A matrix-geometric analysis of queueing systems with periodic service interruptions
Eenige, van, M.J.A.; Resing, J.A.C.; van der Wal, J.

Published: 01/01/1993

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the author's version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

Citation for published version (APA):
Memorandum COSOR 93-32

A Matrix-Geometric Analysis of Queueing Systems with Periodic Service Interruptions

M.J.A. van Eenige
J.A.C. Resing
J. van der Wal

Eindhoven, November 1993
The Netherlands
A Matrix-Geometric Analysis of Queueing Systems with Periodic Service Interruptions

M.J.A. van Eenige, J.A.C. Resing and J. van der Wal
Eindhoven University of Technology
Department of Mathematics and Computing Science
P.O. Box 513
5600 MB Eindhoven
The Netherlands

Abstract

In this paper, two queueing models with periodic (cyclic) service interruptions are studied, one in discrete time and one in continuous time. For both models, the matrix-geometric approach is used to obtain the equilibrium distribution of the number of customers in the system. From this equilibrium distribution, one can compute the stationary sojourn time distribution and study the effects of interruptions on the probability a customer receives service before some specific due date. Examples show the influence on this probability of balancing interruptions over a cycle.

Keywords: Queueing systems, periodic service interruptions, matrix-geometric approach, sojourn times, tail probabilities, due dates.
1 Introduction

In production systems and real-time computer systems, the servicing of jobs can be interrupted. These interruptions may occur fairly random, for example, server breakdowns, rush orders, deficiencies of material or information. Or they may follow a more periodic pattern, in the case of periodic production schemes, safety maintenance or clocked software schedules.

An exact analysis of these systems seems to be possible only in a few exceptional cases (see, e.g., Jaiswal [7]). Therefore, most attention has been focused on approximating performance measures like the mean waiting time or the mean sojourn time. See for example Fischer [5], Federgruen and Green [3] and Bhat [1].

In this paper, we will analyze two queueing systems with interruptions. In both models, there is a periodic sequence of alternating time intervals during which the server can and cannot serve customers. Henceforth, a time interval during which the server is available (not available) to customers will be called an on-period (off-period). Furthermore, one such periodic sequence of on- and off-periods will be called a cycle.

The first model we will discuss is a discrete time model with scheduled interruptions in which each interruption will take a fixed amount of time. This model can be seen as an extension of the traffic-light queue discussed by Darroch (cf. [2]). Darroch considered a traffic-light queue with one green phase (on-period) and one red phase (off-period) per cycle, at which cars arrive at constant rate. By using time-discretization he obtained an expression for the generating function of the number of cars waiting in front of the traffic-light at the start of the green phase. We will extend this model to multiple on- and off-periods per cycle and varying arrival rates. However, instead of using the generating function technique, we will use the matrix-geometric approach introduced by Neuts (cf. [10]) to analyze this model.

The second model concerns a cycle with multiple on- and off-periods and varying arrival rates in continuous time. Moreover, the length of each on- and off-period will not be fixed, but will be generalized Erlang distributed. Just as for the discrete time model, we will use the matrix-geometric approach to analyze this model. Neuts (cf. [8] and [9]), and Federgruen and Green (cf. [4]) already studied a similar kind of model. Neuts considered an M|M|1 queue with varying arrival and service rates, which depend on the state of an underlying finite Markov chain. By applying the matrix-geometric approach, he obtained the stationary joint distribution of the queue length and the state of the Markov chain which regulates the arrival and service rates. Federgruen and Green studied a model with only one on- and off-period per cycle and without varying (arrival) rates, but with the length of an off-period generally distributed. They gave an algorithm to approximate the limiting joint distribution of the queue length and the phase in the on-period at service completion epochs.

The goal of this paper is to analyze the influence of interruptions on the sojourn time of a customer for these models. More specifically, we are interested in questions like:

- What are the effects of interruptions on the probability a customer receives service before some specific due date?
• What is the influence of balancing the off-periods in the cycle (e.g., by splitting an off­period into smaller ones which are rescheduled in the cycle) ?

The organization of the paper is as follows. In Section 2 we will describe the two models. Section 3 will be devoted to the analysis of the queue length process in these models. In Section 4, we will answer the questions stated above for some numerical examples. Some generalizations of the models are given in Section 5. Finally, in Section 6 we will give a summary.

2 The Models

In this section we will describe the two models in detail. The discrete time model with fixed lengths of the on- and off-periods (DT-model) will be given in subsection 2.1, the continuous time model with generalized Erlang distributions for the lengths of the on- and off-periods (CT-model) in subsection 2.2.

2.1 The DT-Model

We consider a cycle which is divided into \( N \), \( N \in \mathbb{N} \), intervals of equal length. Henceforth, one such interval will be called a slot and these slots will be numbered \( 1, 2, \ldots, N \). During each of these slots, the server is either available for service or not. The set \( A \) will denote the set of on-slots in the cycle, i.e., the set of slots in the cycle during which the server is Available to customers for service. The set of off-slots in the cycle, i.e., the set of slots in the cycle during which the service of customers is Blocked, will be denoted by \( B \).

We assume that in each slot at most one customer can arrive at the system and that arrivals in successive slots are independent. The probability of an arrival in a slot may depend on the number of the slot. With \( p_n \), \( 0 \leq p_n \leq 1 \), we denote the probability of an arrival in the \( n \)-th slot of the cycle. Since we can choose the length of a slot arbitrarily small, the assumption of at most one arrival per slot is not very restrictive (see also Section 5, where we comment on the situation with more than one arrival per slot).

The service time of a customer will be expressed in slots. This service time will have a discrete phase type distribution, as defined by Neuts (cf. [10]). More specifically, it is defined as the number of transitions until absorption in a finite Markov chain with \( M + 1 \) states, transition probability matrix \( \hat{P} \)

\[
\hat{P} = \begin{pmatrix}
\hat{p}_{1,1} & \hat{p}_{1,2} & \cdots & \hat{p}_{1,M+1} \\
\hat{p}_{2,1} & \hat{p}_{2,2} & \cdots & \hat{p}_{2,M+1} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{p}_{M,1} & \hat{p}_{M,2} & \cdots & \hat{p}_{M,M+1} \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]

and initial distribution \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M, 0) \), with \( \hat{p}_{i,j} \geq 0 \), \( \alpha_i \geq 0 \), \( \sum_{i=1}^{M+1} \hat{p}_{i,j} = 1 \) and \( \sum_{i=1}^{M} \alpha_i = 1 \) for all \( i \) and \( j \). The service times of different customers are assumed to be independent of each other. Moreover, the arrival process and the service times are assumed to be mutually independent.
Customers will be served in the order of their arrival. Clearly, it is possible that the service of a customer is interrupted by one or more off-periods. In that case, after an off-period, the service of the interrupted customer will be resumed at the place it was interrupted.

All events, i.e., customer arrivals, phase transitions and service completions, occur at slot boundaries. More specifically, for convenience, we assume that phase transitions and service completions occur just before a slot boundary and customer arrivals occur just after a slot boundary, see Figure 1. In the sequel, we will say that a customer departing (arriving) at the slot boundary between slot \((k - 1)\) and slot \(k\) (i.e., the \(k\)-th slot boundary) is departing in slot \((k - 1)\) (arriving in slot \(k\)). Furthermore, the service of a customer arriving at an empty system during an on-period can start in the slot in which he arrives.

Finally, we will assume that the system is stable. In other words, it will be assumed that the average amount of work brought into the system per cycle is strictly less than the service capacity of the system per cycle. Thus, when \(v^i_j\) denotes the expected number of visits to the transient state \(j\) in the Markov chain with probability transition matrix \(\hat{P}\), given the chain starts in state \(i\) we will demand that

\[
\sum_{n=1}^{N} \sum_{i=1}^{M} \alpha_i \sum_{j=1}^{M} v^i_j < |\mathcal{A}| = N - |\mathcal{B}|, \tag{1}
\]

where \(| \cdot |\) denotes the cardinality function.

### 2.2 The CT-Model

As in the previous subsection, we consider a cycle consisting of \(N\) intervals which are numbered 1, 2, \ldots, \(N\). These intervals are now called phases and the length of phase \(n\), \(n = 1, 2, \ldots, N\), is exponentially distributed with parameter \(\nu_n\). During each of these phases, the server is either available to customers or not. Once again, \(\mathcal{A}\) denotes the set of phases in the cycle during which the server can serve customers (i.e., the set of on-phases) and \(\mathcal{B}\) denotes the set of phases in the cycle during which the service of customers is blocked (i.e., the set of off-phases).

The arrival rate of customers may depend on the phase in the cycle. More specifically, during the \(n\)-th phase the arrival process is given by a Poisson process with rate \(\lambda_n\).

The service time of a customer will have a continuous phase type distribution, as defined by Neuts (cf. [10]). Thus, it is defined as the time until absorption in a finite, continuous time.
Markov chain with $M + 1$ states, infinitesimal generator $\hat{Q}$

$$\hat{Q} = \begin{pmatrix} \hat{q}_{1,1} & \hat{q}_{1,2} & \cdots & \hat{q}_{1,M+1} \\ \hat{q}_{2,1} & \hat{q}_{2,2} & \cdots & \hat{q}_{2,M+1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{q}_{M,1} & \hat{q}_{M,2} & \cdots & \hat{q}_{M,M+1} \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

and initial distribution $\beta = (\beta_1, \beta_2, \ldots, \beta_M, 0)$, with $\hat{q}_{i,j} \geq 0$ (i.e., $\hat{q}_{i,i} < 0, \sum_{i=1}^{M+1} \hat{q}_{i,j} = 0$) and $\sum_{i=1}^{M} \beta_i = 1$ for all $i$ and $j$.

Like in the DT-model, customers are served according to the FCFS-discipline and an interrupted service will be resumed at the place of interruption. Moreover, the arrival process, the service times and the lengths of the phases in the cycle are assumed to be independent.

Finally, we again assume that the system is stable. Thus, we demand that the average amount of work brought into the system per cycle is strictly less than the average service capacity of the system per cycle, i.e.,

$$\sum_{n=1}^{N} \nu_n \sum_{i=1}^{M} \beta_i \sum_{j=1}^{M} v_j^i \left( \frac{1}{-\hat{q}_{i,j}} \right) < \sum_{n \in \mathbb{N}} \frac{1}{\nu_n},$$

where $v_j^i$ denotes the expected number of visits to the transient state $j$ in the Markov chain with infinitesimal generator $\hat{Q}$, given this chain starts in state $i$.

### 3 The Queue Length Process

In this section we will analyse the queue length processes in the two models. Just as in the previous section, we start with the DT-model.

#### 3.1 The DT-Model

Let $(X_k, Y_k, Z_k)$ denote the number of customers in the system, the phase of the service of the customer in service and the number of the forthcoming slot in the cycle, respectively, at the $k$-th slot boundary (i.e., just after a possible service phase transition and just before a possible arrival (see Figure 1)). When the system is empty, we set $Y_k := 0$. Then, the process $\{(X_k, Y_k, Z_k)\}_{k \in \mathbb{N}}$ is a Markov chain with state space

$$S = \{(l, m, n) | l, m, n \in \mathbb{N}, 1 \leq m \leq M, 1 \leq n \leq N\} \cup \{(0, 0, n) | n \in \mathbb{N}, 1 \leq n \leq N\}.$$

This chain is periodic with period $N$ (due to the periodic pattern of the slots), irreducible and positive recurrent (because of assumption (1)), so there exists a stationary distribution of this process, which will be determined below.

The state space of this Markov chain is ordered lexicographically. Moreover, define for $l = 0, 1, 2, \ldots$, level $l$ as the set $S_l$ of all states for which there are $l$ customers in the system, i.e.,

$$S_0 = \{(0, 0, n) | n \in \mathbb{N}, 1 \leq n \leq N\}$$
and for $l = 1, 2, \ldots$

$$\mathcal{S}_l = \{(l, m, n) | m, n \in \mathbb{N}, 1 \leq m \leq M, 1 \leq n \leq N\}.$$ (4)

Then first we note that, for $l \geq 1$, $|\mathcal{S}_l| = M \cdot N$ and thus, independent of $l$. Second, we observe that for $l \geq 2$ the transition probability matrix from level $l$ to level $l'$, $l' = l - 1, l, l + 1$, is equal to the transition probability matrix from level $(l + h)$ to level $(l' + h)$ thus independent of $l$. Therefore, when we partition the state space $\mathcal{S}$ into the sets $\mathcal{S}_l$, $l = 0, 1, 2, \ldots$, we can write the transition probability matrix $P$ of this Markov chain into the following canonical form

$$P = \begin{pmatrix}
B_0 & B_1 & 0 & \cdots & 0 \\
B_2 & A_1 & A_0 & 0 & \cdots \\
0 & B_1 & A_0 & 0 & \cdots \\
& 0 & 0 & A_0 & \cdots \\
& & & & \ddots & \ddots & \ddots
\end{pmatrix},$$

where $A_0$, $A_1$ and $A_2$ are all $(M \cdot N) \times (M \cdot N)$ matrices and $B_0$, $B_1$ and $B_2$ are $N \times N$, $N \times (M \cdot N)$ and $(M \cdot N) \times N$ matrices, respectively. So, these matrices represent the transition probabilities from a level either to itself or to a neighbouring level. Since the determination of the elements of these matrices is straightforward, we will not give these elements here explicitly.

When $\pi$ denotes the row vector consisting of the stationary probabilities of the Markov chain, we can now apply the matrix-geometric approach to obtain this vector $\pi$, i.e., to solve the balance equations

$$\pi = \pi \cdot P$$

and normalize $\pi$ by

$$\pi \cdot e = 1,$$

where $e$ denotes the column vector with all its components equal to one. Let $\pi_l$, $l = 0, 1, 2, \ldots$, denote the row vector consisting of the stationary probabilities of the states $(l, m, n) \in \mathcal{S}_l$. Then, since the Markov chain is positive recurrent, we have by Theorem 1.3.2 in Neuts [10]

$$\pi_l = \pi_1 \cdot R^{l-1} \quad l \geq 1,$$

where the $(M \cdot N) \times (M \cdot N)$ matrix $R$ is the minimal nonnegative solution of the quadratic matrix equation

$$A_0 + R \cdot A_1 + R^2 \cdot A_2 = R.$$

The vectors $\pi_0$ and $\pi_1$ can be obtained from the boundary conditions

$$\pi_0 = \pi_0 \cdot B_0 + \pi_1 \cdot B_2$$

and

$$\pi_1 = \pi_0 \cdot B_1 + \pi_1 \cdot A_1 + \pi_2 \cdot A_2,$$

and the normalization.

We can use this stationary distribution $\pi$ to determine the performance measures of interest, such as the distribution function of the sojourn time of a customer arriving in a certain slot. For a specific example (see also Section 4) this will be done in Appendix A.
3.2 The CT-Model

The analysis of the CT-model is similar to the analysis of the DT-model. Let \((X_t, Y_t, Z_t)\) denote the number of customers in the system, the phase of the service of the customer in service and the number of the current phase in the cycle, respectively, at time \(t\), where we set \(Y_t := 0\) if the system is empty. Then, the process \(\{(X_t, Y_t, Z_t)\}_{t \geq 0}\) is a Markov chain with state space

\[
S = \{(l, m, n) | l, m, n \in \mathbb{N}, 1 \leq m \leq M, 1 \leq n \leq N\} \cup \{(0, 0, n) | n \in \mathbb{N}, 1 \leq n \leq N\}.
\]

Since this chain is irreducible and positive recurrent (due to assumption (2)), it contains a stationary distribution, which will be determined below.

Just as for the previous model, we order the state space lexicographically and define for \(l = 0, 1, 2, \ldots\), level \(l\) as the set \(S_l\) as defined in (3) and (4). We again observe for \(l \geq 2\) that the number of states at level \(l\) as well as the transition rate matrix from level \(l\) to level \(l'\), \(l' = l - 1, l, l + 1\), are independent of \(l\). Hence, when we partition the state space into the sets \(S_l, l = 0, 1, 2, \ldots\), the infinitesimal generator \(Q\) of the Markov chain can also be written in canonical form

\[
Q = \begin{pmatrix}
B_0 & B_1 \\
B_2 & A_1 & A_0 \\
A_2 & A_1 & A_0 \\
& & & \ddots & \ddots & \ddots
\end{pmatrix},
\]

where \(A_0, A_1\) and \(A_2\) are all \((M \cdot N) \times (M \cdot N)\) matrices and \(B_0, B_1\) and \(B_2\) are \(N \times N, N \times (M \cdot N)\) and \((M \cdot N) \times N\) matrices, respectively. Since the elements of these matrices can be easily determined, we omit this here.

Let \(\pi\) denote the row vector containing the stationary probabilities of this Markov chain. Now, by applying the matrix-geometric approach, we can obtain this vector \(\pi\), i.e., solve the balance equations

\[
\pi \cdot Q = 0
\]

and normalize this solution by

\[
\pi \cdot e = 1.
\]

For this purpose, we define \(\pi_l, l = 0, 1, 2, \ldots\), as the row vector consisting of the stationary probabilities of the states in the set \(S_l\). Since the Markov chain is positive recurrent, we can apply Theorem 1.7.1 in Neuts [10], i.e., we have

\[
\pi_l = \pi_1 \cdot R^{l-1} \quad l \geq 1,
\]

with \(R\) the minimal nonnegative solution of the quadratic matrix equation

\[
A_0 + R \cdot A_1 + R^2 \cdot A_2 = 0.
\]

The vectors \(\pi_0\) and \(\pi_1\) follow from the boundary conditions

\[
\pi_0 \cdot B_0 + \pi_1 \cdot B_2 = 0
\]
\begin{equation}
\pi_0 \cdot B_1 + \pi_1 \cdot A_1 + \pi_2 \cdot A_2 = 0,
\end{equation}
and the normalization.

As before we can use this stationary distribution \( \pi \) to obtain interesting quantities, for example the sojourn time distribution of a customer arriving in a certain phase in the cycle. For a specific example (see also Section 4) this will be done in Appendix B.

4 Numerical Results

In this section, we will evaluate for some numerical examples the change in performance when the off-periods of the server are more balanced in the cycle. As already mentioned in the introduction, we will use the probability that the service of an arbitrary arriving customer is completed before some specific due date as the performance measure.

4.1 Numerical Results for the DT-Model

Consider a machine which is used for internal production as well as production to order for customers. For internal production the machine is needed 50% of the time. It has to be decided which of the available 40 hours per week are used for internal production and which for the customer orders. Each hour (of the 40) at most one customer order arrives. The probability of an arrival is \( p \). Servicing of a customer job can only start at the hour. The service times are geometrically distributed with mean 1 hours.

We will focus on the performance for the customer orders and thus treat the hours used for internal production as the off-hours. We will compare 4 schedules for the on- and off-hours.

Let DT1 denote the schedule in which the machine is available for production during the first half of the week. In DT2 the machine operates at Monday, Wednesday-morning and Thursday. DT3 and DT4 are the schedules in which the machine is available every weekday-morning and every other hour, respectively. In Figure 2 we summarize these schedules graphically.

Given a certain due date, it is hard to predict for which of these schedules the machine will perform best. When considering relatively small due dates, a machine with less balanced off-periods might perform better than a machine with more balanced off-periods. For example, the probability a job, which needs 5 hours service, is served within 5 hours since its arrival is zero when the off-periods of the machine are governed by the schedules DT3 or DT4, but might be positive when the off-periods are governed by schedule DT1 or DT2. For larger due dates, however, one might expect that a machine with more balanced off-periods will perform better than a machine with less balanced off-periods.

For this model (i.e., Bernoulli arrivals and geometric service times), we derive in Appendix A the distribution function of the sojourn time of a job arriving in slot \( n \). The distribution function of an arbitrary job can be easily obtained from these distribution functions. Moreover, other performance measures, e.g., the waiting time distribution of an arbitrary job, can be obtained in a similar manner. In Table 1 and Table 2, we present for the four schedules the expected sojourn
Figure 2: Graphical representation of the on- and off-hours of the machine.

time and tail probabilities of the sojourn time, respectively, for an arbitrary job for different values of the load of the system $\rho$ (which is defined as the quotient of the expected amount of work arriving at the machine per week and the service capacity of this machine per week). In

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>DT1</th>
<th>DT2</th>
<th>DT3</th>
<th>DT4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>10.596</td>
<td>7.155</td>
<td>6.376</td>
<td>6</td>
</tr>
<tr>
<td>0.75</td>
<td>15.613</td>
<td>12.176</td>
<td>11.375</td>
<td>11</td>
</tr>
<tr>
<td>0.90</td>
<td>30.593</td>
<td>27.189</td>
<td>26.374</td>
<td>26</td>
</tr>
<tr>
<td>0.95</td>
<td>55.582</td>
<td>52.194</td>
<td>51.374</td>
<td>51</td>
</tr>
</tbody>
</table>

Table 1: Values for the expected sojourn time $E[S]$. 

these examples, we have fixed $q$ at 0.50 and chosen $p$ such that the indicated load of the system is obtained. We remark that the last column of Table 1 has been obtained by an exact analysis.

Table 1 shows that if the customer load increases the relative difference in mean sojourn time between the schedules decreases as could be expected. However, the absolute difference is almost, but not completely, constant. From Table 2 we see that the balanced schedules perform better than the unbalanced ones, i.e., for all $s$ and $\rho$, $Pr[S > s]$ for schedule DTi is larger than $Pr[S > s]$ for schedule DTj, if $i < j$. Remark that, for this example, the previous mentioned effect of unbalanced schedules performing better for small values of $s$ and balanced schedules performing better for large values of $s$ does not occur. This is probably due to the geometric service times. Furthermore, Table 2 shows that there is little difference between the schedules DT3 and DT4. The difference between DT1 and DT4 is larger but still moderate in the relevant areas (note that DT1 is not a serious option if the customers ask for due dates of 20 hours or less).
<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$s$</th>
<th>Pr[$S &gt; s$]</th>
<th>DT1</th>
<th>DT2</th>
<th>DT3</th>
<th>DT4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>2</td>
<td>0.7696</td>
<td>0.7604</td>
<td>0.7511</td>
<td>0.6939</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.6654</td>
<td>0.6089</td>
<td>0.5521</td>
<td>0.4806</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.4638</td>
<td>0.2166</td>
<td>0.1730</td>
<td>0.1588</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.1385</td>
<td>0.0326</td>
<td>0.0270</td>
<td>0.0249</td>
<td></td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.0023</td>
<td>0.0008</td>
<td>0.0007</td>
<td>0.0006</td>
<td></td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.0001</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>2</td>
<td>0.8713</td>
<td>0.8659</td>
<td>0.8602</td>
<td>0.8284</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.7996</td>
<td>0.7650</td>
<td>0.7291</td>
<td>0.6850</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.6230</td>
<td>0.4430</td>
<td>0.4016</td>
<td>0.3855</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.2899</td>
<td>0.1677</td>
<td>0.1531</td>
<td>0.1474</td>
<td></td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.0383</td>
<td>0.0245</td>
<td>0.0224</td>
<td>0.0215</td>
<td></td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.0055</td>
<td>0.0036</td>
<td>0.0033</td>
<td>0.0031</td>
<td></td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>0.0008</td>
<td>0.0005</td>
<td>0.0005</td>
<td>0.0005</td>
<td></td>
</tr>
<tr>
<td>0.90</td>
<td>2</td>
<td>0.9446</td>
<td>0.9423</td>
<td>0.9397</td>
<td>0.9261</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.9091</td>
<td>0.8937</td>
<td>0.8771</td>
<td>0.8568</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.8054</td>
<td>0.7133</td>
<td>0.6879</td>
<td>0.6771</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.5727</td>
<td>0.4800</td>
<td>0.4637</td>
<td>0.4568</td>
<td></td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.2546</td>
<td>0.2184</td>
<td>0.2110</td>
<td>0.2078</td>
<td></td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.1157</td>
<td>0.0994</td>
<td>0.0960</td>
<td>0.0946</td>
<td></td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>0.0526</td>
<td>0.0452</td>
<td>0.0437</td>
<td>0.0430</td>
<td></td>
</tr>
<tr>
<td>0.95</td>
<td>2</td>
<td>0.9716</td>
<td>0.9704</td>
<td>0.9691</td>
<td>0.9621</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.9525</td>
<td>0.9445</td>
<td>0.9358</td>
<td>0.9251</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.8930</td>
<td>0.8428</td>
<td>0.8281</td>
<td>0.8216</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.7491</td>
<td>0.6904</td>
<td>0.6789</td>
<td>0.6738</td>
<td></td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.4990</td>
<td>0.4643</td>
<td>0.4565</td>
<td>0.4531</td>
<td></td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.3354</td>
<td>0.3122</td>
<td>0.3070</td>
<td>0.3047</td>
<td></td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>0.2256</td>
<td>0.2099</td>
<td>0.2064</td>
<td>0.2049</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Tail probabilities of the sojourn time $S$.  

9
4.2 Numerical Results for the CT-Model

A production center consists of 2 machines. For internal use jobs are processed first on machine 1 then on machine 2. There is one operator for the two machines and the production requires the constant attention of the operator. On both machines the processing time of a job is exponential with mean 1. The operator works as follows. When $d$ jobs are ready at machine 1 he takes them to machine 2 and finishes them there. Then he returns to machine 1 and starts producing $d$ internal jobs again. So both machines are used only 50% of the time. Part of the remaining capacity at machine 1 is used for the production to order of relatively simple jobs that require only a negligible amount of attention from the operator. These jobs arrive according to a homogeneous Poisson process with rate $\lambda$ and have exponential service times with mean $\frac{1}{\mu}$. The internal production has priority, so if the operator returns from machine 2 and finds machine 1 busy with a customer job he interrupts this job. The processing is resumed when he leaves for machine 2 again.

We are interested in the performance of the customer jobs. Therefore we consider the time needed to produce $d$ internal jobs on machine 1 as $d$ off-phases and the time the operator spends at machine 2 as $d$ on-phases.

First, let CT1 denote the schedule of on- and off-periods of machine 1 for customer jobs with $d = 20$. For the second schedule (CT2), we consider the case $d = 10$. Finally, the schedules with $d = 4$ and $d = 1$ will be denoted by CT3 and CT4, respectively.

Just as for the numerical examples of the DT-model, it is hard to predict for which of these schedules the production center will perform best. For small due dates, the production center with relatively unbalanced on- and off-periods (i.e., high values of $d$) might outperform the center with highly balanced on- and off-periods (i.e., small values of $d$). On the other hand, when considering larger due dates, one might expect that balancing gives a better performance.

In Appendix B, we derive for this CT-model (i.e., homogeneous Poisson arrivals, exponential service times and the lengths of the phases identically distributed with mean 1) the sojourn time distribution for a customer job arriving in phase $n$. The sojourn time distribution for an arbitrary customer job can be obtained from these distribution functions. In Table 3 and Table 4, we present for the schedules above the expected sojourn time and tail probabilities of the sojourn time, respectively, for an arbitrary customer job for different values of the load of the production center $\rho$ (which is defined as the quotient of the expected amount of work of the customer jobs arriving at this center per cycle and the average service capacity of this production center for the service of these jobs per cycle). For these numerical examples, we have fixed $\mu$ at 0.50. The arrival rate $\lambda$ of the customer jobs is chosen such that the indicated load of the center is obtained. The last column of Table 3 has been obtained by an exact analysis.

Tables 3 and 4 show similar results as Tables 1 and 2, but the differences between the schedules are smaller.
Table 3: Values for the expected sojourn time $E[S]$.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>CT1</th>
<th>CT2</th>
<th>CT3</th>
<th>CT4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>12.680</td>
<td>10.236</td>
<td>9.228</td>
<td>9</td>
</tr>
<tr>
<td>0.75</td>
<td>21.576</td>
<td>19.192</td>
<td>18.221</td>
<td>18</td>
</tr>
<tr>
<td>0.90</td>
<td>48.507</td>
<td>46.167</td>
<td>45.218</td>
<td>45</td>
</tr>
<tr>
<td>0.95</td>
<td>93.483</td>
<td>91.159</td>
<td>90.217</td>
<td>90</td>
</tr>
</tbody>
</table>

Table 4: Tail probabilities of the sojourn time $S$.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$t$</th>
<th>$\Pr[S &gt; t]$</th>
<th>CT1</th>
<th>CT2</th>
<th>CT3</th>
<th>CT4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>2</td>
<td>0.8201</td>
<td>0.8168</td>
<td>0.8085</td>
<td>0.7965</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.7183</td>
<td>0.6998</td>
<td>0.6601</td>
<td>0.6387</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.5229</td>
<td>0.4118</td>
<td>0.3392</td>
<td>0.3293</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.2317</td>
<td>0.1285</td>
<td>0.1125</td>
<td>0.1092</td>
<td></td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.0212</td>
<td>0.0143</td>
<td>0.0124</td>
<td>0.0120</td>
<td></td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.0024</td>
<td>0.0016</td>
<td>0.0014</td>
<td>0.0013</td>
<td></td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>0.0003</td>
<td>0.0002</td>
<td>0.0001</td>
<td>0.0001</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td>2</td>
<td>0.9045</td>
<td>0.9028</td>
<td>0.8984</td>
<td>0.8923</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.8422</td>
<td>0.8318</td>
<td>0.8104</td>
<td>0.7987</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.6968</td>
<td>0.6273</td>
<td>0.5805</td>
<td>0.5729</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.4383</td>
<td>0.3540</td>
<td>0.3337</td>
<td>0.3293</td>
<td></td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.1384</td>
<td>0.1173</td>
<td>0.1102</td>
<td>0.1088</td>
<td></td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.0459</td>
<td>0.0338</td>
<td>0.0364</td>
<td>0.0359</td>
<td></td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>0.0152</td>
<td>0.0128</td>
<td>0.0120</td>
<td>0.0119</td>
<td></td>
</tr>
<tr>
<td>0.90</td>
<td>2</td>
<td>0.9604</td>
<td>0.9596</td>
<td>0.9579</td>
<td>0.9554</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.9322</td>
<td>0.9278</td>
<td>0.9188</td>
<td>0.9139</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.8576</td>
<td>0.8250</td>
<td>0.8039</td>
<td>0.7999</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.7051</td>
<td>0.6580</td>
<td>0.6439</td>
<td>0.6407</td>
<td></td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.4473</td>
<td>0.4224</td>
<td>0.4131</td>
<td>0.4110</td>
<td></td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.2870</td>
<td>0.2710</td>
<td>0.2650</td>
<td>0.2637</td>
<td></td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>0.1842</td>
<td>0.1739</td>
<td>0.1700</td>
<td>0.1692</td>
<td></td>
</tr>
<tr>
<td>0.95</td>
<td>2</td>
<td>0.9799</td>
<td>0.9795</td>
<td>0.9786</td>
<td>0.9774</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.9652</td>
<td>0.9630</td>
<td>0.9584</td>
<td>0.9559</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.9247</td>
<td>0.9081</td>
<td>0.8965</td>
<td>0.8943</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.8372</td>
<td>0.8107</td>
<td>0.8023</td>
<td>0.8000</td>
<td></td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>0.6673</td>
<td>0.6495</td>
<td>0.6425</td>
<td>0.6409</td>
<td></td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.5344</td>
<td>0.5201</td>
<td>0.5145</td>
<td>0.5133</td>
<td></td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>0.4280</td>
<td>0.4165</td>
<td>0.4121</td>
<td>0.4110</td>
<td></td>
</tr>
</tbody>
</table>
5 Generalizations of the Models

In this section we will briefly discuss three generalizations for the DT-model and two generalizations for the CT-model.

First, the DT-model can be generalized to a model in which the number of arrivals in a slot is generally distributed. As in the DT-model, the distribution of this number of arrivals in a slot may depend on the number of this slot. When defining the process \( \{ (X_k, Y_k, Z_k) \}_{k \in \mathbb{N}} \) as in Section 3.1, it can be easily seen that this process is Markovian with the same state space as defined in that section. By partitioning this state space into the levels \( S_l \) as defined in (4), we can write the transition probability matrix of this process into the following canonical form

\[
P = \begin{pmatrix}
B_0 & B_1 & B_2 & B_3 & \ldots \\
C_0 & A_1 & A_2 & A_3 & \ldots \\
A_0 & A_1 & A_2 & \ldots \\
A_0 & A_1 & \ldots \\
& & & \ddots & \ddots
\end{pmatrix}
\]

By applying the matrix-geometric approach for systems of the M|G|1-type as introduced by Neuts in [11], the stationary distribution of this process can be obtained. For the determination of this stationary distribution, we have to solve the matrix equation \( G = \sum_{i=0}^{\infty} A_i \cdot G^i \), which will take more effort than solving the quadratic matrix equation for the DT-model. However, when the stationary distribution is known, the determination of other performance measures is identical to the determination of these performance measures for the DT-model.

The second generalization for the DT-model is a model in which the phase transitions of the service of the customer in service may depend on the slot in the cycle. In this case, the matrix \( \hat{P} \) depends on the slot, but the stationary distribution of this process can be obtained in exactly the same manner as described in Section 3.1. The derivation of other performance measures, however, becomes more complex. A similar generalization can be applied to the CT-model.

As the third generalization of the DT-model, the length of the off-periods in the cycle may be generally distributed instead of deterministic, like in the model discussed by Federgruen and Green (cf. [4]). From the arrival rate during an off-period and the distribution function of the length of an off-period, one can derive the distribution function of the number of customers arriving during an off-period. Suppose that the \( i \)-th off-period is between the \( a_i \)-th on-slot and the \( (a_i + 1) \)-st on-slot in the cycle. For the analysis, the customers arriving during the \( i \)-th off-period are considered to arrive in the \( (a_i + 1) \)-st on-slot and the \( i \)-th off-period is removed from the cycle, for all \( i \). Now, introduce the process \( \{ (X_k, Y_k, Z_k) \}_{k \in \mathbb{N}} \) as defined in Section 3.1, where \( Z_k \) now denotes the number of the slot in the reduced cycle (i.e., the cycle without off-periods). This process is Markovian and the state space of this process can again be partitioned into levels \( l \) which are the states for which the number of customers is equal to \( l \). Then, we can
write the transition probability matrix of this process into the following canonical form

\[
P = \begin{pmatrix}
B_0 & B_1 & B_2 & B_3 & \cdots \\
C_0 & A_1 & A_2 & A_3 & \cdots \\
A_0 & A_1 & A_2 & \cdots \\
& & & & \ddots
\end{pmatrix}
\]

By applying the matrix-geometric approach as introduced in [11], we can obtain the stationary distribution of the process. Just as for the first generalization, the derivation of this stationary distribution takes more effort than the derivation of the stationary distribution for the DT-model. But, since the lengths of the off-periods are now random, the determination of other performance measures becomes far more complex. For the CT-model we can consider a similar generalization. Again, for the analysis we remove the off-phases from the cycle. However, the customers arriving during an off-period are now considered as a batch arrival at the phase transition of the cycle from the on-phase just before this off-period to the on-phase just after this on-period.

6 Summary

We have analysed two queueing models, one in discrete and one in continuous time, in which the server availability follows a periodic pattern. In the discrete time model a cycle consists of a fixed number of slots with Bernoulli arrivals and phase type service times. In the continuous time model the cycle contains a fixed number of exponential phases, the arrival process is Poisson and the service times are phase type. The server is available in a fixed subset of the slots or phases only. Using the matrix-geometric approach the equilibrium distribution of the number of customers in the system is obtained. From this performance quantities such as the sojourn time distribution can be computed. Two examples are presented that show how changes in the periodic pattern influence the performance. Various extensions concerning the arrivals and the off-times are possible.

Appendix A: The Sojourn Time Distribution for the DT-Model

In this appendix, we will derive an expression for the distribution function of the sojourn time of a customer arriving in a specific slot for the DT-model of Section 4.1. Let \( \Pr[S(n) = s] \) denote the probability that the sojourn time of a customer arriving in slot \( n \) is equal to \( s \). Because the service of a customer can not be completed in an off-period, this probability is obviously zero if \((n + s - 1) \mod N \in \mathcal{B}\). So, henceforth, we will assume that \((n + s - 1) \mod N \in \mathcal{A}\).

Let \( v(n, s) \) denote the number of on-slots between the start of slot \( n \) and the start of the \( s \)-th slot after slot \( n \). Due to the assumption \((n + s - 1) \mod N \in \mathcal{A}\), the sojourn time of a customer arriving in slot \( n \) is equal to \( s \) if and only if the amount of work in the system just after his
arrival (i.e., the sum of the amount of work in the system upon arrival and his service time) is equal to $v(n, s)$. So, when $V(n)$ denotes the amount of work just after the arrival in slot $n$ we have

$$\Pr[S(n) = s] = \Pr[V(n) = v(n, s)]$$

(5)

Let $x(n)$ denote the number of customers present in the system at the $n$-th slot boundary. Then, conditioned on $x(n) = l$, the total amount of work in the system just after the arrival in slot $n$ is the $(l+1)$-fold convolution of the geometric service time (with mean $\frac{1}{\varrho}$), i.e.,

$$\Pr[V(n) = v|x(n) = l] = \left(\frac{v - 1}{l}\right)^{l+1}(1 - q)^{v-(l+1)}.$$  

(6)

Further, applying the Bernoulli-arrivals-see-time-average property (cf. [6]), we have

$$\Pr[x(n) = l] = N\pi_l(n)$$

where $\pi_l(n)$ denotes the stationary probability of the state $(l, n) := (l, 1, n)$ as obtained in Section 3.1.

Now, by conditioning (5) on the number of customers present upon arrival in slot $n$, we have

$$\Pr[S(n) = s] = \sum_{l=0}^{v(n, s)-1} \left(\frac{v(n, s) - 1}{l}\right)^{l+1}(1 - q)^{v(n, s)-(l+1)}N\pi_l(n).$$

The distribution function of the sojourn time of a customer arriving in slot $n$ can now be easily obtained from

$$\Pr[S(n) \leq D] = \sum_{s=1}^{D} \Pr[S(n) = s] \quad D = 0, 1, \ldots$$

Appendix B: The Sojourn Time Distribution for the CT-Model

In this appendix we will derive an expression for the distribution function of the sojourn time of a customer for the CT-model of Section 4.2. Let $s_n(\cdot)$ denote the probability density function (p.d.f.) of the sojourn time of a customer arriving in phase $n$. We obtain this p.d.f. by conditioning on $x(n)$, the number of customers present upon arrival, on $v(n)$, the amount of work in the system just after the arrival (i.e., the sum of the amount of work in the system upon arrival and the service time of the arriving customer), and on $A(n)$, the number of on-phases needed to complete the amount of work $v(n)$.

By using the PASTA property and because the lengths of the phases are identically distributed, we have

$$\Pr[x(n) = l] = N\pi_l(n),$$

where $\pi_l(n)$ denotes the stationary probability of the state $(l, n) := (l, 1, n)$ as obtained in Section 3.2.
Furthermore, let $v_n(\cdot)$ denote the p.d.f. of $v(n)$. If $x(n) = l$ then, since the service time of a customer is exponentially distributed with parameter $\mu$, the amount of work in the system just after the arrival is Erlang distributed with $(l + 1)$ phases and parameter $\mu$, i.e.,

$$v_n(v|\nu = l) = \frac{\mu(\nu)^l}{l!} \cdot e^{-\nu}.$$

Finally, since the lengths of the on-phases are exponentially distributed with mean 1, we have, for $a \geq 1$,

$$\Pr[A(n) = a|v(n) = v, x(n) = l] = \frac{v^{a-1}}{(a-1)!} \cdot e^{-v}.$$

Then, we have

$$s_n(s) = \sum_{a=0}^{\infty} \sum_{a=1}^{\infty} s_n(s|A(n) = a, v(n) = v, x(n) = l) = \frac{\frac{v^{a-1}}{(a-1)!} \cdot e^{-v} \mu(\nu)^l}{l!} e^{-\nu} N \pi_1(n) \, dv. \quad (7)$$

Clearly, $s_n(s|A(n) = a, v(n) = v, x(n) = l) = 0$ if $v > s$ and hence we only have to integrate from 0 to $s$ in (7).

Just as for the DT-model, the arrival phase $n$ and the number of on-phases $A(n)$ completely determine the number of off-phases during the sojourn time. Let $B(n, A(n))$ denote this number of off-phases. Clearly, if $B(n, A(n)) = 0$, the sojourn time is just equal to the amount of work in the system just after the arrival in phase $n$, i.e., $v(n)$. On the other hand, if $B(n, A(n)) \geq 1$, the sojourn time is equal to the sum of $v(n)$ and the total length of these $B(n, A(n))$ off-phases.

In order to make a distinction between values of $A(n)$ for which $B(n, A(n)) = 0$ and for which $B(n, A(n)) \geq 1$, we introduce $A^+_n$ as the value of $A(n)$ for which

$$B(n, A(n)) = 0 \quad \text{and} \quad B(n, A(n) + 1) > 0,$$

if $n \in A$ and set $A^+_n = 0$ if $n \in B$. Note that this value $A^+_n$ is unique since $B(n, a) = 0$ implies $B(n, c) = 0$ for $c = 1, 2, \ldots, a - 1$. Since the length of an off-phase is exponentially distributed with mean 1, the total length of $B(n, A(n)) \geq 1$ off-phases is Erlang distributed with $B(n, A(n))$ phases and parameter 1. Hence, when $b(\cdot|B(n, A(n)))$ denotes the p.d.f. of this total length, we have

$$b(s|B(n, A(n))) = \frac{s^{B(n, A(n))-1}}{(B(n, A(n)) - 1)!} \cdot e^{-s}. \quad (8)$$

Since $B(n, a) = 0$ for $a \leq A^+_n$, we have in that case that the sojourn time is equal to $v(n)$. For $a \geq A^+_n + 1$ we have that $B(n, a) > 0$. Hence, when $v(n) = v \leq s$ is given, the sojourn time can only be equal to $s$ if the total length of these $B(n, a)$ off-phases is equal to $(s - v)$. Thus, when using all these observations, equation (7) becomes

$$s_n(s) = e^{-(1+\nu)s} \cdot N \sum_{l=0}^{\infty} \frac{\mu(\nu)^l}{l!} \cdot \pi_1(n) \sum_{a=1}^{A^+_n} \frac{s^{a-1}}{(a-1)!} + \int_0^s \sum_{l=0}^{\infty} \frac{\mu(\nu)^l}{l!} e^{-\nu} N \pi_1(n) \sum_{a=A^+_n+1}^{\infty} \frac{v^{a-1}}{(a-1)!} e^{-v} \frac{(s-v)^{B(n,a)-1}}{(B(n,a) - 1)!} e^{-(s-v)} \, dv.$$
The distribution function $S_n(t)$ of the sojourn time of a customer arriving in phase $n$ can now be obtained by integrating the p.d.f. $s_n(s)$ from 0 to $t$. By using Newton’s binomial formula and the identity

$$
\int_0^t \frac{\theta \cdot (\theta \cdot x)^h}{h!} \cdot e^{-\theta x} \, dx = \sum_{j=h+1}^{\infty} \frac{(\theta \cdot t)^j}{j!} \cdot e^{-\theta t},
$$

we are able to remove the integrals in the expression for $S_n(t)$ and write it into summations only. This seems useful, because it avoids numerical integration. However, when implementing this expression with only summations, it appears to be unstable due to the alternating terms which arise from applying Newton’s binomial formula to $(s - v)^{B(n,a)-1}$. For this reason, we have chosen for another, but stable way of implementing the expression for $S_n(t)$.

Instead of considering $S_n(t)$, we now consider $1 - S_n(t)$, i.e.,

$$
1 - S_n(t) = \int_t^{\infty} e^{-(1+\mu)x} \cdot N \sum_{l=0}^{\infty} \frac{\mu^l}{l!} \cdot \pi_l(n) \sum_{a=1}^{A^*_n} \frac{x^{a-1}}{(a-1)!} \, dx +
\int_t^{\infty} \int_0^{x} \frac{\mu^l}{l!} e^{-\mu v} N \pi_l(n) \sum_{a=1}^{A^*_n+1} \frac{v^{a-1}}{(a-1)!} (B(n,a) - 1)! \cdot \frac{(z - v)^{B(n,a)-1}}{(a-1)!} \cdot e^{-x} \, dv \, dx.
$$

Since we do not have to use Newton’s binomial formula for the first part of the right hand side of this equation, we write this part into summations only by using identity (9). The second part of the right hand side is split into two parts. More specifically, the integral from 0 to $x$ is split into an integral from 0 to $t$ and one from $t$ to $x$. We can write the first of these parts also in summations only by interchanging the integrals and using identity (9). For the second part, we also interchange the integrals, but now use (9) only to remove the inner integral. The remaining expression is obtained by chopping of the infinite sums and by numerical integration. In this way, we remove the numerical instability.

References


<table>
<thead>
<tr>
<th>Number</th>
<th>Month</th>
<th>Author(s)</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>93-01</td>
<td>January</td>
<td>P. v.d. Laan, C. v. Eeden</td>
<td>Subset selection for the best of two populations: Tables of the expected subset size</td>
</tr>
<tr>
<td>93-03</td>
<td>February</td>
<td>Jan Beirlant, John H.J. Einmahl</td>
<td>Asymptotic confidence intervals for the length of the shortest under random censoring.</td>
</tr>
<tr>
<td>93-04</td>
<td>February</td>
<td>E. Balas, J. K. Lenstra, A. Vazacopoulos</td>
<td>One machine scheduling with delayed precedence constraints</td>
</tr>
<tr>
<td>93-05</td>
<td>March</td>
<td>A.A. Stoorvogel, J.H.A. Ludlage</td>
<td>The discrete time minimum entropy $H_\infty$ control problem</td>
</tr>
<tr>
<td>93-06</td>
<td>March</td>
<td>H.J.C. Huijberts, C.H. Moog</td>
<td>Controlled invariance of nonlinear systems: Nonexact forms speak louder than exact forms</td>
</tr>
<tr>
<td>93-07</td>
<td>March</td>
<td>Marinus Veldhorst</td>
<td>A linear time algorithm to schedule trees with communication delays optimally on two machines</td>
</tr>
<tr>
<td>93-08</td>
<td>March</td>
<td>Stan van Hoesel, Antoon Kolen</td>
<td>A class of strong valid inequalities for the discrete lot-sizing and scheduling problem</td>
</tr>
<tr>
<td>93-09</td>
<td>March</td>
<td>F.P.A. Coolen</td>
<td>Bayesian decision theory with imprecise prior probabilities applied to replacement problems</td>
</tr>
<tr>
<td>93-10</td>
<td>March</td>
<td>A.W.J. Kolen, A.H.G. Rinnooy Kan, C.P.M. van Hoesel, A.P.M. Wagelmans</td>
<td>Sensitivity analysis of list scheduling heuristics</td>
</tr>
<tr>
<td>93-11</td>
<td>March</td>
<td>A.A. Stoorvogel, J.H.A. Ludlage</td>
<td>Squaring-down and the problems of almost-zeros for continuous-time systems</td>
</tr>
<tr>
<td>93-12</td>
<td>April</td>
<td>Paul van der Laan</td>
<td>The efficiency of subset selection of an $\varepsilon$-best uniform population relative to selection of the best one</td>
</tr>
<tr>
<td>93-13</td>
<td>April</td>
<td>R.J.G. Wilms</td>
<td>On the limiting distribution of fractional parts of extreme order statistics</td>
</tr>
<tr>
<td>Number</td>
<td>Month</td>
<td>Author</td>
<td>Title</td>
</tr>
<tr>
<td>--------</td>
<td>--------------</td>
<td>-------------------------------</td>
<td>----------------------------------------------------------------------</td>
</tr>
<tr>
<td>93-14</td>
<td>May</td>
<td>L.C.G.J.M. Habets</td>
<td>On the Genericity of Stabilizability for Time-Day Systems</td>
</tr>
<tr>
<td>93-15</td>
<td>June</td>
<td>P. van der Laan C. van Eeden</td>
<td>Subset selection with a generalized selection goal based on a loss function</td>
</tr>
<tr>
<td>93-16</td>
<td>June</td>
<td>A.A. Stoorvogel A. Saberi B.M. Chen</td>
<td>The Discrete-time $H_\infty$ Control Problem with Strictly Proper Measurement Feedback</td>
</tr>
<tr>
<td>93-17</td>
<td>June</td>
<td>J. Beirlant J.H.J. Einmahl</td>
<td>Maximal type test statistics based on conditional processes</td>
</tr>
<tr>
<td>93-18</td>
<td>July</td>
<td>F.P.A. Coolen</td>
<td>Decision making with imprecise probabilities</td>
</tr>
<tr>
<td>93-19</td>
<td>July</td>
<td>J.A. Hoogeveen J.K. Lenstra B. Veltman</td>
<td>Three, four, five, six or the Complexity of Scheduling with Communication Delays</td>
</tr>
<tr>
<td>93-20</td>
<td>July</td>
<td>J.A. Hoogeveen J.K. Lenstra B. Veltman</td>
<td>Preemptive scheduling in a two-stage multiprocessor flow shop is NP-hard</td>
</tr>
<tr>
<td>93-21</td>
<td>July</td>
<td>P. van der Laan C. van Eeden</td>
<td>Some generalized subset selection procedures</td>
</tr>
<tr>
<td>93-22</td>
<td>July</td>
<td>R.J.G. Wilms</td>
<td>Infinite divisible and stable distributions modulo 1</td>
</tr>
<tr>
<td>93-23</td>
<td>July</td>
<td>J.H.J. Einmahl F.H. Ruymgaart</td>
<td>Tail processes under heavy random censorship with applications</td>
</tr>
<tr>
<td>93-24</td>
<td>August</td>
<td>F.W. Steutel</td>
<td>Probabilistic methods in problems of applied analysis</td>
</tr>
<tr>
<td>93-25</td>
<td>August</td>
<td>A.A. Stoorvogel</td>
<td>Stabilizing solutions of the $H_\infty$ algebraic Riccati equation</td>
</tr>
<tr>
<td>93-26</td>
<td>August</td>
<td>R. Perelaer J.K. Lenstra M. Savelsbergh F. Soumis</td>
<td>The Bus Driver Scheduling Problem of the Amsterdam Transit Company</td>
</tr>
<tr>
<td>93-28</td>
<td>September</td>
<td>H.J.C. Huijberts H. Nijmeijer</td>
<td>Dynamic disturbance decoupling of nonlinear systems and linearization</td>
</tr>
<tr>
<td>Number</td>
<td>Month</td>
<td>Author</td>
<td>Title</td>
</tr>
<tr>
<td>--------</td>
<td>-----------</td>
<td>-------------------------------</td>
<td>-----------------------------------------------------------------------</td>
</tr>
<tr>
<td>93-29</td>
<td>September</td>
<td>L.C.G.J.M. Habets</td>
<td>Testing Reachability and Stabilizability of Systems over Polynomial Rings using Gröbner Bases</td>
</tr>
<tr>
<td>93-30</td>
<td>September</td>
<td>Z. Liu, J.A.C. Resing</td>
<td>Duality and Equivalencies in Closed Tandem Queueing Networks</td>
</tr>
<tr>
<td>93-31</td>
<td>November</td>
<td>A.G. de Kok, J. van der Wal</td>
<td>Assigning identical operators to different machines</td>
</tr>
<tr>
<td>93-32</td>
<td>November</td>
<td>M.J.A. van Eenige, J.A.C. Resing, J. van der Wal</td>
<td>A Matrix-Geometric Analysis of Queueing Systems with Periodic Service Interruptions</td>
</tr>
</tbody>
</table>