S_{1,g} \oplus S_{2,g}) \cap \ker [C_g, C_g]) \subseteq (V_{1,g} \oplus V_{2,g}) + \im \begin{bmatrix} B_g \\ B_g \end{bmatrix}.$$  

To that extent denote

$$\tilde{A}_0 = \begin{bmatrix} A_g & 0 \\ 0 & -A_g \end{bmatrix}, \quad \tilde{B}_0 = \begin{bmatrix} B_g \\ B_g \end{bmatrix}, \quad \tilde{C}_0 = [C_g, C_g],$$

$$\tilde{S} = \begin{bmatrix} D_1 X & 0 \\ R X & 0 \\ 0 & D_2 X \\ 0 & -R X \end{bmatrix} \quad \text{and} \quad \tilde{T} = \begin{bmatrix} TE_1 & TP & 0 & 0 \\ 0 & 0 & TE_2 & -TP \end{bmatrix}.$$  

Then the above inclusion reads $\tilde{A}_0(\im \tilde{S} \cap \ker \tilde{C}_0) \subseteq \ker \tilde{T} + \im \tilde{B}_0$. It is easy to see that the latter subspace inclusion is equivalent to $\tilde{A}_1 \ker \tilde{C}_1 \subseteq \im \tilde{B}_1$ where

$$\tilde{A}_1 = \tilde{T}\tilde{A}_0\tilde{S} = \begin{bmatrix} TE_1 AD_1 X & 0 \\ 0 & -TE_2 AD_2 X \end{bmatrix}, \quad \tilde{B} = \tilde{T}\tilde{B}_0 = \begin{bmatrix} TE_1 B & TP \\ TE_2 B & -TP \end{bmatrix}.$$
\[
\tilde{C}_1 = \tilde{C}_0 \tilde{s} = \begin{bmatrix} CD_1 X & CD_2 X \\ RX & -RX \end{bmatrix}.
\]

Now let \( \tilde{U}_1 = \begin{bmatrix} 1 & 0 \\ Q_X & I \end{bmatrix} \) and \( \tilde{V}_1 = \begin{bmatrix} I & Q_T \\ 0 & I \end{bmatrix} \). Then \( \tilde{A}_1 \ker \tilde{C}_1 \subseteq \text{im} \tilde{B}_1 \) if and only if \( \tilde{A}_2 \ker \tilde{C}_2 \subseteq \text{im} \tilde{B}_2 \) where \( \tilde{A}_2 = \tilde{V}_1 \tilde{A}_1 \tilde{U}_1 \), \( \tilde{B}_2 = \tilde{V}_1 \tilde{B}_1 \) and \( \tilde{C}_2 = \tilde{C}_1 \tilde{U}_1 \).

A straightforward calculation using \( TP = TP \), \( RX = RX \), \( D_1 X + D_2 \check{X} = X \) and \( TE_1 + TE_2 = T \) shows that
\[
\tilde{A}_2 = \begin{bmatrix} T(AD_1 + E_1 A - A)X & -\check{TE}_2 AD_2 X \\ -\check{TE}_2 AD_2 \check{X} & -\check{TE}_2 AD_2 X \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} TB & 0 \\ TE_2 B & -TP \end{bmatrix}
\]

and
\[
\tilde{C}_2 = \begin{bmatrix} CX & CD_2 X \\ 0 & -RX \end{bmatrix}.
\]

Recall that \( \begin{bmatrix} D_2 X \\ -RX \end{bmatrix} = \begin{bmatrix} X_0 & 0 & X_2 & D_{23} \\ 0 & 0 & 0 & Y_3 \end{bmatrix} \) with the matrix \( Y_3 \) injective and note that any vector in \( \ker \tilde{C}_2 \) consists of 8 "components".

Because of the injectivity of the matrix \( Y_3 \) the 8-th component of any vector in \( \ker \tilde{C}_2 \) is zero. This means that the 8-th column of \( \tilde{A}_2 \) does not play a role in the description of \( \tilde{A}_2 \ker \tilde{C}_2 \).

Furthermore there follows that the 6-th component of any vector in \( \tilde{C}_2 \) can be chosen completely arbitrarily. However, the contribution of this 6-th component to the description of \( \tilde{A}_2 \ker \tilde{C}_2 \) is annihilated because the 6-th column of \( \tilde{A}_2 \) is a zero column. Hence, we can delete the 6-th as well as the 8-th column in both \( \tilde{A}_2 \) and \( \tilde{C}_2 \) and still have a description of the subspace \( \tilde{A}_2 \ker \tilde{C}_2 \).

By dual arguments it can be shown that we can delete the 6-th as well as the 8-th row in both \( \tilde{A}_2 \) and \( \tilde{B}_2 \) and still we are left with a subspace inclusion equivalent to \( \tilde{A}_2 \ker \tilde{C}_2 \subseteq \text{im} \tilde{B}_2 \). In view of this all denote
\[
\tilde{A}_3 = \begin{bmatrix} T(AD_1 + E_1 A - A)X & -\check{TE}_2 AD_2 X' \\ -T'AD_2 \check{X} & -T'AX' \end{bmatrix}, \quad \tilde{B}_3 = \begin{bmatrix} TB \\ T'B \end{bmatrix}
\]

and \( \tilde{C}_3 = [CX, CX'] \)

where \( X' = [X_0, X_2] \) and \( T' = \begin{bmatrix} T_0 \\ T_2 \end{bmatrix} \).

Now there holds \( \tilde{A}_2 \ker \tilde{C}_2 \subseteq \text{im} \tilde{B}_2 \) if and only if \( \tilde{A}_3 \ker \tilde{C}_3 \subseteq \text{im} \tilde{B}_3 \). Let \( Q_X' \) and \( Q_T' \) be matrices of suitable dimensions such that \( XQ_X' = X' \) and \( Q_T'T = T' \) and let \( \tilde{U}_3 = \begin{bmatrix} I & Q_X' \\ 0 & I \end{bmatrix} \) and \( \tilde{V}_3 = \begin{bmatrix} I & 0 \\ -Q_T' & I \end{bmatrix} \). Then it follows that \( \tilde{A}_3 \ker \tilde{C}_3 \subseteq \text{im} \tilde{B}_3 \) if and only if
\[
\tilde{A}_4 \ker \tilde{C}_4 \subseteq \text{im} \tilde{B}_4 \quad \text{where} \quad \tilde{A}_4 = \tilde{V}_3 \tilde{A}_3 \tilde{U}_3, \quad \tilde{B}_4 = \tilde{V}_3 \tilde{B}_3 \quad \text{and} \quad \tilde{C}_4 = \tilde{C}_3 \tilde{U}_3.
\]

Again straightforward calculation shows that
\[ \tilde{A}_4 = \begin{bmatrix} T(AD_1 + E_1A - A)X & T''X' \\ T''AX & 0 \end{bmatrix}, \quad \tilde{B}_4 = \begin{bmatrix} TB \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{C}_4 = [CX, 0] \]

where \( X' = [X_0, 0] \) and \( T'' = \begin{bmatrix} T_0 \\ 0 \end{bmatrix} \).

Because of the structure of the matrices \( \tilde{A}_4, \tilde{B}_4 \) and \( \tilde{C}_4 \) it follows that \( \tilde{A}_4 \ker \tilde{C}_4 \subseteq \im \tilde{B}_4 \) is equivalent to \( T(E_1A + AD_1 - A)X \ker CX \subseteq \im TB, \quad \ker CX \subseteq \ker T''AX \) and \( \im T''AX \subseteq \im TB \).

Translated into the original subspaces the latter three conditions mean that

\[ (AD_1 + E_1A - A)(S^* \cap \ker C) \subseteq V^* + \im B, \quad A(S^* \cap \ker C) \subseteq V_1^* + V_2^* \quad \text{and} \quad A(S_1^* \cap S_2^*) \subseteq V^* + \im B. \]

Now of the latter three the first subspace inclusion holds by assumption while the second and third subspace inclusion hold because it can be shown that \( S^* \subseteq V_1^* + V_2^* \) and \( S_1^* \cap S_2^* \subseteq V^* \). Indeed, \( S^* = (D_1 + D_2 - (D_1 + D_2 - I))S^* \subseteq \subseteq D_1S^* + D_2S^* + (D_1 + D_2 - I)S^* \subseteq V_1^* + V_2^* + S_1^* \cap S_2^* = V_1^* + V_2^* + (D_1S_1^* \cap D_2S_2^*) \subseteq V_1^* + V_2^* + (D_1S_1^* \cap D_2S_2^*) = V_1^* + V_2^* \).

By dual arguments it can be shown that \( S_1^* \cap S_2^* \subseteq V^* \). Hence, the conditions of Theorem 3.3 are fulfilled and we have completed the proof of the (if)-part of Theorem 4.1.

(only if)

In order to prove the (only if)-part of Theorem 4.1 we assume that \( \text{(NICPM)} \) is solvable. That is, we assume that there exists a measurement feedback compensator \( \Sigma_c \) such that in the closed loop system \( \Sigma_c \) there holds \( H_{1,e}(sl - A_e)^{-1}G_{2,e} = 0 \) and \( H_{2,e}(sl - A_e)^{-1}G_{1,e} = 0 \).

Recall that \( R^{n+k} \) is the state space of \( \Sigma_c \) and that we denote

\[ A_e = \begin{bmatrix} A + BNC & BM \\ LC & K \end{bmatrix}, \quad G_i,e = \begin{bmatrix} G_i \\ 0 \end{bmatrix} \quad (i = 1, 2) \quad \text{and} \quad H_{i,e} = [H_{i,0}] \quad (i = 1, 2). \]

Let \( W_{1,e} \) and \( W_{2,e} \) be linear subspaces in \( R^{n+k} \) defined as

\[ W_{1,e} = <A_e \mid \im G_{1,e}> = \im G_{1,e} + A_e \im G_{1,e} + \cdots + A_e^{n+k-1} \im G_{1,e} \quad \text{and} \]

\[ W_{2,e} = <A_e \mid \im G_{2,e}> \]

Then we have \( \im G_{1,e} \subseteq W_{1,e} \subseteq \ker H_{2,e}, \im G_{2,e} \subseteq W_{2,e} \subseteq \ker H_{1,e}, \quad A_e W_{1,e} \subseteq W_{1,e} \quad \text{and} \quad A_e W_{2,e} \subseteq W_{2,e}. \)

But also there holds \( A_e(W_{1,e} + W_{2,e}) \subseteq (W_{1,e} + W_{2,e}) \) and \( A_e(W_{1,e} \cap W_{2,e}) \subseteq (W_{1,e} \cap W_{2,e}). \)
Let $S_1, S_2, S, V_1, V_2$ and $V$ be linear subspaces in $\mathbb{R}^n$ defined as

$$S_i = \{ x \in \mathbb{R}^n \mid \begin{bmatrix} x \\ 0 \end{bmatrix} \in W_{i,e} \}, \quad (i = 1,2),$$

$$S = \{ x \in \mathbb{R}^n \mid \begin{bmatrix} x \\ 0 \end{bmatrix} \in (W_{1,e} + W_{2,e}) \},$$

$$V_i = \{ x \in \mathbb{R}^n \mid \exists w \in \mathbb{R}^k : \begin{bmatrix} x \\ w \end{bmatrix} \in W_{i,e} \}, \quad (i = 1,2) \text{ and}$$

$$V = \{ x \in \mathbb{R}^n \mid \exists w \in \mathbb{R}^k : \begin{bmatrix} x \\ w \end{bmatrix} \in (W_{1,e} \cap W_{2,e}) \}.$$

Note that $S_i$ is the intersection of $W_{i,e}$ with $\mathbb{R}^n$, $(i = 1,2)$, $S$ is the intersection of $(W_{1,e} + W_{2,e})$ with $\mathbb{R}^n$, $V_i$ is the projection of $W_{i,e}$ onto $\mathbb{R}^n$ along $\mathbb{R}^k$, $(i = 1,2)$ and $V$ is the projection of $(W_{1,e} \cap W_{2,e})$ onto $\mathbb{R}^n$ along $\mathbb{R}^k$. From the latter it is immediate that

$$S_i \oplus \{0\} \subseteq W_{i,e} \subseteq V_i \oplus \mathbb{R}^k, \quad (i = 1,2),$$

$$(S_1 \cap S_2) \oplus \{0\} \subseteq (S_1 \oplus \{0\}) \cap (S_2 \oplus \{0\}) \subseteq (W_{1,e} \cap W_{2,e}) \subseteq V \oplus \mathbb{R}^k$$

and

$$S \oplus \{0\} \subseteq (W_{1,e} + W_{2,e}) \subseteq (V_1 \oplus \mathbb{R}^k) + (V_2 \oplus \mathbb{R}^k) = (V_1 + V_2) \oplus \mathbb{R}^k.$$

Clearly, there follows that $S_i \subseteq V_i$ $(i = 1,2)$, $S_1 \cap S_2 \subseteq V$ and $S \subseteq V_1 + V_2$.

Because the linear subspaces $W_{1,e}$, $W_{2,e}$ and $(W_{1,e} + W_{2,e})$ are $A_e$-invariant, it follows that the subspaces $S_1, S_2$ and $S$ are $(C, A)$-invariant subspaces. Indeed, let $x \in S_1 \cap \ker C$. Then

$$\begin{bmatrix} x \\ 0 \end{bmatrix} \in W_{1,e} \text{ and } A_e \begin{bmatrix} x \\ 0 \end{bmatrix} = A_e \begin{bmatrix} x \\ 0 \end{bmatrix} \in W_{1,e}. \text{ Hence } A(S_1 \cap \ker C) \subseteq S_1. \text{ Similarly, it can be shown that also } S_2 \text{ and } S \text{ are } (C, A)-\text{invariant subspaces.}$$

By dual reasoning it can be shown that due to the $A_e$-invariance of the linear subspaces $W_{1,e}$, $W_{2,e}$ and $(W_{1,e} \cap W_{2,e})$ the subspaces $V_1$, $V_2$ and $V$ are $(A, B)$-invariant subspaces.

Next observe that $\im G_1 \subseteq S_1$, $\im G_2 \subseteq S_2$, $V_1 \subseteq \ker H_2$, $V_2 \subseteq \ker H_1$, $\im [G_1, G_2] \subseteq S$ and $V \subseteq \ker \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$.

From the previous we may conclude that

$$S_1^* \subseteq V_1^*, \quad S_2^* \subseteq V_2^*, \quad S^* \subseteq V_1^* + V_2^* \quad \text{and} \quad S_1^* \cap S_2^* \subseteq V^*,$$

where we have used the notation as given in the beginning of the present section. If, in addition, we assume that the linear subspaces $S_1^*, S_2^*, S^*, V_1^*, V_2^*$ and $V^*$ have representation as indicated in Remark 4.2, then there follows that
\[
\begin{bmatrix}
X_0 & X_1 \\
0 & 0
\end{bmatrix} \subseteq W_{1,e}, \quad \begin{bmatrix}
X_0 & X_2 \\
0 & 0
\end{bmatrix} \subseteq W_{2,e}, \quad \begin{bmatrix}
X_0 & X_1 & X_2 & X_3 \\
0 & 0 & 0 & 0
\end{bmatrix} \subseteq (W_{1,e} + W_{2,e}),
\]

\[
W_{1,e} \subseteq \ker \begin{bmatrix}
T_0 & 0 \\
T_1 & 0
\end{bmatrix}, \quad W_{2,e} \subseteq \ker \begin{bmatrix}
T_0 & 0 \\
T_2 & 0
\end{bmatrix} \quad \text{and} \quad (W_{1,e} \cap W_{2,e}) \subseteq \ker \begin{bmatrix}
T_0 & 0 \\
T_1 & 0 \\
T_2 & 0 \\
T_3 & 0
\end{bmatrix}.
\]

So there exist matrices \(D_{13}, D_{23}, Y_3, E_{31}, E_{32}\) and \(U_3\) of suitable dimensions such that

\[
\begin{bmatrix}
X_3 \\
0
\end{bmatrix} = \begin{bmatrix}
D_{13} \\
Y_3
\end{bmatrix} + \begin{bmatrix}
D_{23} \\
-Y_3
\end{bmatrix} \quad \text{with} \quad \begin{bmatrix}
D_{13} \\
-Y_3
\end{bmatrix} \subseteq W_{1,e}, \quad \begin{bmatrix}
D_{23}
\end{bmatrix} \subseteq W_{2,e}
\]

and

\[
[T_3, 0] = [E_{31}, U_3] + [E_{32}, -U_3] \quad \text{with} \quad W_{1,e} \subseteq \ker [E_{31}, U_3], \quad W_{2,e} \subseteq \ker [E_{32}, -U_3].
\]

Hence, the matrices \(D_{13}, D_{23}, Y_3, E_{31}, E_{32},\) and \(U_3\) are such that

\[
\begin{bmatrix}
X_0 & X_1 & D_{13} \\
0 & 0 & Y_3
\end{bmatrix} \subseteq W_{1,e} \subseteq \ker \begin{bmatrix}
T_0 & 0 \\
T_1 & 0 \\
E_{31} & U_3
\end{bmatrix}
\]

\[
\begin{bmatrix}
X_0 & X_2 & D_{23} \\
0 & 0 & -Y_3
\end{bmatrix} \subseteq W_{2,e} \subseteq \ker \begin{bmatrix}
T_0 & 0 \\
T_2 & 0 \\
E_{32} & -U_3
\end{bmatrix}
\]

with \(D_{13} + D_{23} = X_3\) and \(E_{31} + E_{32} = T_3.\)

Let \(D_1, D_2, E_1, E_2 \in \mathbb{R}^{n \times n}\) and \(R, P^T \in \mathbb{R}^{k \times n}\) be matrices determined by

\(D_1 [X_0, X_1, X_2, X_3] = [X_0, X_1, 0, D_{13}],\) \quad \(D_2 [X_0, X_1, X_2, X_3] = [X_0, 0, X_2, D_{23}],\)

\(R [X_0, X_1, X_2, X_3] = [0, 0, 0, Y_3],\)

\[
\begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3
\end{bmatrix} E_1 = \begin{bmatrix}
T_0 \\
T_1 \\
0 \\
T_3
\end{bmatrix}, \quad \begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3
\end{bmatrix} E_2 = \begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
E_{32}
\end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix}
0 \\
0 \\
0 \\
U_3
\end{bmatrix}.
\]

It is clear that \((D_1 - I) [X_0, X_1] = 0,\) \((D_2 - I) [X_0, X_2] = 0\) and \((D_1 + D_2 - I) [X_0, X_1, X_2, X_3] = [X_0, 0, 0, 0]\) which means that \((D_1 - I) S_1^* = \{0\},\)

\((D_2 - I) S_2^* = \{0\}\) and \((D_1 + D_2 - I) S^* = S_1^* \cap S_2^*.\) Hence, \((D_1, D_2) \in \Phi(S_1^*, S_2^*, S^*).\)

Dually there follows that \((E_1^T, E_2^T) \in \Phi(V_1^{*L}, V_2^{*L}, V^{*L}).\) From the previous it is clear that
which imply \( T(E_1 D_1 + PR)X = 0 \) and \( T(E_2 D_2 + PR)X = 0 \).

In its turn the latter two expressions imply that for instance \( \omega = \[\cos \Theta_1 \sin \Theta_1 \cos \Theta_2 \cos \Theta_2 \sin \Theta_2 \] \( E, D, \) and \( PR \) is such that \( D_1 X \), \( D_2 X \) which means that \( D_1 S^* \subset V^*_R \).

Analogously it can be shown that \( D_2 S^* \subset V^*_R \), \( E_1 S_1^* \subset V^*_R \) and \( E_2 S_2^* \subset V^*_R \). Also there follows that \( Q_1 T(E_2 D_2 + PR)X Q_X = T(E_1 D_1 + PR)X = 0 \). Recall that \( D_2 X = X - D_1 X \), \( T(E_2 D_2 + PR)X = T(E_1 D_1 + PR)X + T(D_1_e + E_1 D_1 X) \). Hence, \( T(D_1_e + E_1 D_1 X) = 0 \) which means that \( (D_1 + E_1 D_1 X) S^* \subset V^*_R \).

Finally we recall that the subspaces \( W_{1,e} \) and \( W_{2,e} \) are \( A_e \)-invariant from which it is immediate that

\[
A_e \text{ im} \left[ \begin{array}{c} D_1 X \\ RX \end{array} \right] \subset \ker [TE_1, TP] \quad \text{and} \quad A_e \text{ im} \left[ \begin{array}{c} D_2 X \\ -RX \end{array} \right] \subset \ker [TE_2, TP].
\]

In turn these two inclusions are equivalent to

\[
[TE_1, TP] A_e \left[ \begin{array}{c} D_1 X \\ RX \end{array} \right] = T(E_1 A_1 D_1 X + [TE_1 B, TP]) \left[ \begin{array}{c} N \\ M \\ L \\ K \end{array} \right] \left[ \begin{array}{c} CD_1 X \\ RX \end{array} \right] = 0
\]

and

\[
[TE_2, TP] A_e \left[ \begin{array}{c} D_2 X \\ -RX \end{array} \right] = T(E_2 A_2 D_2 X + [TE_2 B, TP]) \left[ \begin{array}{c} N \\ M \\ L \\ K \end{array} \right] \left[ \begin{array}{c} CD_2 X \\ -RX \end{array} \right] = 0.
\]

Observe that the matrix \( \left[ \begin{array}{c} N \\ M \\ L \\ K \end{array} \right] \) is a common solution to two linear matrix equations. Therefore, by Theorem 3.1, it follows that \( \tilde{A}_1 \ker \tilde{C}_1 \subset \text{im} \tilde{B}_1 \) where we have denoted

\[
\tilde{A}_1 = \left[ \begin{array}{c} TE_1 A_1 D_1 X \\ 0 \\ 0 \\ -TE_2 A_2 D_2 X \end{array} \right], \quad \tilde{B}_1 = \left[ \begin{array}{c} TE_1 B \\ TP \\ TE_2 B \\ -TP \end{array} \right] \quad \text{and} \quad \tilde{C}_1 = \left[ \begin{array}{c} CD_1 X \\ CD_2 X \\ RX \\ -RX \end{array} \right].
\]

As in the (if)-part of the proof of Theorem 4.1 let

\[
\tilde{U}_1 = \left[ \begin{array}{c} I \\ 0 \\ Q_X \\ I \end{array} \right] \quad \text{and} \quad \tilde{V}_1 = \left[ \begin{array}{c} I \\ Q_T \\ 0 \\ I \end{array} \right].
\]

Then there holds that \( \tilde{A}_2 \ker \tilde{C}_2 \subset \text{im} \tilde{B}_2 \) where \( \tilde{A}_2 = \tilde{V}_1 \tilde{A}_1 \tilde{U}_1, \tilde{B}_2 = \tilde{V}_1 \tilde{B}_1 \) and \( \tilde{C}_2 = \tilde{C}_1 \tilde{U}_1 \).

Straightforward calculation shows that

\[
\text{im} \left[ \begin{array}{c} D_1 X \\ RX \end{array} \right] \subset W_{1,e} \subset \ker [TE_1, TP] \quad \text{and} \quad \text{im} \left[ \begin{array}{c} D_2 X \\ -RX \end{array} \right] \subset W_{2,e} \subset \ker [TE_2, TP].
\]
\[ \tilde{A}_2 = \begin{bmatrix} T(AD_1 + E_1A - A)X & -\tilde{T}E_2AD_2X \\ -\tilde{T}E_2AD_2X & -\tilde{T}E_2AD_2X \end{bmatrix}, \tilde{B}_2 = \begin{bmatrix} TB & 0 \\ TE_2B & -TP \end{bmatrix} \text{ and} \]

\[ \tilde{C}_2 = \begin{bmatrix} CX & CD_2X \\ 0 & -RX \end{bmatrix}. \]

Now note that

\[ \begin{bmatrix} T(AD_1 + E_1A - A)X \\ -\tilde{T}E_2AD_2X \end{bmatrix} \ker CX \subseteq \tilde{A}_2 \ker \tilde{C}_2 \subseteq \im \tilde{B}_2 \]

from which it is immediate that

\[ T(AD_1 + E_1A - A)X \ker CX \subseteq \im TB. \]

The latter means that

\[ (AD_1 + E_1A - A)(S^* \cap \ker C) \subseteq V^* + \im B. \]

This completes the proof of the (only if)-part of Theorem 4.1. \qed
5. Conclusions and Remarks

5.1. In this paper we studied systems that apart from a control input and a measurement output have two exogenous inputs and two exogenous outputs.

For this kind of systems we derived verifiable necessary and sufficient conditions for the existence of a measurement feedback compensator such that the resulting closed loop system has off-diagonal blocks equal to zero.

5.2. To end this section we shall give a conceptual algorithm that, if it exists, determines a compensator $\Sigma_c$ that achieves non interaction.

To that extent let the system $\Sigma$ be given.

Then the algorithm consists of the following steps where we use the notation as introduced in Remark 4.2.

1. Calculate the subspaces $S_1^*, S_2^*, S^*, V_1^*, V_2^*$, for instance by means of the algorithms mentioned in Section 3, and let these subspaces have representations as indicated in Remark 4.2.

2. Check whether or not there exist matrices $D_1, D_2, E_1$ and $E_2$ such that the conditions of Theorem 4.1 are fulfilled. See Remark 4.2. If these matrices do not exist, then (NICPM) is not solvable. If they do exist the algorithm continues as follows.

3. Determine matrices $P$ and $R$ as indicated in the proof of the (if)-part of Theorem 4.1.

4. Let $A_g, B_g, C_g, G_1,g, G_2,g, H_1,g$ and $H_2,g$ be matrices and let $S_{1,g}, S_{2,g}, V_{1,g}$ and $V_{2,g}$ be linear subspaces as described in the proof of the (if)-part of Theorem 4.1.

5. Compute matrices $F_g, J_g$ and $N_g$ such that $(A_g + B_g F_g) V_{i,g} \subseteq V_{i,g}$ ($i = 1, 2$), $(A_g + J_g C_g) S_{i,g} \subseteq S_{i,g}$ ($i = 1, 2$) and $(A_g + B_g N_g C_g) S_{i,g} \subseteq V_{i,g}$ ($i = 1, 2$).

The computation of for instance $N_g$ can be performed as follows. Let $\bar{X}_i$ ($i = 1, 2$) and $\bar{T}_i$ ($i = 1, 2$) be matrices such that $\text{im} \bar{X}_i = S_{i,g}$ ($i = 1, 2$) and $\ker \bar{T}_i = V_{i,g}$ ($i = 1, 2$).

Then it suffices to compute $N_g$ such that $\bar{T}_i A_g \bar{X}_i + \bar{T}_i B_g N_g C_g \bar{X}_i = 0$ ($i = 1, 2$). Using Kronecker products the latter two linear matrix equations can be written as one linear equation that can be solved using standard techniques. See also Van der Woude [7].

Next, rearrangement of the obtained solution provides $N_g$.

Similar remarks can be made with respect to the computation of $F_g$ and $J_g$.

6. Define $K_g = A_g + B_g F_g + J_g C_g - B_g N_g C_g$, $L_g = B_g N_g - J_g$, $M_g = F_g - N_g C_g$ and partition

$$N_g = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}, \quad M_g = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}, \quad L_g = [L_1, L_2]$$

where $M_1$, $[N_{11}, N_{12}]$ consist of $m$ rows and $L_1$, $[N_{11}]$ consist of $p$ columns.
(7) Define

\[ K = \begin{bmatrix} N_{22} & M_2 \\ L_2 & K_g \end{bmatrix}, \quad L = \begin{bmatrix} N_{21} \\ L_1 \end{bmatrix}, \quad M = [N_{12}, M_1] \quad \text{and} \quad N = N_{11}. \]

Then the matrices \( K, L, M \) and \( N \) constitute a measurement feedback compensator that achieves non interaction.
References


