Markov decision processes and quasi-martingales

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Markov decision processes and
Quasi-Martingales

by

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Eindhoven, February 1976

The Netherlands
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0. Abstract

It is showed in this paper that quasi-(super)martingales play an important role in the theory of Markov decision processes.

For excessive functions (with respect to a charge) it is proved that the value of the state at time \( t \) converges almost surely under each Markov strategy, which implies that the value function in the state at time \( t \) converges to zero (a.s), if an optimal strategy is used. At last a characterization of the conserving and equalizing properties is formulated using martingale theory.

1. Introduction

In this section the framework of convergent dynamic programming, see Hordijk (1974a, 1974b), is sketched.

A Markov decision process will be a triple \((S, P, r)\) where \( S \) is a countable set, called the state space, \( r \) is a real measurable function on \( S \times P \), called the reward function and \( P \) is a Borel subset of \( E \) where \( E \) is the set of all Markov transition functions on \( S \), i.e.

\[
P \in E, \text{ implies } P : S \times S \to [0,1], \sum_{j \in S} P(i,j) \leq 1 \text{ for all } i \in S.
\]

It is assumed that \( E \) is endowed with a metric such that \( E \) is a Polish space.

We will use the following notational convention for functions \( g \) on \( S \times P \):

\[
g_p(i) := g(i,P) \text{ and we assume that all functions on } S \times P \text{ are measurable on } P.
\]

It is assumed that \( P \) and \( r \) have the following (product) properties:

let \( P_1, P_2, P_3, \ldots \in P \) and \( A_1, A_2, A_3, \ldots \subseteq S \) then there is a \( P \in P \) such that for all \( i \in S \):

\[
P(i, \cdot) = P_k(i, \cdot) \text{ if } i \in A_k \text{ and } r_p(i) = r_{P_1}(i) \text{ if } P_1(i, \cdot) = P_2(i, \cdot).
\]

A Markov strategy is a sequence \((P_0, P_1, P_2, \ldots)\) with \( P_i \in P_i \); the set
of all Markov strategies is denoted by $M$. The state space $S$ is extended to $S^*$ by adding a state $p$ such that $S^* := S \cup \{p\}$, and all $P \in P$ are extended to $S^*$ by: $P(i,p) := 1 - \sum_{j \in S} P(i,j)$ and $P(p,p) := 1$.

All functions on $S$ are extended to $S^*$ by defining them 0 in $p$.

Let $F_n$ be the usual $\sigma$-field on $(S^*)^{\infty}$ generated by the first $n+1$ coordinates of the paths, $n = 0, 1, 2, \ldots$, and $F$ is the $\sigma$-field generated by $\bigcup_{n=0}^{\infty} F_n$.

Now $(X_n, n = 0, 1, 2, \ldots)$ is a stochastic process on $((S^*)^{\infty}, F_{i,R})$ for each $(i,R)$, and for all $\omega \in (S^*)^{\infty}$, $X_n(\omega)$ selects the $(n+1)$-th coordinate of $\omega$.

The expectation with respect to $P_{i,R}$ is denoted by $E_{i,R}$. Any $(\Omega,F)$-measurable function $f$ is said to be integrable w.r.t. $F$ if at least one of the terms $E_{i,R}f^+$ and $E_{i,R}f^-$ is finite, and summable if both are finite.

If $f(X_n)$ is integrable w.r.t. $P_{i,R}$, the expectation may be evaluated as follows: $E_{i,R}[f(X_n)] = P_0 \cdots P_{n-1} f(i)$ for $R = P_0, P_1, \ldots$ (an empty product of Markov transition functions is defined as the identity operator).

**Definition 1.1.**

i) A function $g : S \times P \to \mathbb{R}$ is called a charge iff

$$E_{i,R} \left[ \sum_{n=0}^{\infty} |g_{P_n}(X_n)| \right] < \infty$$

for all $i \in S$, $R \in M$.

Let $g$ be such a charge.

ii) A function $f : S \to \mathbb{R}$ is called superharmonic w.r.t. (a charge) $g$ iff $f(X_n)$ is integrable w.r.t. $P_{i,R}$ for all $i,R,n$ and

$$f \geq g_{P} + Pf$$

for all $P \in P$.

iii) A function $f : S \to \mathbb{R}$ is called excessive w.r.t. (a charge) $g$ iff $f$ is superharmonic w.r.t. $g$ and

$$f(i) \geq \sum_{n=0}^{\infty} E_{i,R}[g_{P_n}(X_n)]$$

for all $i,R$. 
Assumption 1.2.

i) The reward function \( r \) is a charge and

\[
\sup_{R \in M} \mathbb{E}_{i, R}[\sum_{n=0}^{\infty} r_{P_n}^+ (X_n)] < \infty
\]

(recall: \( x^+ := \max(0, x) \), \( x^- := (-x)^+ \)).

Definition 1.3.

i) The value function \( v \) of \((S, P, r)\) is a real function on \( S \):

\[
v(i) := \sup_{R \in M} \mathbb{E}_{i, R}[\sum_{n=0}^{\infty} r_{P_n} (X_n)] \quad \text{for all } i \in S
\]

ii) A strategy \( R \in M \) is called optimal if this supremum is attained for \( R \) in all \( i \in S \).

In this paper it will be shown, using some theory on supermartingales, that each superharmonic function \( f \) with w.r.t. charge \( g \) has the property that \( f(X_n) \) converges \( \mathbb{P}_{i, R} \) almost surely for all \( i \) and \( R \). Especially the value function \( v(X) \) converges almost surely to zero under each optimal strategy. As last result we give a slight extension of the following theorem of Hordijk (1974a), which is based on a result of Dynkin and Juschkewitsch (1969): if \( \tau_1 \) and \( \tau_2 \) are stopping times for the sequence of \( \sigma \)-fields \( \{F_n, n = 0, 1, \ldots\} \) then \( \tau_1 \leq \tau_2 \mathbb{P}_{i, R} \)-a.s. implies:

\[
\mathbb{E}_{i, R}[v(X_{\tau_1}) + \sum_{k=0}^{\tau_1} r_{P_k} (X_k)] \geq \mathbb{E}_{i, R}[v(X_{\tau_2}) + \sum_{k=0}^{\tau_2} r_{P_k} (X_k)].
\]

In section 2 some theory on quasi-martingales is developed and in section 3 this theory is applied to Markov decision processes. Most of the lemmas used in this paper are well-known but the authors do not know of any place in the literature where the facts were combined to get the results mentioned above.
2. Quasi-martingales

Quasi-martingales have been introduced by Fisk (1965) as continuous time stochastic processes having a decomposition into the sum of a martingale and a process having almost all sample functions of bounded variation. In this paper we give essentially the same definition for the discrete time case.

Let $\mathbb{N}$ be the set $\{0, 1, 2, \ldots\}$ and let $(\mathcal{G}, \mathcal{A}, \mathbb{P})$ be a probability space and $(\mathcal{A}_t, t \in \mathbb{N})$ an increasing sequence of $\sigma$-fields contained in $\mathcal{A}$. All stochastic processes in this section are defined on $(\mathcal{G}, \mathcal{A}, \mathbb{P})$ and have values in the set of real numbers with the Borel-$\sigma$-field on it. Moreover they are adapted to $(\mathcal{A}_t, t \in \mathbb{N})$, i.e. the $\sigma$-field generated by the first $n+1$ coordinates of the paths is a subset of $\mathcal{A}_n$, $n \in \mathbb{N}$ (The conditional expectation w.r.t. $\mathcal{A}_t$ is denoted by $\mathbb{E}^{\mathcal{A}_t}$).

**Definition 2.1.**

Let $\{B_t, t \in \mathbb{N}\}$ be a stochastic process such that $\sum_{t \in \mathbb{N}} |B_t| < \infty \mathbb{P}$-a.s.

A stochastic process $\{v_t, t \in \mathbb{N}\}$ is called a quasi-(super)martingale (QSPM) if there exists a (super)martingale $\{S_t, t \in \mathbb{N}\}$ such that

$$v_t = S_t + \sum_{k=0}^{t-1} B_k \quad \mathbb{P}\text{-a.s.}$$

In Fisk's paper the process $\{B_t, t \in \mathbb{N}\}$ of definition 2.1 is called a process of bounded variation. $\left(\{v_t, t \in \mathbb{N}\}\right)$ is said to be a QSPM w.r.t. $\{B_t, t \in \mathbb{N}\}$.

**Lemma 2.2.**

Let $\{B_t, t \in \mathbb{N}\}$ and $\{v_t, t \in \mathbb{N}\}$ be stochastic processes with $\sum_{t \in \mathbb{N}} |B_t| < \infty \mathbb{P}$-a.s. Then:

$\left(\{v_t, t \in \mathbb{N}\}\right)$ is a QSPM w.r.t. $\{B_t, t \in \mathbb{N}\}$ iff $\mathbb{E}^{\mathcal{A}_t} v_{t+1} \leq B_t + V_t$, $\mathbb{P}$-a.s.

**Proof:**

Define $S_t := V_t - \sum_{k=0}^{t-1} B_k$, $t \in \mathbb{N}$. 


i) Suppose
\[ A_t V_{t+1} \leq V_t + B_t \]
Then
\[ A_t S_{t+1} = A_t V_{t+1} - \sum_{k=0}^{t} B_k \leq V_t - \sum_{k=0}^{t-1} B_k = S_t \]
Hence \( \{S_t, t \in \mathbb{N}\} \) is a supermartingale.

ii) Conversely, suppose \( \{V_t, t \in \mathbb{N}\} \) is a QSPM w.r.t. \( \{B_t, t \in \mathbb{N}\} \).
Then
\[ A_t V_{t+1} = A_t S_{t+1} + \sum_{k=0}^{t} B_k \leq S_t + \sum_{k=0}^{t} B_k = V_t + B_t \quad \text{P-a.s.} \]

For a quasi-martingale the same characterization holds with equality. In lemma 2.3 it is shown that quasi-(super)martingales converge \( \text{P-a.s.} \) under a condition analogous to that for supermartingales.

**Lemma 2.3.**

Let \( \{B_t, t \in \mathbb{N}\} \) and \( \{V_t, t \in \mathbb{N}\} \) be as in lemma 2.2 with \( A_t V_{t+1} \leq B_t + V_t \), \( \text{P-a.s.} \).
Assume furthermore

i) \( \limsup_{t \in \mathbb{N}} \mathbb{E} V_t^- < \infty \)

ii) \( \mathbb{E} \sum_{k=0}^{\infty} B_k^+ < \infty \)

then \( V_t \) converges \( \text{P-a.s.} \).

**Proof.**

Let
\[ S_t := V_t - \sum_{k=0}^{t-1} B_k, \ t \in \mathbb{N}. \]

\[ \limsup_{t \in \mathbb{N}} \mathbb{E} S_t^- \leq \limsup_{t \in \mathbb{N}} \mathbb{E} V_t^- + \limsup_{t \in \mathbb{N}} \mathbb{E} \sum_{k=0}^{t-1} B_k^+ < \infty. \]
Since $\{S_t, t \in \mathbb{N}\}$ is a supermartingale it follows from the convergence theorem on supermartingales (see e.g. Neveu (1972). IV -1-2) that $S_{t-1}$ converges $\mathbb{P}$-a.s. Also $\sum_{k=0}^{t-1} B_k$ converges a.s., hence $V_t$ does. 

Remark.

From the proof of the cited theorem of Neveu it can be seen that Neveu's condition: $\sup_{t \in \mathbb{N}} \mathbb{E} V^-_t < \infty$ can be replaced by $\limsup_{t \in \mathbb{N}} \mathbb{E} V^-_t < \infty$

The next lemma is not really used in the rest of the paper, but it shows that the requirements of lemma 2.3 almost imply that $\mathbb{E} \sum_{n=0}^{\infty} B_n < \infty$.

Lemma 2.4.

Assume in addition to the assumptions of lemma 2.3 that $V_0$ is summable, then it holds that

$$\mathbb{E} \sum_{n=0}^{\infty} |B_n| < \infty.$$ 

Proof.

Let $S_t$ be defined as in the proof of lemma 2.3.

Note that

$$S_0 \geq E S_t = E V_t - E \sum_{k=0}^{t-1} B_k.$$ 

Let $M := \limsup_t \mathbb{E} V^-_t$ and note that $\limsup_t \mathbb{E} V_t + M \geq 0$.

Then

$$\mathbb{E} S_0 + \mathbb{E} \sum_{k=0}^{t-1} B_k^+ + M \geq \mathbb{E} (V_t + M) + \mathbb{E} \sum_{k=0}^{t-1} B_k^-$$

hence, by taking the limsup of both sides we have:

$$\mathbb{E} S_0 + \mathbb{E} \sum_{k=0}^{\infty} B_k^+ + M \geq \mathbb{E} \sum_{k=0}^{\infty} B_k^- \text{ for all } t \in \mathbb{N}.$$
Since \( \mathbb{E} S_0 = \mathbb{E} V_0 < \infty \) it holds that \( \mathbb{E} \sum_{k=0}^{\infty} B_k < \infty \) which proves the lemma.

This section ends with a property on regular supermartingales. The definition of regularity given below is equivalent to the usual one (see e.g. Neveu (1972) IV-5-24).

Let \( \{S_t, t \in \mathbb{N}\} \) be a supermartingale and let \( \tau_1 \) and \( \tau_2 \) be two stopping times w.r.t. \( \{A_t, t \in \mathbb{N}\} \). \( A_\infty \) is the \( \sigma \)-field generated by \( \cup_{n \in \mathbb{N}} A_n \) and
\[
A_\tau := \{ B \in A_\infty | B \cap \{ \tau = n \} \in n, n \in \mathbb{N} \}
\]

Definition 2.5.

The supermartingale \( \{S_t, t \in \mathbb{N}\} \) is called regular iff the sequence \( \{S_t, t \in \mathbb{N}\} \) converges in \( L^1 \)-sense.

Property 2.6. Let \( \{S_t, t \in \mathbb{N}\} \) be a regular supermartingale and let \( \tau_1, \tau_2 \) be stopping times. It holds that \( S_{\tau_1} \) and \( S_{\tau_2} \) are integrable and
\[
A_{\tau_1} \geq \mathbb{E}^{\tau_1} S_{\tau_2} \text{ P-a.s. on } \{\tau_1 \leq \tau_2\}
\]

(for a proof see Neveu (1972) IV-5-25).


In this section we return to the model described in section I. We first give a quick survey of the properties of this model which are relevant to our exposition here.

Properties.

3.1. \( \sup_{\mathbb{R} \times \mathbb{M}} \sup \mathbb{E}_{i,\pi} \left[ \sum_{n=0}^{\infty} r_p(X_n) \right] = \sup \mathbb{E}_{i,\pi} \left[ \sum_{n=0}^{\infty} r_p(X_n) \right] \)
where \( \Pi \) is the set of all strategies (see Van Hee (1975) for the definition of \( \Pi, E_{i,n} \), and the proof).

For this property assumption 1.2 (ii) is required.

3.2. The value function \( v \) satisfies Bellman's optimality equation:

\[
v(i) = \sup_{P \in \mathcal{P}} \{ r_p(i) + P v(i) \}.
\]

This statement is a standard consequence of 3.1: the proof is similar to those in Ross (1970), th. 6.1 or Hordijk (1974\(^a\)) th. 3.1 and 3.5.

3.3. Let \( g \) be a charge and \( f \) a superharmonic function w.r.t. \( g \) then \( \lim_{n \to \infty} E_{i,R} f(X_n) \) exists for all \( R \in M, i \in S \) and the following assertions are equivalent:

i) \( f \) is excessive w.r.t. \( g \)

ii) \( \lim_{n \to \infty} E_{i,R} f(X_n) = 0 \) for all \( R \in M, i \in S \)

iii) \( \lim_{n \to \infty} E_{i,R} f(X_n) \geq 0 \) for all \( R \in M, i \in S \).

For a proof see Hordijk (1974\(^a\)) th. 2.17

Remark 3.4.

It is obvious from 3.2 that the value function \( v \) is superharmonic w.r.t \( r \) and by its definition it is clear that

\[
v(i) \geq E_{i,R} \left[ \sum_{n=0}^{\infty} r_p(X_n) \right] \quad \text{for all } i \in S, R \in M,
\]

hence \( v \) is excessive w.r.t. \( r \).

Remark 3.5.

If \( f \) is excessive w.r.t. a charge \( g \) it holds that

\[
\lim_{n \to \infty} E_{i,R} [f(X_n)] = \lim_{n \to \infty} E_{i,R} [f(X_n)] \quad \text{for all } i \in S, R \in M
\]

by 3.3 (ii).
Lemma 3.6.
Let $f$ be a superharmonic function w.r.t. $g$. Then for all $R \in M, t \in \mathbb{N}$ and $i \in S$ it holds that

$$f(X_t) \geq g_p(X_t) + E_{i,R}^{F_t}f(X_{t+1})$$

Proof.
It is clear that $(P_t f)(X_t)$ is $F_t$-measurable and since

$$E_{i,R}[P_t f(X_t)] = P_0 \cdots P_{t-1} P_t f(i) = E_{i,R}[f(X_{t+1})]$$

it holds that

$$E_{i,R}[f(X_{t+1})] = P_t f(X_t) P_{i,R} - a.s.$$.

Hence, by the superharmonicity of $f$ the statement follows.

The main results of this paper are easy to prove now.

Theorem 3.7.
Let $f$ be an excessive function w.r.t a charge $g$.
For any $i \in S, R \in M \{ f(X_t), t \in \mathbb{N} \}$ is a quasi-supermartingale w.r.t. $\{g_p(X_k), k \in \mathbb{N} \}$ and $f(X_t)$ converges $P_{i,R} - a.s.$ (for $t \to \infty$).

Proof.
Fix $i \in S, R \in M$. By lemma 3.6, we have $f(X_t) \geq g_p(X_t) + E_{i,R}^{F_t}f(X_{t+1})$.

Since $g$ is a charge we have

$$\sum_{k=0}^{\infty} |g_p(X_k)| < \infty \quad P_{i,R} - a.s.$$.

So lemma 2.2 shows that $\{ f(X_t), t \in \mathbb{N} \}$ is QSPM w.r.t. $\{g_p(X_k), k \in \mathbb{N} \}$. From $g$ being a charge and property 3.3 ii) it follows that all conditions of lemma 2.3 are fulfilled, which proves the theorem.
Theorem 3.8.

Let $f$ be an excessive function w.r.t. a charge $g$.

The supermartingale

$$
\{ f(X_t) + \sum_{k=0}^{t-1} g_p(X_k), \ t \in \mathbb{N} \}
$$

is regular.

Proof.

Fix $i \in S$, $R \in \mathcal{M}$ and let $S_t := f(X_t) + \sum_{k=0}^{t-1} g_p(X_k)$.

By theorem 3.7 we have: $\{ S_t, \ t \in \mathbb{N} \}$ is a supermartingale, so we only have to check that $S_t^-$ converges in $L^1$-sense.

$$
S_t^- \leq f^-(X_t) + \sum_{k=0}^{t-1} g_p(X_k)
$$

hence

$$
S_t^- - f^-(X_t) \leq \sum_{k=0}^{t-1} g_p(X_k) \leq \sum_{k=0}^{\infty} |g_p(X_k)|
$$

On the other hand, since $(a + b)^- \geq a^-- b^+$

$$
S_t^- = [f(X_t) + \sum_{k=0}^{t-1} g_p(X_k)]^- \geq f^-(X_t) - [\sum_{k=0}^{t-1} g_p(X_k)]^+
$$

$$
\geq f^-(X_t) - \sum_{k=0}^{t-1} g^+_p(X_k)
$$

hence

$$
S_t^- - f^-(X_t) \geq - \sum_{k=0}^{\infty} |g_p(X_k)|
$$

By the dominated convergence theorem we have the $L^1$-convergence of $S_t^- - f^-(X_t)$. By property 3.3 ii) we have the $L^1$-convergence of $f^-(X_t)$
to zero. This implies the $L^1$-convergence of $S^-_t$. \[ \square \]

**Corollary 3.9.**

Let $f$ be an excessive function w.r.t. a charge $g$, and let $\tau_1$ and $\tau_2$ be stopping times w.r.t. $\{F_t, t \in \mathbb{N}\}$ then

\[
\sum_{k=0}^{\tau_1-1} g_{F_{\tau_1}}(X_k) + f(X_{\tau_1}) \geq \mathbf{E}_{\tau_1, R} \left[ \sum_{k=0}^{\tau_2-1} g_{F_{\tau_2}}(X_k) + f(X_{\tau_2}) \right]
\]

$\mathbf{P}_{i,R}$-a.s. on $\{\tau_1 \leq \tau_2\}$ for all $i \in S$, $R \in M$.

Note that 3.9 is a direct consequence of property 2.6 and theorem 3.8.

If $\mathbf{P}_{i,R}[\tau_1 \leq \tau_2] = 1$ integration w.r.t. $\mathbf{P}_{i,R}$ gives the theorem of Hordijk mentioned in the introduction.

4. Some remarks.

1). A strategy $R = (P_0, P_1, \ldots) \in M$ is called *conserving* if $v = r_{P_t} + P_t v$ for all $t \in \mathbb{N}$ ($v$ is the value function) and $R$ is called *equalizing* if

\[
\lim_{t \to \infty} \mathbf{E}_{i,R}[v(X_t)] = 0 \quad \text{for all } i \in S.
\]

It is well-known (see the proof of th. 4.6 in Hordijk (1974)) that $R \in M$ is optimal iff $R$ is equalizing and conserving.

For each equalizing strategy $R$ we may conclude that $v(X_n) \to 0$ $\mathbf{P}_{i,R}$-a.s. (for all $i \in S$) since by th. 3.7 $v(X_n)$ converges $\mathbf{P}_{i,R}$-a.s. and by 3.5 we know that this limit must be zero.

In Groenewegen (1975) this result has been proved for optimal strategies.

2). For a conserving strategy $R = (P_0, P_1, \ldots)$ it holds that

\[
v(X_t) = r_{P_t}(X_t) + P_t v(X_t)
\]

hence $\{v(X_t), t \in \mathbb{N}\}$ is a quasi-martingale in this case.
3). Let $g : S \times \mathbb{P} \to \mathbb{R}$ and $f_t : S \to \mathbb{R}$, $t \in \mathbb{N}$; suppose we do not know whether $g$ is a charge or not. Assume

i) $f_t(X_t) \geq g_p(X_t) + \mathbb{E}^{F_t}_{i,R} f_{t+1}(X_{t+1}) P_{i,R}$ for all $i \in S$, $t \in \mathbb{N}$, $R \in \mathcal{M}$ and with $\mathbb{E}^{F_t}_{i,R} f_{t+1}(X_{t+1})$ well-defined.

ii) $\limsup_{t \to \infty} \mathbb{E}^{F_t}_{i,R} f(X_t) < \infty$ for all $i \in S$, $R \in \mathcal{M}$ and $t \in \mathbb{N}$.

iii) $\sum_{k=0}^{\infty} |g_{p_k}(X_k)| < \infty$ for all $i \in S$, $R \in \mathcal{M}$.

iv) $\mathbb{E}^{F}_{i,R} \sum_{k=0}^{\infty} |g_{p_k}(X_k)| < \infty$ for all $i \in S$, $R \in \mathcal{M}$.

Under these conditions, similar to those in lemma 2.3, we have that $f_t(X_t)$ converges $\mathbb{P}_{i,R}$-a.s. for each $i \in S$, $R \in \mathcal{M}$. If in addition $f_0$ is finite, this implies also by lemma 2.4 that $g$ is a charge. So if $f_t = f$ for all $t \in \mathbb{N}$, $f$ is superharmonic w.r.t. $g$.

4). Let $N \in \mathbb{N}$ and $Q_0, Q_{N+1}, \ldots \in \mathbb{P}$ and let $g$ be a charge. Define $R := \{ R = P_0 P_1 \ldots \in \mathcal{M} | P_k = Q_k \text{ for } k \geq N \}$ and

$$v_k := \left\{ \begin{array}{ll} \sum_{n=0}^{k} Q_k Q_{k+1} \cdots Q_{n-1} g_{Q_n} & k \geq N \\ \sup_{P \in \mathcal{P}} \{ g_{P} + P v_{k+1} \} & 0 \leq k \leq N - 1 \end{array} \right\}$$

It is easy to check that the assumptions i), iii) and iv) in the above remark are satisfied for $R \in R$. And assumption ii) is satisfied by the observation that $v(n, X_n) := v_n(X_n)$ is excessive w.r.t. $g$ for the space-time process, where the only allowed strategies are elements of $R$. Hence $v_n(X_n)$ converges $\mathbb{P}_{i,R}$-a.s.
5). Let $f$ be an excessive function w.r.t. a charge $g$. From the 3.8 we know that $f(X_t)$ converges $\mathbb{P}_{i,R}$-a.s. for $t \to \infty$ and from property 3.3 (ii) we know that $f^-(X_t)$ converges in $L^1$-sense for $t \to \infty$.

The following counterexample shows that in general $f^+(X_t)$ does not converge in $L^1$-sense for $t \to \infty$.

Example:

$S := \{0, 1, 2, \ldots \}$. $P$ and $Q$ are Markov transition functions with $P(0, 0) = 1$, $P(i, i + 1) = i / (i + 1)$ for $i \geq 1$, $P(i, 0) = 1 / (i + 1)$ for $i \geq 1$, $Q(i, 0) = 1$. $P$ is the collection of Markov transition functions which can be generated from $P$ and $Q$ by using the product property. Furthermore $r_P \equiv 0$ and $r_Q(i) = i$.

It can be verified easily that the conditions 1.2 i) and ii) are fulfilled, that $v(i) = i$, that $\lim v(X_t) = 0$ $\mathbb{P}_{i,R}$-a.s. for all $R$ and that

$$\lim_{t \to \infty} \mathbb{E}_{i,R} v(X_t) = 1 \text{ for } R = PPP \ldots$$

But it is well-known that the $L^1$-limit and the a.s.-limit should be equal, if both exist. So $v(X_t)$ does not converge in $L^1$-sense for $t \to \infty$.

6). In Mandl (1974) a martingale is considered in connection with the average cost criterion for the optimal control of a Markov chain. Using his construction for the total return criterion we get the following. Define a real function on $S \times P$:

$$\Phi(i, P) := r_P(i) + P v(i) - v(i)$$

and a random variable

$$Y_n := r_P(i) + v(X_{n+1}) - v(X_n) - \Phi(X_n, P_n)$$

It is easy to see that

$$\mathbb{E}^n_n Y_n = 0, \text{ so } M_n := \sum_{k=0}^{n-1} Y_k$$

is a martingale. Now this martingale becomes

$$M_n = \sum_{m=0}^{n-1} r_P(X_m) + v(X_n) - v(X_0) - \sum_{m=0}^{n-1} \Phi(X_m, P_m).$$

Note that

1. $v(X_0) = v(i) \mathbb{P}_{i,R}$-a.s. for all $R \in M$
2. by (3.2) $\Phi(X_m, P_m) \leq 0 \mathbb{P}_{i,R}$-a.s.
Hence

\[ \sum_{m=0}^{n-1} r_p (X_m) + v(X_n) \]

is the supermartingale treated in section 3. In view of the conserving and equalizing strategies, mentioned in remark 4.1 it is worthwhile to note that

\[ v(i) - E_{i,R} \left[ \sum_{n=0}^{\infty} r_p (X_n) \right] = \lim_{n \to \infty} E_{i,R} \left[ v(X_n) \right] + E_{i,R} \left[ \sum_{n=0}^{\infty} \phi(X_n, P_n) \right] \]

for all \( R \in M \).

From this equality it is easy to see that a \( R \in M \) is optimal if and only if it is equalizing and conserving.

Literature


