A theory of generalized functions based on holomorphic semi-groups

de Graaf, J.

Published: 01/01/1979

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the author's version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

Citation for published version (APA):
A THEORY OF GENERALIZED FUNCTIONS BASED ON

HOLOMORPHIC SEMI-GROUPS

by

J. de Graaf

T.H.-Report 79-WSK-02

March 1979
Contents

Abstract 1

Chapter 0. Introduction

Chapter 1. The space $S_{X,A}$ 12

Chapter 2. The space $T_{X,A}$ 26

Chapter 3. The pairing of $S_{X,A}$ and $T_{X,A}$ 33

Chapter 4. Characterization of continuous linear mappings between the spaces $S_{X,A}$, $S'_{X,A}$, $S_{Y,B}$ and $S'_{Y,B}$ 39

Chapter 5. Topological tensor products of spaces of type $S_{X,A}$, $S'_{X,A}$ 48

Chapter 6. Kernel theorems 59

Appendix A. Functions of self-adjoint operators 65

Appendix B. Holomorphic semi-groups 67

Appendix C. A Banach-Steinhaus theorem 69

Acknowledgement 70

References 71
Abstract

In a Hilbert space $X$ consider the evolution equation

$$\frac{du}{dt} = -Au$$

with $A$ a non-negative unbounded self-adjoint operator. $A$ is the infinitesimal generator of a holomorphic semi-group. Solutions $u(\cdot) : (0, \infty) \to X$ of this equation are called trajectories. Such a trajectory may or may not correspond to an "initial condition at $t = 0"$ in $X$. The set of trajectories is considered as a space of generalized functions. The test function space is defined to be $S_{X,A} = \bigcap_{t>0} e^{-tA}(X)$. For the spaces $S_{X,A}, S'_{X,A}$ I discuss a pairing, topologies, morphisms, tensor products and Kernel theorems. Examples are given.

AMS Classifications 46F05 46F10
CHAPTER 0. Introduction

In a very inspiring paper, De Bruijn [B] has proposed a theory of generalized functions based on a specific one-parameter semigroup of smoothing operators. This semigroup constitutes what is known as a holomorphic semigroup of self-adjoint bounded operators on $L^2(\mathbb{R})$.

This observation has enabled me to generalize De Bruijn's theory and to place it in a wider context of functional analysis. By this approach, we can introduce De Bruijn type generalized functions and corresponding spaces of test functions on manifolds and also on duals of nuclear spaces. We can construct the spaces of test functions even so as to be invariant under one or more given operators. In [B] such a given operator is the Fourier transform.

Even so, there are many possibilities, allowing us to impose further conditions. In the sequel I study extensively the spaces of test functions and distributions that arise in this fashion. I characterize their topologies and morphisms. Moreover, I give necessary and sufficient conditions for the validity of Kernel Theorems.

In this introduction I illustrate the ideas by a few examples. The first example is very simple and I discuss it in some detail. I use it to give an outline of the general theory. The remaining examples merely illustrate how to apply the general theory; they are not worked out in detail. They could be the subject of further investigation.
Example 1

Let us consider the elementary diffusion equation

\[
\frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial x^2}, \quad x \in \mathbb{R}, \quad t > 0.
\]

A classical solution \( F(x,t) \) with the property \( \forall t > 0 \ F(\cdot,t) \in L^2(\mathbb{R}) \)
will be called a trajectory. Corresponding to any "initial condition" \( f \in L^2(\mathbb{R}) \) there is a trajectory given by the well-known formula

\[
F(x,t) = \langle M_t f \rangle (x) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^\infty f(\xi) \exp\left(-\frac{(x-\xi)^2}{4t}\right) d\xi, \quad t > 0.
\]

From this expression it follows that the special trajectory \( F(\cdot,t) = M_t f \), \( f \in L^2(\mathbb{R}) \) has the properties

(i) \( \forall t > 0 \ \forall \tau > 0 \ M_t F(\cdot,\tau) = F(\cdot, \tau + t) = M_t F(\cdot,t) \).

(ii) \( \forall t > 0 \ F(x,t) \) is an entire analytic function of \( x \).

With the aid of the maximum principle for parabolic equations, [PW] p.160, it follows that (i) is true for each trajectory and hence also (ii) is true. From the positivity and linearity of the integral operator \( M_t \) it follows that for a given trajectory \( F(\cdot,t) \) there exists at most one \( g \in L^2(\mathbb{R}) \) such that \( F(\cdot,t) = M_t g \). In general no such \( g \) does exist at all, consider e.g.

\[
G(x,t) = 2(\pi t)^{-\frac{1}{2}} \exp\left(-\frac{(x-a)^2}{4t}\right).
\]

Note that \( G(x,t) \) "tends" to the \( \delta \)-function \( \delta(x-a) \) for \( t \to 0 \). Any derivative of \( G \) is again a trajectory.

Following De Bruijn we gather all trajectories of (1) in a complex vector space \( S' \). The elements of \( S' \) are to be interpreted as generalized functions. As we just have seen, a generalized function sometimes corresponds to an element of \( L^2(\mathbb{R}) \) and, if so, the correspondence is unique. The test function space \( S \) is defined to be the dense linear subspace of \( L^2(\mathbb{R}) \) consisting of smooth elements of the form \( M_t h \), \( h \in L^2(\mathbb{R}), \quad t > 0 \). The densely defined inverse of \( M_t \) is denoted by \( M_{-t} \). For each \( \varphi \in S \) there exists \( \tau > 0 \) such that \( M_{-\tau} \varphi \) makes sense. The pairing between \( S \) and \( S' \) is defined by
Here \((\cdot,\cdot)\) denotes the inner product in \(L_2(\mathbb{R})\).

The definition makes sense for \(\epsilon\) positive and sufficiently small.

Since \(M_t\) is a symmetric operator and \(M_t M_t^* = M_t^* M_t\),
the definition (2) does not depend on the specific choice of \(\epsilon\).
In the following chapters we provide \(S\) with suitable topologies and we prove

(i) For each fixed \(F \in S'\) the map \(\phi \mapsto \langle \phi, F \rangle\) is a continuous linear
functional on \(S\).

(ii) For each continuous linear functional \(\ell\) on \(S\) there exists exactly
one trajectory \(F_\ell\) such that for all \(\phi \in S\) we have \(\ell(\phi) = \langle \phi, F_\ell \rangle\).

Suppose \(P\) is a densely defined linear operator in \(L_2(\mathbb{R})\) with a densely
defined adjoint \(P^*\) which leaves \(S\) invariant, i.e. \(P^*(S) \subseteq S\). Then \(P\)
can be extended to a continuous mapping from \(S'\) into itself by

\[
\langle \phi, P_F \rangle = \langle P^* \phi, F \rangle.
\]

Examples of such operators \(P\) are \(M_\alpha, R_a, T_a, Z_\lambda, D\) and compositions of
these. Here \((R f)(x) = e^{iax} f(x)\), \((T_b f)(x) = f(x + b)\), \((Z f)(x) = f(\lambda x)\),
\((Df)(x) = \frac{df(x)}{dx}\) with \(a, b, \lambda \in \mathbb{R}\).

Finally we want to show that certain strongly divergent Fourier integrals
can be interpreted as elements of \(S'\).

This interpretation is closely related to the Gauss-Weierstrass summation
method. Let \(g\) be a measurable function on \(\mathbb{R}\) such that for each \(\epsilon > 0\) the
function \(g(y)e^{-\epsilon y^2}\) is in \(L_1(\mathbb{R})\). The possibly divergent Fourier integral
\[
\int_{-\infty}^{\infty} g(y)e^{iyx} \, dy
\]
can be considered as an element of \(S'\). The trajectory \(G\) is given by \(G(x,t) = \int_{-\infty}^{\infty} g(y)e^{-ty^2} e^{iyx} \, dy\) .
This simple illustration of our theory certainly has some elegant features, but in some respects
it is too simple. Since \(M_t\) is not a Hilbert-Schmidt operator there is
no Kernel Theorem in this case. That is to say, there exist continuous
linear mappings from \(S\) into \(S'\) which do not arise from a trajectory of
the diffusion equation in \(\mathbb{R}^2\). Cf. Chapter 6 case b.
Partial sketch of the general theory

We consider an evolution equation in a Hilbert space \( H \)

\[
\frac{du}{dt} = -Au
\]

Here \( A \) is a densely defined unbounded operator such that the exponential \( e^{-At} \) can be given a meaning. A trajectory of (3) is defined to be a mapping \( F : (0,\infty) \to H \) with the property \( \forall t > 0 \ \forall \tau > 0 \ e^{-A\tau}F(t) = F(t + \tau) \).

Examples of trajectories are \( F(t) = e^{-At}h, \ h \in H \).

Further we introduce a test function space \( S = \bigcup_{t \geq 0} e^{-At}(H) \).

In order to make all this into a non-trivial interesting theory of generalized functions which generalizes the situation of example 1, we must meet some requirements.

(i) \(-A\) must be an infinitesimal generator. In other words \( e^{-At} \) must have a meaning and it is an everywhere defined bounded operator.

(ii) \( e^{-At} \) must be injective for each \( t > 0 \).

(iii) The inverse \( e^{At} \) of \( e^{-At} \) has to be unbounded in order that \( e^{-At} \) is "smoothing".

These conditions are fulfilled if \(-A\) is the infinitesimal generator of a one-parameter holomorphic semi-group \( e^{-At}, \ t \geq 0 \). Here holomorphic means holomorphic in \( t \) (as a bounded operator-valued function). This is not related to any kind of smoothness or analyticity of the "test functions" in \( S \). The next chapters deal with the case that \( A \) is a non-negative unbounded self-adjoint operator in a separable Hilbert space \( H \). In example 1 we had \( H = L^2(\mathbb{R}), \ A = -\frac{d^2}{dx^2} \) with domain \( D(A) = H^2(\mathbb{R}) \), the 2\textsuperscript{nd} order Sobolev space.

**Example 2**

\[
H = L^2(\mathbb{R}), \quad A = (-\frac{d^2}{dx^2})^\alpha, \quad \alpha > 0 \quad D(A) = H^{2\alpha}(\mathbb{R}).
\]

In this case
\[ (M_t u)(x) = (e^{-At} u)(x) = \int_{-\infty}^{\infty} K_t(a;x,y) u(y) \, dy \]

with
\[ K_t(a;x,y) = \frac{1}{\pi} \int_{0}^{\infty} \exp(-|k|^2 a t) \cos k (x - y) \, dk . \]

For \( \alpha = 1 \) the kernel \( K_t \) has been calculated in example 2.

For \( \alpha = \frac{1}{2} \) we have
\[ K_t(\frac{1}{2};x,y) = \frac{1}{\pi} \frac{t}{t^2 + (x-y)^2} \]

which is just the Poisson kernel for solving the Dirichlet problem in a half plane \( t \geq 0 \). This is not surprising, since, at least formally,
\[ \left( \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} \right) = \left( \frac{\partial}{\partial t} - \sqrt{-\frac{\partial^2}{\partial x^2}} \right) \left( \frac{\partial}{\partial t} + \sqrt{-\frac{\partial^2}{\partial x^2}} \right) . \]

The following properties of the corresponding test function spaces \( S \) are easy to verify
\( \alpha > \frac{1}{2} : S \) consists of entire analytic functions.
\( \alpha = \frac{1}{2} : S \) consists of functions which are analytic in a strip around the real axis.
\( 0 < \alpha < \frac{1}{2} : \) The elements in \( S \) are \( C^\infty \) but non-analytic in general.

Finally we want to show that for each \( \alpha > 0 \) certain strongly divergent Fourier integrals can be interpreted as elements of \( S' \). Cf. Example 1.

Let \( g \) be a measurable function on \( \mathbb{R} \) such that for each \( \varepsilon > 0 \) the function \( g(y) \exp - \varepsilon |y|^{2\alpha} \) is in \( L_1(\mathbb{R}) \). The possibly divergent integral \( \int_{-\infty}^{\infty} g(y) e^{iyx} \, dy \) is in \( S' \). The trajectory \( G \) is given by
\[ G(x,t) = \int_{-\infty}^{\infty} g(y) \exp [-t|y|^{2\alpha}] e^{-iyx} \, dy . \]

For \( \alpha = \frac{1}{2} \) this corresponds to Abel's summation method.
Example 3

\[ H = L_2([0,2\pi]) \quad A = (-\frac{d^2}{dx^2}) \quad \alpha > 0 \]

\[ D(-\frac{d^2}{dx^2}) = \{ u | u \in H^2([0,2\pi]) \quad u(0) = u(2\pi) \quad u'(0) = u'(2\pi) \} \]

In this case

\[ (M_t u)(x) = (e^{-At} u)(x) = \int_0^{2\pi} K_t(\alpha;x,y) u(y) dy \]

with

\[ K_t(\alpha;x,y) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp{-|n|^{2\alpha} t + in(x-y)} = \frac{1}{2\pi} \left( 1 + 2 \sum_{n=1}^{\infty} \exp{-n^{2\alpha} t} \cos n(x-y) \right) \]

For \( \alpha = 1 \), \( K_t(1;x,y) = \frac{1}{2\pi} \theta_3(x-y,e^{-t}) \).

Here \( \theta_3 \) is one of Jacobi's theta functions [WW] p. 464.

For \( \alpha = \frac{1}{2} \), \( K_t(\frac{1}{2};x,y) = \frac{1}{2\pi} \frac{\sinh t}{\cosh t - \cos(x-\xi)} \).

In the latter case the smoothing integral operator is the Poisson kernel for the solution of the Dirichlet problem for the unit disc. For a general discussion of this phenomenon see Example 5.

For each \( \alpha > 0 \), \( K_t(\alpha;\cdot,\cdot) \in L_2([0,2\pi]^2) \). In other words \( e^{-At} \) is a Hilbert-Schmidt operator. This implies that for generalized functions of this type a Kernel Theorem holds true. See Chapter 6. Finally, we remark that certain strongly divergent Fourier series do have a meaning within this theory.

Let \( \alpha > 0 \) be fixed. Consider a sequence of complex numbers \( \{c_n\}_{n=-\infty}^{\infty} \) such that \( \{|c_n| \exp{-\varepsilon |n|^{2\alpha}}\} \) is a bounded sequence for each \( \varepsilon > 0 \). Then \( \sum_{n=-\infty}^{\infty} c_n e^{inx} \) can be viewed as a generalized function. The corresponding trajectory is given by \( G(x,t) = \sum_{n=-\infty}^{\infty} c_n \exp{-|n|^{2\alpha} t + inx} \).
Example 4

\[ H = L^2(\mathbb{R}), \quad A = -\frac{d^2}{dx^2} \frac{1}{2\alpha} + \frac{1}{2\beta}, \quad \alpha > 0, \quad \beta > 0, \]

\[ D(A) = H^\alpha (\mathbb{R}) \cap F(H^\beta (\mathbb{R})), \]

here \( F \) is the Fourier operator.

For \( \alpha = \beta = \frac{1}{2} \) an explicit representation of \( e^{-At} \) by means of an integral operator can be found in [B].

This special case has been extensively investigated by De Bruijn and Janssen [B], [J]. A part of our results, especially in chapters 3 and 4 consists of generalizations and adaptations of their work.

Whenever \( \alpha = \beta \) the test function space is invariant under Fourier transformation. We conjecture that the Gelfand-Silov test function classes \( S^\beta, [GS] \) Vol. II Ch. 4 are the same as the test function spaces \( S \) of this example.

The preceding examples can all be generalized to \( \mathbb{R}^n \) in a direct obvious way. These generalizations can also be obtained by application of the theory of Chapter 5 where topological tensor products of the spaces \( S \) and \( S' \) are formed. Repeated formation of tensor products of the \( S \) and \( S' \) spaces of example 3 leads to distribution theories on \( n \)-dimensional tori. Our type of distributions can be introduced on any, not too bad, differentiable manifold \( M \) by taking for \( A \) a positive elliptic differential operator which is self-adjoint in an \( L^2 \)-space over \( M \). On a compact Riemannian manifold one may take the Laplace-Beltrami operator. This can be done on the \( q \)-dimensional unit sphere in \( \mathbb{R}^{q+1} \) for example. In the latter case however a very nice semi-group of smoothing operators can be chosen which is closely related to the Poisson-integral solution of the Dirichlet-problem for the unit ball in \( \mathbb{R}^{q+1} \). This is the subject of the last example.
Example 5

(4) \[ H = L_2(S^q), \quad A = -\lambda(q - 1)I + \sqrt{\lambda(q - 1)^2 - \Delta_{LB}} \]

Here \( \Delta_{LB} \) denotes the Laplace-Beltrami operator, \( I \) denotes the identity operator.

After introducing orthogonal spherical coordinates \( x_i = r F_i(\theta_1, \ldots, \theta_q), \) 
\( 1 \leq i \leq q + 1 \) in \( \mathbb{R}^{q+1} \) we obtain for the Laplacian

(5) \[ \Delta = \sum_{i=1}^{q+1} \frac{\partial^2}{\partial x_i^2} + \frac{q}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{LB} \]

From this it follows, see [M] p. 4, that an \( m \)-th order spherical harmonic is an eigenvector of \( \Delta_{LB} \) with eigenvalue equal to \( -m(m + q - 1) \). Then a simple calculation shows that each \( m \)-th order spherical harmonic is an eigenvector with eigenvalue \( m \) of our operator \( A \).

Introduction of \( r = e^{-t} \) in (5) transforms the Laplacian into

\[ \Delta = e^{2t} \left[ \frac{\partial^2}{\partial t^2} - (q - 1) \frac{\partial}{\partial t} + \Delta_{LB} \right] \]

The expression between \( \{ \} \) can be factored into two evolution equations. Thus

\[ \Delta = e^{2t} \left[ \frac{\partial}{\partial t} + \frac{4}{3}(q - 1)I + \{ \frac{4}{3}(q - 1)^2 - \Delta_{LB} \} \right] \]

\[ + \left[ \frac{\partial}{\partial t} - \frac{4}{3}(q - 1)I + \{ \frac{4}{3}(q - 1)^2 - \Delta_{LB} \} \right] \]

The second factor can be written as \( \frac{\partial}{\partial t} + A \), with the \( A \) from (4).

From these considerations it follows that substitution of \( r = e^{-t} \) in the Poisson integral formula, cf. [M] p. 41, leads to an integral expression for \( e^{-At} \).
Here \( \xi \) and \( \eta \) denote elements of \( S^q \), i.e. unit vectors in \( \mathbb{R}^{q+1} \). The "surface measure" of \( S^q \) is denoted by \( d\omega \), while \( \xi \cdot \eta \) denotes the inner product of \( \xi \) and \( \eta \).

For fixed \( \eta \) the kernel in (6) denotes the trajectory of the \( \delta \)-function centered at \( \eta \). For a representation of the kernel involving zonal harmonics, see [G Se]. Here the operator \( e^{-At} \) is obviously Hilbert-Schmidt. Then from Chapter 6 it follows that for the distribution theory arising from the operator \( A \) in (4) Kernel Theorems are available. Similar to example 3 certain strongly divergent sequences of spherical harmonics can be "summed".

I conclude this introductory chapter by summarizing briefly the contents of the next chapters.

In Chapter I the test function space \( S_{X,A} \) is discussed. Here \( X \) is a separable Hilbert space, \( A \) is a non-negative unbounded self-adjoint operator in \( X \). \( S_{X,A} \) is provided with a non-strict inductive limit topology. A base of the neighbourhood system of 0, is given. Bounded and compact sets are characterized. A necessary and sufficient condition for the convergence of a sequence is given. In Theorem 1.11 the properties of \( S_{X,A} \) are summarized using the standard terminology of the theory on locally convex topological vector spaces.

In Chapter 2 the program of Chapter 1 is repeated for the space of trajectories \( T_{X,A} \). A characterization of the trajectories is given. In Chapter 3 the pairing between elements of \( S_{X,A} \) and \( T_{X,A} \) is discussed. Weak topologies are introduced. Banach-Steinhaus theorems are proved. Strong and weak convergence of sequences are compared. It is proved that \( S_{X,A} \) and \( T_{X,A} \) are both strong and weak duals of each other. This motivates the choice of the notation \( T_{X,A} = S_{X,A}' \).

In Chapter 4 continuous linear mappings between the spaces \( S_{X,A}, S_{X,A}' \), \( S_{Y,B}, S_{Y,B}' \) are extensively discussed and characterized. The problem of extending linear operators is considered.

Chapter 5 deals with completions of topological tensor products of the spaces of Chapter 4.
Chapter 6 is devoted to Kernel Theorems. We give necessary and sufficient conditions on \( A \) and \( B \) in order that all linear mappings of Chapter 4 are themselves element of spaces \( S_{X \otimes Y, A \oplus B} \), etc.
CHAPTER 1. The space \( S_{X,A} \)

In a complex separable Hilbert space \( X \) with inner product \((\cdot,\cdot)_X\) we consider an unbounded non-negative self-adjoint operator \( A \). This operator will be fixed throughout chapters 1, 2, 3. Since \( A \) is self-adjoint it admits a spectral resolution

\[
A = \int_0^\infty \lambda \, dE_\lambda,
\]

For details and conventions on such spectral resolutions we refer to Appendix A. We define

\[
M_t = e^{-At} = \int_0^\infty e^{-\lambda t} \, dE_\lambda, \quad t \in \mathbb{C}.
\]

For \( t \in \mathbb{R} \) the operator \( M_t \) is self-adjoint, \( M_t \) is bounded iff \( \text{Re} \, t > 0 \). \( M_t \) is unitary iff \( \text{Re} \, t = 0 \). Further \( M_t \) is invertible and \( \forall t \in \mathbb{C} \) \( M_t^{-1} = M_{-t} \).

On \( \mathbb{C}^+ \), i.e. the set of \( t \in \mathbb{C} \) with \( \text{Re} \, t > 0 \), the operators \( M_t \) establish a one-parameter semi-group of bounded injective operators on \( X \). For more details on such holomorphic semi-groups see Appendix B.

We now introduce our space of test functions \( S_{X,A} \).

**Definition 1.1.**

\[
S_{X,A} = \bigcup_{\text{Re} \, t > 0} \{ M_t x | x \in X \} = \bigcup_{\text{Re} \, t > 0} M_t(X)
\]

\( S_{X,A} \) is a dense linear subspace of \( X \). From the semi-group properties the following equalities immediately follow for any \( \delta > 0 \)

\[
S_{X,A} = \bigcup_{t>0} M_t(X) = \bigcup_{0 < t < \delta} M_t(X) = \bigcup_{n \in \mathbb{N}, 0 < \frac{1}{n} < \delta} M_t(X).
\]

Each of the spaces \( M_t(X) \), \( t > 0 \), can be considered as a Hilbert space. The inner product is
The completeness of \( M_t(X) \) follows from the closedness of \( M_{-t} \).
\( M_t(X) \) consists of exactly those \( u \in X \) for which
\[
\int_0^\infty e^{2\lambda t} d(E_\lambda u, u) < \infty.
\]
If locally no confusion is likely to arise we suppress as many subscripts as possible. In chapters 1, 2 and 3 we shall write consistently: \( S \) for \( S_{X,\mathcal{A}} \), \( X_t \) for \( M_t(X) \), \((\cdot,\cdot)_0\) or \((\cdot,\cdot)_X\).

Definition 1.2.

The strong topology on \( S \) is the finest locally convex topology on \( S \) for which the natural injections \( i_t : X_t \rightarrow S \), \( t > 0 \), are all continuous.

In other words \( S \) is made into a locally convex topological vector space by imposing the inductive limit topology with respect to the family \( \{X_t\}_{t>0} \). Cf. [SCH] Ch. II.6. Since the natural injections \( X_t \rightarrow X_{\tau} \), \( \tau > t > 0 \), are all continuous the inductive limit topology is already brought about by the family \( \{X_{1/n}\}_{n \in \mathbb{N}} \). A subset \( U \subset S \) is open iff every \( t > 0 \) the set \( i_t^{-1}(U) = U \cap X_t \) is open in \( X \) or, equivalently, iff for every \( n \in \mathbb{N} \) the set \( U \cap X_{1/n} \) is open in \( X_{1/n} \). The inductive limit here is more complicated than in the case of the Schwarz test function spaces because our inductive limit is not strict! A closed subset of \( X_t \) when considered as a set in \( X_t \), \( 0 < t < \tau \), is not necessarily closed.

We shall see that open sets in \( S \) are always unbounded.

We introduce some notations:
- \( B \) denotes the set of everywhere finite real valued Borel functions on \( \mathbb{R} \) such that \( \forall t > 0 \) the function \( \psi(x)e^{-tx} \) is bounded on \([0,\infty)\).
- \( B_+ \subset B \) contains those \( \psi \) with \( \psi(x) > 0 \) \( \forall x \geq 0 \).
- \( T \subset B_+ \) consists of those \( \psi \) which are zero for \( x < 0 \) and piece-wise constant on the intervals \([0,1], (n,n+1], n \in \mathbb{N} \). The value of \( \psi \) on such an interval is a natural number.
- \( \mathcal{B}(A), \mathcal{B}_+(A), T(A) \) denote the corresponding sets of self-adjoint operators in \( X \) defined by
The domain of $\psi(A)$ consists of exactly those $x \in X$ with $\int_0^\infty \psi^2(\lambda) d(E_{\lambda} x, x) < \infty$. It follows that $S$ is contained in the domain of each operator $\psi(A)$.

Further: $\forall \psi \in B \ \forall t > 0$ the operators $\psi(A)e^{-At} = \psi(A)M_t$ are bounded and self-adjoint. The operators in $B_+(A)$ are all strictly positive.

**Definition 1.3.**

For each $\psi \in B_+$ we introduce the following semi-norm (which is actually a norm)

$$p_\psi(u) = (\psi(A)u, u)^{1/2} = \left\{ \int_0^\infty \psi(\lambda) d(E_{\lambda} u, u) \right\}^{1/2}, \quad u \in S.$$ 

Further we define the sets

$$U_{\psi, \epsilon} = \{ u \in S | (\psi(A)u, u) < \epsilon^2, \psi \in B_+, \epsilon > 0 \}.$$ 

The next theorem is very fundamental.

**Theorem 1.4.**

I. $\forall \psi \in B_+ \ \forall \epsilon > 0 \ U_{\psi, \epsilon}$ is a convex, balanced, absorbing open set in the strong topology. In other words the semi-norms $p$ are continuous.

II. Let a convex set $\Omega \subset S$ be such that for each $t > 0$ $\Omega \cap X_t$ contains a neighbourhood of 0 in $X_t$.

Then $\Omega$ contains a set $U_{\psi, \epsilon}$ with $\psi \in B_+$ and $\epsilon > 0$.

III. The set of $U_{\psi, \epsilon}, \psi \in T, \epsilon > 0$, establishes a base of the neighbourhood system of 0 in $S$. In other words the strong topology in $S$ is generated by the sets $U_{\psi, \epsilon}$.

IV. For any open set $V \subset S$, $0 \in V$ and for each $t > 0$ the intersection $X_t \cap V$ is an unbounded set in $X_t$. 

Proof

I. A standard inner product argument shows that $U_{\psi, \varepsilon}$ is convex, balanced and absorbing. For the terminology see [1] Ch. I.1. We now show that for each $t > 0$ $U_{\psi, \varepsilon} \cap X_t$ is an open set in $X_t$.

By definition

$$U_{\psi, \varepsilon} \cap X_t = \{ u | u \in X_t, (\psi(A)u, u) < \varepsilon^2 \}.$$ 

Because of $(\psi(A)u, u) = (\psi(A)M_{2t} - M_t u, M_t u) \leq \|\psi(A)M_{2t}\|\|M_t u\|^2 = \|\psi(A)M_{2t}\|\|u\|^2$ and the boundedness of $\psi(A)M_{2t}$ the sesquilinear form $(\psi(A)u, u)$ is continuous on $X_t$. Therefore the set of $u$'s in $X_t$ for which $(\psi(A)u, u) < \varepsilon^2$ is open in $X_t$.

II. We proceed in six steps

a) Introduce the projection operator $P_n = \int_{n-1}^n dE_x$, $n \in \mathbb{N}$. Let $r_{n,t}$ be the radius of the largest open ball in $P_n(X_t)$ which fits in $U \cap P_n(X_t)$. Thus

$$r_{n,t} = \sup \{ \rho \left[ u \in P_n(X) \wedge \|P_n u\|_t^2 = \int_{n-1}^n e^{2\lambda t} d(\lambda u, u) < \rho^2 \right] \}$$

$$[u \in U \cap P_n(X) \cap X_t]$$

b) For $t > \tau \geq 0$ we have $e^{(n-1)(t-\tau)} r_{n,\tau} \leq r_{n,t} \leq e^{n(t-\tau)} r_{n,\tau}$.

The proof of this runs as follows

$$\|P_n u\|_t^2 = \int_{n-1}^n e^{2\lambda t} d(\lambda u, u) \leq e^{-2(n-1)(t-\tau)} \int_{n-1}^n e^{2\lambda t} d(\lambda u, u) = e^{-2(n-1)(t-\tau)} \|P_n u\|_t^2.$$ 

Therefore if we take $u$ such that
\[ \|P_n u\| \leq e^{(n-1)(t-\tau)} r_{n,\tau} \] it follows that \( \|P_n u\| < r_{n,\tau} \). Consequently \( r_{n,\tau} = e^{(n-1)(t-\tau)} r_{n,\tau} \). The proof of the other inequality runs similarly.

c) For any fixed \( p > 0 \) and \( t > 0 \) the sequence \( \sum_{n=1}^{\infty} n^p (r_{n,\tau})^{-1} \) is convergent. Indeed, there exists an open ball in \( X_\tau \), \( \tau < t \), with sufficiently small radius \( \epsilon \) and centered at 0 which lies entirely within \( U \cap X_\tau \). Then \( \forall n \in \mathbb{N} \quad r_{n,\tau} \geq \epsilon \) and with the first inequality in b) it follows that \( (r_{n,\tau})^{-1} \leq (r_{n,\tau})^{-1} e^{-(n-1)(t-\tau)} \leq e^{-1} e^{-(n-1)(t-\tau)} \).

d) We define a function \( \chi \) on \( \mathbb{R} \) by
\[
\chi(x) = \begin{cases} 4n^2 e^{-2n} (r_{n,1})^{-2} & \text{for } x \in (n-1,n], \quad n \in \mathbb{N} \\ 0 & \text{for } x < 0 \end{cases}
\]
and claim that \( \chi \in B_+ \). For the proof we have to verify that \( \chi(x) e^{-\epsilon x} \) is bounded on \([0,\infty)\) for every \( \epsilon > 0 \). On \((n-1,n]\) and for every \( \tau, \ 0 \leq \tau \leq 1 \) applying b) with \( t = 1 \) we find
\[
\chi(x) e^{-\epsilon x} \leq 4n^2 e^{-2n} (r_{n,1})^{-2} \leq 4n^2 e^{2(n-1)(2\tau-\epsilon)} (r_{n,\tau})^{-2}.
\]
Taking \( \tau < \frac{\epsilon}{2} \) and invoking c) the result follows.

e) We now prove
\[ (*) \quad (\chi(A)u, u) < 1 \Rightarrow u \in \Omega. \]
Suppose \( u \in X_\tau \) for some \( t > 0 \). Then \( \sum_{n=1}^{\infty} \|P_n u\|_t^2 < \infty \) and for some \( \tau, \ 0 < \tau < t \) it follows with b)
\[ (*) \quad \|P_n u\|_t^2 \leq e^{-2(n-1)(t-\tau)} \|u\|_t^2. \]
Further because of our assumption \((*)\) and b)
\[ \|P_n u\|_t \leq e^{n\tau} \|P_n u\|_0 < \frac{n^2}{2n^2} r_{n,1} e^{n\tau-n} \leq \frac{n^2}{2n^2} r_{n,\tau}. \]
Therefore \( 2n^2 P_n u \in \Omega \cap X_\tau \) for every \( n \in \mathbb{N} \).
In $X_t$ we represent $u$ by

$$u = \sum_{n=1}^{N} \frac{1}{2n} (2n^2 P_n u) + (\sum_{n=N}^{\infty} \frac{1}{2n^2} u_n)$$

with

$$u_N = (\sum_{j=N}^{\infty} \frac{1}{2j^2})^{-1} \sum_{n=N}^{\infty} \frac{1}{2n} P_n u.$$ 

With (*) we calculate

$$\|u_N\|_\tau^2 \leq 4N^4 \sum_{n=N}^{\infty} \|P_n u\|_\tau^2 \leq 4N^4 e^{(2N+2)(t-1)} \|u\|_\tau^2.$$ 

Since $\Omega \cap X_t$ contains an open neighbourhood of $0$ for $N$ sufficiently large $u_N \in \Omega \cap X_t$.

Finally we gather that $u$ is a sub-convex combination of elements in $\Omega \cap X_t$ which is a convex set.

(A posteriori it is clear that $u \in \Omega \cap X_t$.)

f) We finish the proof of II by taking $\psi \in T$ such that on $(n-1,n]$ it is equal to the smallest natural number greater than $4n^2 e^{2n} (r_n,1)^{-2}$.

III) Any convex open neighbourhood of $0$ satisfies the conditions of the set $\Omega$ in II. Therefore it contains a set $U_{\psi,\varepsilon}$ with $\psi \in T$ and $\varepsilon > 0$.

IV) $V$ contains a set $U_{\psi,\varepsilon}$. The results IIb and IIc show that $U_{\psi,\varepsilon} \cap X_t$ contains arbitrary large elements.

Remark. Similar to the proof of part I one proves that the sets

$$\bar{U}_{\psi,\varepsilon} = \{ u \in S \mid (\psi(A)u, u) \leq \varepsilon^2 , \psi \in B_+ , \varepsilon > 0 \}$$

are closed.

Definition 1.5.

A subset $W \subset S$ is called bounded if for each $0$-neighbourhood $U$ in $S$ there exists a complex number $\lambda$ such that $W \subset \lambda U$. Cf. [SCH] 1.5.
In the next theorem we characterize bounded sets in $S$.

**Theorem 1.6.**

A subset $W \subset S$ is bounded iff

$$\exists t > 0 \ \exists M > 0 \ \forall u \in W \quad \|u\|_t \leq M.$$ 

**Proof.**

$W$ is bounded iff $\forall \psi \in B, \ \exists M \geq 0 \ \forall u \in W \quad (\psi (A) u, u) \leq M$. Therefore by taking $\psi = 1$ we observe that $W$ is a bounded set in $X$. So we have

$$\exists \rho > 0 \ \forall u \in W \quad (u, u) \leq \rho$$

and

$$\forall u \in W \ \forall M > 0 \ \forall t > 0 \ \int_0^M e^{2 \lambda t} d(E_{\lambda} u, u) \leq e^{2 M t} \rho$$

$\Rightarrow \forall u \in W \quad (\psi (A) u, u) = (\psi (A) M_{2s} u, M_{-s} u) \leq \|\psi (A) M_{2s} \| \| M_{-s} u \|^2 \leq c_1 \| u \|_s^2 = c_1 M^2$

which is a constant only depending on $\psi$.

$\Rightarrow$ Suppose the statement were not true then

$$\forall t > 0 \ \forall M > 0 \ \exists u \in W \ |M u| > M.$$ 

By induction we define two sequences of real numbers $\{t_n\}, \{N_n\}, t_n \to 0$, $N_n \to \infty$ as $n \to \infty$ and a sequence $\{u_n\} \subset W$ as follows

$n = 1$: Choose $t_1 > 0$ and $M = 2$. Then take $N_1 > 0$ and $u_1 \in W$ such that

$$\int_0^{N_1} e^{2 t_1 \lambda} d(E_{\lambda} u_1, u_1) > 1.$$ 

$n = k + 1$: Suppose $\forall t \leq t_k \forall u \in W \ \forall k > 0 \ \int_{N_k}^{N_k + k} e^{2 t \lambda} d(E_{\lambda} u, u) \leq k + 1$ is true.

Then $W$ is a bounded set in $X_t$, $t \leq t_k$, because with the aid of ($\ast$)

$$\int_0^{N_k} e^{2 t \lambda} d(E_{\lambda} u, u) = \int_0^{N_k} + \int_{N_k}^{\infty} e^{2 t \lambda} \rho \leq e^{2 N_k \rho} + k + 1.$$
If our sequences terminate for a certain value of \( n \) then \( W \) is a bounded set.

If not, define the function \( \chi(\lambda) \) on \((0,\infty)\) by

\[
\eta(\lambda) = e^{-\lambda n} \text{ on the interval } (N_{n-1}, N_n], \quad n = 1, 2, \ldots
\]

Since \( \eta(\lambda)e^{-\lambda t} \) is a bounded function for \( t > 0, \eta \in B_+ \) and the sequence \( \{\eta(A)u_n, u_n\} \) should be bounded. However \( \eta(A)u_n, u_n \) = \( \int_0^\infty \eta(\lambda)d(E_{\lambda}u_n, u_n) > n + 1 \).

Contradiction!

In the next theorem we give a characterization for the convergence of sequences \{\( u_n \)\} in the strong topology of \( S \).

**Theorem 1.7.**

\[
\exists t > C \{u_n\} \subset X_t \text{ and } \|u_n\|_t \to 0.
\]

**Proof.**

\( \Leftarrow \) For any \( \psi \in B_+ \) we have \( \psi(A)u_n, u_n = (\psi(A)M_2tM_2-tu_n, M_2-tu_n) \leq \|\psi(A)M_2t\|_t\|M_2-tu_n\|^2 \to 0 \) as \( n \to \infty \) because \( \psi(A)M_2t \) is a bounded operator on \( X_t \).

\( \Rightarrow \) Suppose \( u_n \to 0 \). Then \( \forall \psi \in B_+ \) \( (\psi(A)u_n, u_n) \to 0 \).

We conclude that the sequence \( \{u_n\} \) is a bounded set. So \( \exists \theta > 0 \forall n \in \mathbb{N} \|u_n\| \leq \theta \).

Further, taking \( \psi = 1 \), it is clear that \( \|u_n\|_0 \to 0 \) as \( n \to \infty \). From this we derive

\[
\forall \alpha > 0 \forall t > 0 \int_0^\alpha e^{2\lambda t}d(E_{\lambda}u_n, u_n) \to 0 \text{ as } n \to \infty.
\]

Now we show \( \|u_n\|_t \to 0 \) for any \( t < T \).

\[
\|u_n\|^2_t = \int_0^L e^{2\lambda t}d(E_{\lambda}u_n, u_n) + \int_L^\infty e^{2\lambda t}d(E_{\lambda}u_n, u_n).
\]
The second integral can be estimated uniformly

\[
\int_{L}^{\infty} e^{2\lambda t} d(E, u_n, u_n) = \int_{L}^{\infty} e^{-2\lambda (T-t)} e^{2\lambda t} d(E, u_n, u_n) \leq e^{2L(T-t)} 2^L.
\]

By taking \( L \) sufficiently large the second term in \((\lambda)\) can be made smaller than \( \epsilon \) uniformly in \( n \).

The first term in \((\lambda)\) tends to 0 as \( n \to \infty \) because of \((\ast)\).

\[\Box\]

Theorem 1.8.

I. Suppose \( \{u_n\} \) is a Cauchy sequence in the strong topology of \( S \). Then

\[\exists t > 0 \{u_n\} \subset X_t \text{ and } \{u_n\} \text{ is a Cauchy sequence in } X_t.\]

II. \( S \) is sequentially complete, i.e. every Cauchy sequence converges to a limit point.

\textbf{Proof}

I. An argument similar to the proof of the preceding theorem.

II. Follows from I and the completeness of \( X_t \).

Next we characterize compact sets in \( S \).

Theorem 1.9.

A subset \( K \subset S \) is compact iff

\[\exists t > 0 \ K \subset X_t \text{ and } K \text{ is compact in } X_t.\]

\textbf{Proof}.

\(\Rightarrow\) Let \( \{\Omega_a\} \) be an open covering of \( K \) in \( S \). Then \( \{\Omega_a \cap X_t\} \) is an open covering of \( K \) in \( X_t \). Since \( K \) is supposed to be compact in \( X_t \) there exists a finite subcovering \( \{\Omega_{a_i}\}, \ 1 \leq i \leq N \), with \( \bigcup_{i=1}^{N} (\Omega_{a_i} \cap X_t) \supset K. \) But then certainly \( \bigcup_{i=1}^{N} \Omega_{a_i} \supset K. \)

\(\Rightarrow\) Since \( K \) is compact it is bounded and therefore it is a bounded set with
bound $\theta$ in $X_t$ for some $t > 0$. We show that $K$ is compact in $X_t$ whenever $t < \tau$. Consider a sequence $\{u_n\} \subset K$. There exists a converging subsequence $\{u_{n_j}\}$ with $u_{n_j} \to u \in K$ as $j \to \infty$, convergence in $S$. Then also $u_{n_j} \to u$ in $X_t$-sense. Put $u_j = u_{n_j} - u = v_j$. $\{v_j\}$ is a bounded sequence in $X_t$ and $\|v_j\| \to 0$ as $j \to \infty$. Now we find ourselves in exactly the same position as in the proof of the only-if-part of Theorem 1.7. We conclude $\|v_j\| \to 0$ whenever $0 \leq t < \tau$. Our subsequence $\{u_{n_j}\}$ converges to $u$ in $X_t$-sense. This shows the compactness of $K$ in $X_t$ for $0 \leq t < \tau$.

Lemma 1.10.

Let $u \in X$ be such that $u \in D(\psi(A))$ for all $\psi \in B_+$. Then $u \in S$.

Proof

If not, then for every $t > 0$ the integral $\int_0^\infty e^{2\lambda t} d(E_\lambda u, u)$ is divergent. Pick sequences of real numbers $\{t_i\}, \{N_i\}$ with $t_i \to 0$ and $N_i \to \infty$ such that for all $i \in \mathbb{N}$

$$\int_{N_i-1}^{N_i} e^{2\lambda t} d(E_\lambda u, u) > 1.$$  

Define $\psi$ on $(0, \infty)$ by $\psi(\lambda) = e^{\lambda t_i}$ on $(N_{i-1}, N_i]$. Then $\psi^2 \in B_+$ and $\|\psi(A)u\|^2 = \int_0^\infty \psi^2(\lambda) d(E_\lambda u, u) = \infty$. Contradiction!

In order to make a link with the literature on topological vector spaces we now describe the properties of our space $S$ by using the standard terminology of topological vector spaces. [SCH].

The terminology is explained in the proof.

Theorem 1.11

I $S_{X,A}$ is complete

II $S_{X,A}$ is bornological
III. $S_{X,A}$ is barreled

IV. $S_{X,A}$ is Montel iff for every $t > 0$ the operator $M_t = e^{-At}$ is compact as an operator on $X$.

V. $S_{X,A}$ is Nuclear iff for every $t > 0$ the operator $M_t = e^{-At}$ is a HS (=Hilbert-Schmidt) operator on $X$.

Proof

I. Let $(x_\alpha)$ be a Cauchy net. The $\alpha$'s belong to a directed set $D$. For each neighbourhood $\Omega \ni 0$ there is $\gamma \in D$ such that whenever $\alpha \succ \gamma$ and $\beta \succ \gamma$ one has $x_\alpha - x_\beta \in \Omega$. We now prove that there exists an $x \in S$ such that $x_\alpha \to x$ in the strong topology. Let $x$ be the limit of $x_\alpha$ in $X$-sense. For each $\psi \in B_+$ the net $(\psi(A)x_\alpha)$ converges in $X$-sense to a limit $x_\psi$. Because of the closedness of $\psi(A)$ one has $x_\psi \in D(\psi(A))$ and $x_\psi = \psi(A)x$.

The result follows by applying Lemma 1.10.

II. Every circled convex subset $\Omega \subset S$ that absorbs every bounded subset $W \subset S$ has to be a neighbourhood of $0$. Let $B_t$ be the open unit ball in $X_t$, $t > 0$. $B_t$ is bounded in $S$ therefore for some $\varepsilon > 0$ one has $\varepsilon B_t \subset \Omega \cap X_t$.

We conclude that for every $t > 0$ the set $\Omega \cap X_t$ contains an open neighbourhood of $0$.

But then according to Theorem 1.4.II $\Omega$ contains a set $U_{\psi,\varepsilon}$.

III. A barrel $V$ is a subset which is radical, convex, circled and closed. We have to prove that every barrel contains an open neighbourhood of the origin. Because of the definition of the inductive topology $V \cap X_t$ has to be a barrel in $X_t$ in the $X_t$-topology, for every $t > 0$. Since $X_t$ is a Hilbert space there exists an open neighbourhood of the origin $\Sigma$ with $\Sigma \subset V \cap X_t$. Again the conditions of Theorem 1.4.II are satisfied so that $V$ contains a set $U_{\psi,\varepsilon}$.

IV. We have to prove that every closed and bounded subset of $S$ is compact iff for every $t > 0$ the operator $M_t$ is compact.

$\Rightarrow$ Suppose $M_t$ is compact for every $t > 0$. Let $W$ be a closed and bounded subset in $S$. Then $W \subset X_t$ for some $t > 0$ and $W$ is closed and bounded in $X_t$. See Theorem 1.6. Take $\tau$, $0 < \tau < t$. We claim that $W$ is a compact set in $X_t$. To see this, consider the following commutative diagram where $C_\tau$
denotes the natural injection

\[
\begin{array}{ccc}
X_t & \hookrightarrow & X_t \\
M_t & \uparrow & M_t \\
X_0 & \longrightarrow & X_0 \\
M_t - \tau
\end{array}
\]

The vertical arrows are isomorphisms. So \( \mathcal{G} \) is a compact map and \( \mathcal{W} \) is compact in \( X_t \). Consider an open covering \( \{ C_\alpha \} \) of \( \mathcal{W} \) in \( S \), then \( \{ C_\alpha \cap X_t \} \) is an open covering of \( \mathcal{W} \) in \( X_t \). Because of the compactness of \( \mathcal{W} \) in \( X_t \) there is a finite subcovering:

\[
\mathcal{W} \subset \bigcup_{i=1}^{N} (C_\alpha \cap X_t).
\]

But then certainly \( \mathcal{W} \subset \bigcup_{i=1}^{N} C_\alpha \), which shows the compactness of \( \mathcal{W} \).

\( \Rightarrow \) Suppose \( S \) is Montel, i.e. each closed and bounded set is compact. Let \( \{ u_n \} \) be a bounded sequence in \( X \). Pick any fixed \( t > 0 \). Consider the sequence \( \{ M_t u_n \} \). Consider the closure in \( S \) of this sequence. This closure is a bounded set and, according to our assumption, compact. Thus \( \{ M_t u_n \} \) contains a converging subsequence in \( S \): \( M_t u_n \to v \). This sequence certainly converges in \( X \). So \( M_t \) must be a compact operator.

**V. \( \Rightarrow \)** We proceed in four steps: a,b,c and d.

a) Suppose that for every \( t > 0 \) \( M_t \) is H.S., i.e. for any orthonormal basis \( \{ e_i \} \) in \( X \)

\[
\forall t > 0 \quad \sum_{i=1}^{\infty} \| M_t e_i \|^2 < \infty.
\]

(*)

It follows that \( M_t = e^{-At} \) is compact. Hence \( A \) has a purely discrete spectrum with no finite accumulation point and all eigenspaces are finite dimensional. We order the eigenvalues in a non-decreasing sequence in which each eigenvalue occurs as many times as the
dimension of its eigenspace

\[ 0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots, \quad \lambda_n \to \infty \text{ as } n \to \infty. \]

We choose an orthonormal base of eigenvectors \{e_n\}. Together with (*) we obtain for each \( t > 0 \),

\[ \sum_{k=1}^{\infty} e^{-\lambda_k t} < \infty \text{ and a posteriori } \sum_{k=1}^{\infty} e^{-\lambda_k t} < \infty. \]

b) Let \( B_1 \) and \( B_2 \) be Banach spaces. A bounded operator \( T : B_1 \to B_2 \) is called nuclear if there exists a sequence \( \{f_n\} \subset B_1' \), a sequence \( \{g_n\} \subset B_2 \) and a sequence of complex numbers \( \{c_n\} \) such that

\[ \sup_n (\|f_n\|, \|g_n\|) < \infty, \]

\[ \sum_{n=1}^{\infty} |c_n| < \infty \]

and

\[ \forall x \in B_1 \quad Tx = \lim_{m \to \infty} \sum_{n=1}^{m} c_n \langle x, f_n \rangle g_n \quad \text{in } B_2. \]

Our operator \( e^{-At} \) can be represented by

\[ e^{-At} u = \sum_{n=1}^{\infty} e^{-\lambda_n t} \langle u, e_n \rangle e_n. \]

and is obviously nuclear.

c) Our aim is now to construct an invertible nuclear operator \( v(A) \) of the form

\[ v(A)u = \sum_{k=1}^{\infty} v_k(u, e_k)e_k. \]

in such a way that \( v(A)^{-1} \in B_+(A) \).

Define integers \( N(n)\), \( n = 0, 1, 2, \ldots\), such that \( N(0) = 0\), \( N(n + 1) > N(n)\),

\[ \lambda_{N(n)+1} > \lambda_{N(n)} \text{ and } \]

\[ \sum_{j=N(n)}^{\infty} e^{-\lambda_j 1/n} < \frac{1}{n^2}. \]
Now define the sequence \( \{ \nu_k \} \) by
\[

\nu_k = e^{\frac{-\lambda k}{n}}, \quad N(n-1) < k \leq N(n), \quad n = 0, 1, 2, \ldots.
\]

It is clear that \( \sum_{k=1}^{\infty} \nu_k \) is convergent.

Define a step function \( \sigma \in \mathcal{B} \) by setting \( \sigma(\lambda) = e^{\frac{\lambda}{n}} \) on the interval \((\frac{\lambda - 1}{2}, \frac{\lambda}{2}, \frac{\lambda + 1}{2})\), \( k = 2, 3, \ldots \) and \( \sigma(\lambda) = e^{1} \) on \([0, \frac{\lambda + 1}{2}]\).

It is easily seen that \( \sigma(A)^{-1} = \nu(A) \).

d) \( S \) is a nuclear space iff for every continuous semi-norm \( p \) on \( S \) there is another semi-norm \( q \geq p \) such that the canonical injection \( \hat{S}_q \rightarrow \hat{S}_p \) is a nuclear map. Here the Banach space \( \hat{S}_p \) is defined as the completion of the quotient space \( S/(p^{-1}(0)) \). Since the composition of a nuclear operator with a bounded operator is again nuclear, \( S \) is a nuclear space iff for every semi-norm \( p_\psi \), \( \psi \in \mathcal{B}_+ \), there is a semi-norm \( p_\chi \), \( \chi \in \mathcal{B}_+ \), \( \chi \geq \psi \) such that the canonical injection \( J : \hat{S}_{p_\chi} \rightarrow \hat{S}_{p_\psi} \) is nuclear.

This canonical injection can be written
\[
J u = \sum_{j=1}^{\infty} \chi^{-\lambda_j}(\lambda_j)^{\psi}(\lambda_j)(\chi(A)u, \chi^{-\lambda_j}(A)e_j)\psi(\lambda_j)e_j.
\]

We remind that \( \{ \chi^{-\lambda_j}(A)e_j \} \) and \( \{ \psi^{-\lambda_j}(A)e_j \} \) are orthonormal bases in the Hilbert spaces \( \hat{S}_{p_\chi} \) respectively \( \hat{S}_{p_\psi} \). If we take \( \chi = \sigma^2 \psi \) then (a) becomes
\[
J u = \sum_{j=1}^{\infty} \sigma^{-1}(\lambda_j)(\chi(A)u, \chi^{-\lambda_j}(A)e_j)\psi^{-1}(\lambda_j)e_j.
\]

And this operator is nuclear because \( \sigma^{-1}(\lambda_j) = \nu_j \).

\( \Rightarrow \) Suppose \( S \) is nuclear. See the definition in d) above. Take for \( p \) the \( \chi \)-norm.

Then for some semi-norm \( q \) the injection \( \hat{S}_q \rightarrow \chi \) is nuclear map. Further we can take \( \psi \in \mathcal{B}_+ \) such that \( p_\psi \geq q \). (Apply Theorem 1.4.II). The injection \( \hat{S}_p \rightarrow \chi \) must be nuclear. Therefore \( \psi(A)^{-1} \) must be a nuclear operator and consequently also H.S. But then: \( e^{-At} = \psi(A)^{-1}[\psi(A)e^{-At}] \) must be H.S. since the operator in square brackets is bounded for every \( t > 0 \).
CHAPTER 2. The space $T_{X,A}$

We now introduce our space of trajectories $T_{X,A}$. The elements in this space are candidates for becoming "generalized functions" (or "distributions").

**Definition 2.1.**

$T_{X,A}$ denotes the complex vector space which consists of all mappings $F: \mathbb{C}^+ \to X$ such that

(i) $F$ is holomorphic.

(ii) $\forall t, \tau \in \mathbb{C}^+ \quad M_{t+\tau}F(t) = F(t+\tau)$.

Such a mapping $F$ will be called a trajectory. If for some $\xi \in \mathbb{C}^+$ $F_1(\xi) = F_2(\xi)$ then $F_1 = F_2$ throughout $\mathbb{C}^+$. This follows immediately from the semi-group and analyticity properties. We shall use the notation $M_tF$ for $F(t)$. It is immediate that $\forall t > 0 \quad M_tF \in S$. In chapters 2 and 3 $T_{X,A}$ is abbreviated by $T$.

**Definition 2.2.**

The embedding $\text{emb}: X \to T$ is defined by $(\text{emb} x)(t) = M_t x$ for all $x \in X$ and $t \in \mathbb{C}^+$. Sometimes we shall omit the symbol $\text{emb}$ and loosely consider $X$ as a subset of $T$. Thus $S \subset X \subset T$.

The question arises whether there exist trajectories $F \in T$ such that $F(t)$ does not converge or, even $\|F(t)\| \not\to 0$ if $t \not\to 0$. The answer is affirmative: simply take $x \in X \setminus D(A)$ and $F(t) = M_t x$. More general examples are obtained by taking $\psi \in B, u \in X$ and defining $G(t) = \psi(A)M_t u, t \in \mathbb{C}^+$. The next theorem shows that all elements of $T$ arise in this way.

**Theorem 2.3.**

For every $F \in T$ there exists $w \in X$ and $\psi \in B^+$ such that $F(t) = \psi(A)e^{-At}w$ for every $t \in \mathbb{C}^+$.

**Proof**

If the integral $\int_0^\infty e^{2\lambda}d(E_\lambda F(1),F(1))$ converges then $F(1) \in D(e^A)$. 


Take \( w = e^{A} F(1) \) and \( \psi = 1 \). Now suppose this integral is divergent. Take a sequence \( \{ \tau_n \} \), \( \tau_n \to 0 \). All the integrals \( \int_0^\infty e^{-2\tau_n} d(E, F(1), F(1)) \), \( n = 1, 2, \ldots \), converge. Define a sequence of real numbers \( \{ N(n) \} \), \( n = 0, 1, 2, \ldots \), \( N(n) \to \infty \) such that \( N(0) = 0 \), \( N(n) > N(n - 1) \) and

\[
\int_{N(n)}^\infty e^{2\tau_n} d(E, F(1), F(1)) < \frac{1}{n^2}.
\]

Define \( \psi \) on \([0, \infty)\) by \( e^{1} \) on \([0, N(1)]\) and \( e^{\tau + 1} \) on \((N(n), N(n + 1)]\), \( n = 1, 2, \ldots \). Then \( \psi \in B^+ \) and \( \int_0^\infty e^{2\lambda \psi^{-2}(\lambda)} d(E, F(1), F(1)) < \infty \), which means \( F(1) \in D(\psi^{-1}(A)e^A) \). Take

\[
w = \psi^{-1}(A)e^A F(1).
\]

In \( T \) we introduce the topology of uniform convergence on compacta in \( \mathbb{X}^+ \). Because of the semi-group property \( F_n + F \) in \( T \) iff \( \forall m \in \mathbb{N} F_n \left( \frac{1}{m} \right) \to F \left( \frac{1}{m} \right) \) in the \( X \)-norm. More precisely

**Definition 2.4.**

The strong topology in \( T \) is the topology induced by the semi-norms \( \rho_n \), \( n \in \mathbb{N} \),

\[
\rho_n(F) = \| F \left( \frac{1}{n} \right) \|_X.
\]

In other words a base of open \( 0 \)-neighbourhood is given by

\[
W_{c_1 c_2 \ldots c_N} = \{ F \| F \left( \frac{1}{j} \right) \| < c_j, \ c_j > 0, \ 1 \leq j \leq N, \ N \in \mathbb{N} \}.
\]

**Theorem 2.5.**

I. \( T \) endowed with the strong topology is a Frechet space

II. A base of open sets \( \{ \sigma_{U, t} \} \) is given by \( \sigma_{U, t} = \{ F | F(t) \in U, \ U \text{ open in } X, \ t > 0 \text{ and fixed} \} \).

(In words: the set of trajectories which pass at \( t \) through \( U \)).

Each open set in \( T \) is a denumerable union of sets of this type.
Proof

I. Frechet means metrizable and complete. We define the distance of $F$ from the origin in the standard way by

$$d(F) = \sum_{n=1}^{\infty} 2^{-n} \| F(\frac{1}{n}) \| (1 + \| F(\frac{1}{n}) \|)^{-1}.$$ 

The set of $F$ with $d(F) < 2^{-N-1} \min(1, c_1, \ldots, c_N)$ certainly lies within $\mathcal{W}$. On the other hand each $\mathcal{W}$ lies within a ball centered at the origin with radius

$$\frac{1}{2N} \sum_{n=1}^{N} \frac{c_n}{1 + c_n} + \frac{1}{2}.$$ 

Therefore the sets $\mathcal{W}_n = \{ F | d(F) < \frac{1}{n}, \, n \in \mathbb{N} \}$ establish a base of 0-neighbourhoods. Further, since $\mathcal{Y}$ is metrizable it is complete iff each Cauchy sequence converges. Let $\{ F_m \}$ be a Cauchy sequence in $\mathcal{Y}$.

Then for each $n \in \mathbb{N}$, $\{ F_m(\frac{1}{n}) \}$ is a Cauchy sequence in $X$ which converges to an element $h_{1/n} \in X$. For $\nu > n$ one has $F_m(\frac{1}{\nu}) = e^{((1/n)-(1/\nu))A} \frac{1}{n}$. Since both sides converge and $e^{((1/n)-(1/\nu))A}$ is a closed operator we have $h_{1/n} \in \mathcal{D}(e^{((1/n)-(1/\nu))A})$ and $h_{1/\nu} = e^{((1/n)-(1/\nu))A} h_{1/n}$. Now we define

$$F(t) = e^{-(t-(1/\nu))A} h_{1/\nu}, \quad \text{Re } t \in [\frac{1}{\nu}, \frac{1}{\nu-1}], \, \nu \in \mathbb{N}.$$ 

Clearly $F \in \mathcal{Y}$ and is the limit of $\{ F_m \}$.

II. We proceed in three steps.

a) First we prove that, given $\sigma_{U,t}$, then for each $\tau$, $0 < \tau < t$, there exists an open set $V \subset X$ such that $\sigma_{U,t} = \sigma_{V,\tau}$. For $V$ we take the inverse image of $U$ under the mapping $e^{-A(t-\tau)}$, then $e^{-A(t-\tau)}(V) \subset U$. It is clear that each trajectory which passes at $t$ through $U$, passes at $\tau$ through $V$ and conversely.

b) We prove that $\sigma_{U,1/n}$, $U \subset X$ open, $n \in \mathbb{N}$, is an open set: $U$ can be covered by open balls lying inside $U$, therefore $\sigma_{U,1/n}$ is a union of translates of sets of type $\mathcal{W}_{c_1 \cdots c_N}$. 

c) From a) and b) we conclude that the sets $\sigma_{U,t}$ are open. Further $\sigma_{U,t} \cap \sigma_{V,t}$ with $\tau < t$ can be written as $\sigma_{U,\tau} \cap \sigma_{V,\tau} = \sigma_{U} \cap \sigma_{V}$.

Here $\sigma_{U}$ is the inverse image of $U$ under $e^{-\frac{(t-\tau)A}{n}}$. Now it is easy to see that each translate of a set of type $W_{c_1, \ldots, c_N}$ is a set of type $\sigma_{U,t}$. We are ready if we show that an arbitrary union of sets $\sigma_{U,t}$ can be reduced to a denumerable union of sets of this type. This can be achieved by the construction

$$\bigcup_{\frac{1}{n+1} < t < \frac{1}{n}} \sigma_{U,t} = \sigma_{V, \frac{1}{n+1}}$$

with

$$V = \bigcup_{\frac{1}{n+1} < t < \frac{1}{n}} \sigma_{U}$$

where for each $t \in \left[\frac{1}{n+1}, \frac{1}{n}\right)$ $\sigma_{U}$ denotes the reduction of $U$ at the point $t$ to the point $\frac{1}{n+1}$ in the way we described in a).}

**Theorem 2.6.**

$\text{emb}(S)$ is everywhere dense in $T$.

**Proof**

For any $F \in T$ the sequence $\text{emb}(F(1/n))$ converges to $F$ in the strong topology.

**Theorem 2.7.**

A set $B \subset T$ is bounded iff for every $t > 0$ the set $\{F(t) | F \in B\}$ is bounded in $X$.

**Proof**

$\Rightarrow$ Each continuous semi-norm has to be bounded on $B$.

Therefore $\rho_n(F) = \|F(1/n)\|$ is a bounded function on $B$. 


Because of the boundedness of $M_t$ for each $t > 0$ it then follows that $\{F(t) \mid F \in B\}$ is a bounded set for each fixed $t > 0$.

$\implies$ Suppose for each fixed $t > 0$ the set $\{\|F(t)\| \mid F \in B\}$ is bounded.

We have to prove that each open $0$-neighbourhood can be blown up such that $B$ is contained in it. Let $\alpha_t = \sup_{F \in B} \|F(t)\|$, $t$ fixed. Then $B$ lies within an open ball with radius $\sum_{n=1}^{\infty} 2^{-n} \alpha_1/n (1 + \alpha_1/n)^{-1}$.

**Example**

Let $B$ be a bounded set in $X$. Let $\psi \in B$. Then the set $\{F \mid F(t) = \psi(A)M_{tx}, x \in B\}$ is bounded in $T$.

**Theorem 2.8.**

A set $K \subset T$ is compact iff for each $t > 0$ the set $\{F(t) \mid F \in K\}$ is compact in $X$.

**Proof.**

$\implies$ If $K$ is compact then each sequence $\{F_n\} \subset K$ has a convergent subsequence. This means that in the set $K_t = \{F(t) \mid F \in K, t$ fixed$\}$ each sequence has a convergent subsequence, which says that $K_t$ is compact in $X$.

$\impliedby$ Let $\{F_n\}$ be a sequence in $K$. We must prove the existence of a converging subsequence. Consider the sequence $\{F_n(1)\} \subset K_1 \subset X$. $K_1$ is compact therefore a convergent subsequence in $K_1$ exists. Denote it by $\{F_n^1(1)\}$. The sequence $\{F_n^1(\gamma)\}$ has a convergent subsequence in $K_\gamma$. Denote it by $\{F_n^2(\gamma)\}$.

Proceeding in this way we arrive at sequences $\{F_n^m\} \subset K$ such that $\{F_n^m\} \subset \{F_n^{l}\}$ for $m > l$ and $\{F_n^m(1/m)\}$ converges to an element in $K_{1/m}$.

The "diagonal sequence" $\{F_n^n\}$ has the property that $\{F_n^n(t)\}$ converges to $F(t) \in K_t$. But then also $F_n^n \to F$ in the strong topology.
Example
Let \( K \) be a compact set in \( X \). Let \( \psi \in \mathcal{B} \). Then the set \( \{ F \mid F(t) = \psi(A)M_t x, \ x \in K \} \) is compact in \( T \).

In the last theorem of this chapter we describe the properties of our topological vectorspace \( T \) in the standard terminology of topological vectorspaces. [SCH].

Theorem 2.9.

I. \( T_{X,A} \) is bornological
II. \( T_{X,A} \) is barreled
III. \( T_{X,A} \) is Montel iff for every \( t > 0 \) the operator \( M_t = e^{-At} \) is compact as an operator on \( X \).
IV. \( T_{X,A} \) is Nuclear iff for every \( t > 0 \) the operator \( M_t = e^{-At} \) is a H.S.-operator on \( X \).

Proof

I,II. \( T \) is bornological and barreled because it is metrizable. For a simple proof see [SCH] II.8.

III \( \Rightarrow \) Suppose \( T \) is Montel. Take a bounded sequence \( \{x_n\} \subset X \). The sequence \( \{F_n\} \) defined by \( F_n(t) = M_t x_n \) is a bounded set in \( T \). Because of our assumption the closure of this set is compact. But then, by theorem 2.8, for each \( t > 0 \) the sequence \( F_n(t) \) contains a converging subsequence. This shows the compactness of \( M_t \).

\( \Leftarrow \) Suppose \( M_t \) is compact for every \( t > 0 \).
Let \( B \) be a closed and bounded set. Then the set \( B_t = \{ F(t) \mid F \in B, \ t > 0 \text{ and fixed} \} \) is bounded. Since \( B_{t+t} = M_t B_t \) it follows that \( B_{t+t} \) is precompact in \( X \). Now take any sequence \( \{G_n\} \subset B \). Because of the precompactness of each \( B_t \) the "diagonal procedure" of the proof of theorem 2.8 yields a converging subsequence of \( \{G_n\} \). This subsequence converges to an element in \( B \) because \( B \) is closed. We conclude that \( B \) is compact.
Suppose $T$ is a nuclear space. Take $n \in \mathbb{N}$. There exists a semi-norm $\rho \geq \rho_n$ such that the natural injection $\tilde{T}_\rho \to \tilde{T}_\rho = X$ is nuclear. However there exists $m > n$ and $\alpha > 0$ such that the semi-norm $\alpha \rho_m \geq \rho$. Since the composition of a nuclear and a bounded map is nuclear the natural injection $\tilde{T}_\rho \to \tilde{T}_\rho$ is nuclear.

This natural injection is realized by the map $M_{(1/n)-(1/m)}$ which must therefore be nuclear. It follows that $M_t$ is H.S. for each $t > 0$.

Suppose for each $t > 0$ $M_t$ is H.S. Because of the semi-group property $M_t$ is also nuclear. Let $m > n$. The natural injection $\tilde{T}_\rho \to \tilde{T}_\rho$ is realized by $M_{(1/n)-(1/m)}$ and is therefore nuclear. Now let $\rho$ be an arbitrary semi-norm on $T$.

There exist $\alpha > 0$ and $n \in \mathbb{N}$ such that $\alpha \rho_m > \rho$ on $T$. Then for $m > n$ we have $\alpha \rho_m > \rho$ and the natural injection $\tilde{T}_{\alpha \rho_m} \to \tilde{T}_{\rho}$ is nuclear. \(\square\)
CHAPTER 3. The pairing of $S_{X,A}$ and $T_{X,A}$

We now consider a pairing of $S$ and $T$. It turns out that $S$ and $T$ can be considered as the strong topological dual spaces of each other.

**Definition 3.1.**

On $S_{X,A} \times T_{X,A}$ we introduce a sesquilinear form by

$$\langle u, F \rangle_X = (M_{-t} u, F(t))_X.$$  

Note that this definition makes sense for $t$ sufficiently small. Note also that because of the semi-group property and self-adjointness of $M^t$ the definition does not depend on the choice of $t$. We remark that $\langle u_0, F \rangle = 0$ for all $F \in T$ implies $u_0 = 0$ and $\langle u, F_0 \rangle = 0$ for all $u \in S$ implies $F_0 = 0$. These two facts easily follow by taking $F = \text{emb}(u_0)$, respectively the denseness of $\text{emb}(S)$ in $T$.

**Theorem 3.2.**

I. For each $F \in T$ the linear functional $\langle u, F \rangle$, which maps $S$ onto $\mathbb{C}$, is continuous in the strong topology of $S$.

II. For each strongly continuous linear functional $\ell$ on $S$ there exists $G \in T$ such that $\ell(u) = \langle u, G \rangle$ for all $u \in S$.

III. For each $\nu \in S$ the linear functional $\overline{\langle \nu, G \rangle}$, which maps $T$ onto $\mathbb{C}$, is continuous in the strong topology of $T$.

IV. For each strongly continuous linear functional $m$ on $T$ there exists $w \in S$ such that $m(F) = \langle w, F \rangle$ for all $F \in T$.

**Proof**

I. $\langle u, F \rangle$ is continuous on $S$ iff for all $t > 0$ it is continuous in the $X_t$-norm when restricted to this subspace. Indeed

$$|\langle u, F \rangle| = |(M_{-t} u, F(t))| \leq \|M_{-t} u\| \|F(t)\|_0 = \|F(t)\| \|u\|_X.$$
II. For fixed $t > 0$ the mapping $M_t : X \to S$ is continuous. Therefore the mapping $\ell (M_t x) : X \to \mathbb{C}$ is continuous. From Riess' theorem follows the existence of $b_t \in X$ such that $\ell (M_t x) = \langle x, b_t \rangle$ for all $x \in X$.

If we replace $x$ by $M_t y$, $y \in X$, it follows from the self-adjointness of $M_t$ that $b_{t+\epsilon} = M_t b_t$. We take $G$ such that $G(t) = b_t$. Then $\ell (u) = \langle u, G \rangle$ for all $u \in S$.

III. Since $T$ is metrizable it is sufficient to prove continuity for sequences in $T$. Let $G_n \to 0$ in the strong topology. Then for sufficiently small $t$ we have $\langle v, G_n \rangle = (\overline{M_{-t} v, G_n(t)}) \to 0$ because $G_n(t) \to 0$ in $X$-sense.

IV. Take $\psi \in B^+$. Take a sequence $\{x_n\} \subset X$, $x_n \to 0$ in $X$. Define $\psi(A)x_n \in T$ by $\psi(A)x_n(t) = \psi(A)M_t x_n$. Since $\psi(A)M_t$ is a bounded operator $\psi(A)x_n \to 0$ strongly in $T$. By taking $\psi = 1$ we conclude first that the restriction of $m$ to $X$ has the Riess representation $m(x) = \langle x, \theta \rangle$ for some $\theta \in X$.

Secondly we conclude that $m(\psi(A)x)$ is a continuous function on $X$ for each $\psi \in B^+$. Using the self-adjointness $\psi(A)$ it follows that $\theta \in D(\psi(A))$ for each $\psi \in B^+$. But then, by Lemma 1.10, $\theta \in S$. Finally, as $X$ is dense in $T$ the representation $m(F) = \langle F, \theta \rangle = \overline{\langle \theta, F \rangle}$ is valid for all $F \in T$.

Definition 3.3.

The weak topology on $S$ is the topology induced by the semi-norms $p_F(u) = |\langle u, F \rangle|$, $F \in T$.

The weak topology on $T$ is the topology induced by the semi-norms $p_v(G) = |\langle v, G \rangle|$, $v \in S$.

A simple standard argument, [CH] II § 22, shows that the weakly continuous functionals on $S$ are all obtained by pairing with elements of $T$ and vice versa. Together with Theorem 3.2 it follows then that $S$ and $T$ are reflexive both in the strong and the weak topology.

In the sequel we shall denote $T$ by $S'$.

In the next theorem we show that in both spaces $S$ and $S'$ weakly bounded sets are strongly bounded.
Theorem 3.4. (Banach-Steinhaus)

I. Let \( \Xi \subset S' \) be such that for each \( g \in S \) there exists \( M_g > 0 \) such that for every \( F \in \Xi \) one has \( |\langle g, F \rangle| \leq M_g \), then for each \( t > 0 \) there exists \( C_t > 0 \) such that for every \( F \in \Xi \) one has \( \|F(t)\| \leq C_t \).

II. Let \( \Theta \subset S \) be such that for every \( F \in S' \) there exists \( M_F > 0 \) such that for every \( f \in \Theta \) one has \( |\langle f, F \rangle| \leq M_F \), then there exists \( \tau > 0 \) and \( c > 0 \) such that \( \Theta \subset X_\tau \) and \( \|f\|_\tau < c \) for all \( f \in \Theta \).

\textbf{Proof}

I. From the assumption it follows that for each \( h \in X \) and each \( t > 0 \) there exists a constant \( M_{h,t} > 0 \) such that for every \( F \in \Xi \) \( |\langle M_{h,t}, F \rangle| \leq M_{h,t} \) or \( |(h, F(t))| \leq M_{h,t} \). From the Banach-Steinhaus theorem in Hilbert space it then follows that the set \( \{F(t) | F \in \Xi \}, t \text{ fixed} \) is a bounded set in \( X \).

II. Let \( \psi \in B^+ \). Let \( w \in D(\psi(A)) \). Then \( \psi(A)^2 w \), defined by \( (\psi(A)^2 w)(t) = \psi(A)^2 M_{t,w} \), belongs to \( S' \). From our assumption it follows that for every \( f \in \Theta \) \( |\langle f, \psi(A)^2 w \rangle| = \|\psi(A) f, \psi(A) w\| < M_{\psi,w} \).

From the Banach-Steinhaus theorem in Hilbert space it then follows that \( \Theta \) is a bounded set in the Hilbert space \( X_\psi \), i.e. the completion of \( S \) with respect to the norm \( \|\psi(A) \cdot \| \). But this means that \( \Theta \) is bounded in \( S \), since each semi-norm \( p_{\psi, \psi} \), \( \psi \in B^+ \), is bounded on it.

In the next two theorems we give characterizations of weak convergence of sequences both in \( S \) and \( S' \).

\textbf{Theorem 3.5.}

\( u_n \to 0 \) in the weak topology of \( S \) iff

\[ \exists \tau > 0 \{u_n \} \subset X_\tau \ \text{and} \ \forall w \in X_\tau \ \langle w, u_n \rangle \to 0. \]

\textbf{Proof}

Weak convergence of \( \{u_n \} \) in \( X_\tau \) means weak convergence of \( M_{-\tau} u_n \) in \( X \).
\( \Leftrightarrow \) Let \( F \in S' \). \( \langle u_n, F(t) \rangle = (M_{-t}u_n, F(t)) \). Since \( M_{-t}u_n \to 0 \) weakly in \( X \) and \( F(t) \in X \), it follows that \( \langle u_n, F \rangle \to 0 \).

\( \Rightarrow \) First we remark that weak convergence in \( S \) implies weak convergence in \( X \). Therefore for any \( w \in X \) any \( L > 0 \), \( t > 0 \) \( \int_0^L e^{\lambda t} d(E_{\lambda u_n,w}) \to 0 \) as \( n \to \infty \). \( \{u_n\} \) is a bounded set in \( S \). Therefore \( \exists \tau > 0 \, \exists \theta > 0 \, \forall n \in \mathbb{N} \, \|u_n\|_\tau \leq \theta \). We shall prove that \( u_n \to 0 \) weakly in \( X_\tau \). Denote the projection operator \( \int_0^L dE_{\lambda} \) by \( \Pi_L \). \( \Pi_L \) commutes with \( M_{-\tau} \). Take \( w \in X \) then

\( \langle M_{-\tau} u_n, w \rangle = \int_0^L e^{\lambda t} d(E_{\lambda u_n,w}) + \langle \Pi_L M_{-\tau} u_n, \Pi_L w \rangle_0. \)

We have \( \Pi_L M_{-\tau} u_n \| \leq \theta \) and \( \Pi_L w \| \to 0 \) as \( L \to \infty \). Therefore if we take \( L \) large enough the second term in (*) is smaller than \( \frac{\theta \varepsilon}{2} \) uniformly for all \( n \). As we have just seen the first term in (*) can be made smaller than \( \frac{\theta \varepsilon}{2} \) by taking \( n \) large enough. This finishes the proof. \( \square \)

**Corollary 3.6.**

I. Strong convergence of a sequence in \( S \) implies its weak convergence.

II. Any bounded sequence in \( S \) has a weakly converging subsequence.

**Theorem 3.7.**

\( F_n \to 0 \) in the weak topology of \( S' \) iff

\[ \forall t > 0 \, \forall v \in X \quad (v,F_n(t))_0 \to 0. \]

**Proof**

\( \Rightarrow \) For any \( v \in X \) \( \langle M_{-t}v, F_n \rangle = (v,F_n(t))_0 \to 0 \) as \( n \to \infty \).

\( \Rightarrow \) For any \( \phi \in S \) and \( t \) sufficiently small

\[ \langle \phi, F_n \rangle = (M_{-t}\phi, F_n(t)) \to 0 \]

because \( M_{-t}\phi \in X \).
Corollary 3.8.

I. Strong convergence of a sequence in $S'$ implies its weak convergence.

II. Any bounded sequence in $S'$ has a weakly convergent subsequence. (Use a diagonal argument).

It looks reasonable to conjecture that the weak topology on $S$ is the inductive limit topology with respect to the Hilbert spaces $X_t$, now endowed with the weak topology. It is easily seen that the weak topology on $S'$ is induced by the semi-norms $\rho_{t,v}$, $t > 0$, $v \in X$, and $\rho_{t,v}(F) = |(v,F(t))_0|$. We will not pursue these things further here. The next theorem deals with the question: When does weak convergence of a sequence imply its strong convergence?

Theorem 3.9.

The following three statements are equivalent.

I. For each $t > 0$ $e^{-At}$ is a compact operator on $X$.

II. Each weakly convergent sequence in $S$ converges strongly in $S$.

III. Each weakly convergent sequence in $S'$ converges strongly in $S'$.

Proof

I $\Rightarrow$ II. A weakly convergent sequence in $S$ converges, for some $t > 0$, weakly in $X_t$. See Theorem 3.5. Because of the assumption the natural injection $X_t \subset X_\alpha$, $0 < \alpha < t$, is compact. Cf. the proof of Theorem 1.11. But then our sequence converges strongly in $X_\alpha$.

II $\Rightarrow$ I. Take any sequence $\{f_n\} \subset X$, $f_n \to 0$ weakly in $X$. For each $t > 0$ and each $F \in S'$ we have $\langle M_{t_n} F, f_n \rangle \to 0$. Thus $M_{t_n} f_n \to 0$ weakly in $S$. Because of the assumption $M_{t_n} f_n \to 0$ strongly in $S$. And so $\|M_{t_n} f_n\| \to 0$. This shows that $M_t$ must be compact.

I $\Rightarrow$ III. Let $\{F_n\} \subset S'$. Suppose $\forall g \in S$ $\langle g, F_n \rangle \to 0$.

Then $\forall h \in X$ $\forall \alpha > 0$ $\langle M_{\alpha} h, F_n \rangle = \langle h, F_n(\alpha) \rangle \to 0$. This means that $\forall \alpha > 0$ $F_n(\alpha) \to 0$ weakly in $X$. Using the compactness of $M_\beta$, $\beta > 0$, we find that $F_n(\alpha + \beta) = M_\beta F_n(\alpha) \to 0$ strongly in $X$. Therefore $\forall t > 0$ $\|F_n(t)\| \to 0$. 

- 37 -
Let $\{v_n\} \subseteq X$ be such that $v_n \rightharpoonup 0$ weakly in $X$. Then $v_n \rightharpoonup 0$ weakly in $S'$ because $\forall g \in S \quad <g, v_n> = (g, v_n)_0 \to 0$. Now the assumption says $v_n \to 0$ strongly in $S'$. This means $\forall t > 0 \quad M_t v_n \to 0$ strongly in $X$. The compactness of $M_t$ follows for each $t > 0$.

One might wonder whether in the case that $M_t$ is compact for each $t > 0$ the strong and weak topologies coincide. This however cannot be true since a set which belongs to a weak base of $0$-neighbourhoods always contains a closed subspace of finite codimension. Generally speaking such "large" sets do not fit in a strongly open $0$-neighbourhood.

It is relatively simple to show that both in $S$ and $S'$ the weakly compact sets are just the (weakly) closed and bounded sets. Then it is not difficult to prove that $S$ and $S'$ with their strong topologies are Mackey spaces. That is the strong topology is the finest locally convex topology for which the dual of $S$ (resp. $S'$) is $S'$ (resp. $S$). (All spaces which are either barreled or bornological are Mackey. See [SCH] IV.4).
CHAPTER 4. Characterization of continuous linear mappings between the spaces $S_{X,A}$, $S'_{X,A}$, $S_{Y,B}$ and $S'_{Y,B}$

Let $B$ be a non-negative self-adjoint operator in a separable Hilbert space $Y$. As before $A$ is a non-negative self-adjoint operator in a separable Hilbert space $X$. In this chapter we shall derive conditions implying the continuity of linear mappings $S_{X,A} + S_{Y,B}$, $S_{X,A} \rightarrow S'_{Y,B}$, $S'_{X,A} + S'_{Y,B}$, $S'_{X,A} \rightarrow S'_{Y,B}$. Further we investigate which linear operators, defined on a subset of $X$, can be continuously extended to operators on $S'_{X,A}$. The next theorem is an immediate consequence of the fact that $S_{X,A}$ is bornological. Cf. Theorem 1.11. For completeness we give an ad hoc proof.

Theorem 4.1.

A linear map $Q : S_{X,A} \rightarrow V$, where $V$ is an arbitrary locally convex topological vector space, is continuous

I. iff for each $t > 0$ the map $Q_{M_t} : X \rightarrow V$ is continuous.

II. iff for each null sequence $\{u_n\} \subset S_{X,A}$, $u_n \rightarrow 0$ in $S_{X,A}$, the sequence $\{Q_{u_n}\}$ is a null sequence in $V$.

Proof

1) $M_t$ is an isomorphism from $X$ to $X_t$. By the definition of the inductive limit, $X_t$ is continuously injected in $S_{X,A}$. So if $Q$ is continuous it follows that $Q_{M_t}$ is continuous.

2) Let $Q_t$ denote the restriction of $Q$ to $X_t$. From the continuity of $Q_{M_t}$ on $X$ follows the continuity of $Q_t$ on $X_t$. Now let $\Omega \ni 0$ be open in $V$. For each $t > 0$ $Q_{M_t}^{-1}(\Omega) \cap X_t = Q_t^{-1}(\Omega)$ is an open $0$-neighbourhood in $X_t$. Thus $Q_t^{-1}(\Omega)$ is open in $S_{X,A}$.

II) Follows from I because null sequences in $S$ are always null sequences in some $X_t$, $t > 0$ and vice versa.

In the next theorem we characterize continuous linear mappings from $S_{X,A}$ to $S_{Y,B}$. We denote $N_t = e^{-Bt}$.
Theorem 4.2.

Suppose \( P : S_{X,A} \rightarrow S_{Y,B} \) is a linear mapping. The following six conditions are equivalent.

I. \( P \) is continuous with respect to the strong topologies of \( S_{X,A} \) and \( S_{Y,B} \).

II. \( u_n \rightarrow 0 \), strongly in \( S_{X,A} \), implies \( P u_n \rightarrow 0 \), strongly in \( S_{Y,B} \).

III. For each \( \alpha > 0 \) the operator \( P \alpha \) is a bounded linear operator from \( X \) into \( Y \).

IV. For each \( t > 0 \) there exists \( \beta > 0 \) such that \( P \alpha \) is a bounded linear operator from \( X \) into \( Y \).

V. There exists a dense linear subspace \( E \subset Y \) such that for each fixed \( y \in E \) the linear functional \( L_{P,y}(f) = (Pf,y)_Y \) is continuous on \( S_{X,A} \).

VI. For each \( t > 0 \) \( (P \alpha \) \) is a bounded linear operator from \( Y \) to \( X \).

(Remark: One has \( (P \alpha \) \) is a bounded linear operator from \( Y \) to \( X \).

Proof

We proceed according to the scheme

\[ \text{I} \quad \Rightarrow \quad \text{II} \quad \Rightarrow \quad \text{III} \quad \Rightarrow \quad \text{IV} \quad \Rightarrow \quad \text{V} \quad \Rightarrow \quad \text{VI} \]

I \( \Rightarrow \) II. See Theorem 4.1.

II \( \Rightarrow \) III. If \( \{x_n\} \) is a null sequence in \( X \) then for any \( \alpha > 0 \) \( \{\alpha x_n\} \) is a null sequence in \( S_{X,A} \). By II \( \{P \alpha x_n\} \) is a null sequence in \( S_{X,A} \) and hence in \( Y \).

III \( \Rightarrow \) IV. Let \( \alpha > 0 \). We define a class \( A_{\alpha} \) of continuous linear functionals on \( X \) in the following way

\[ A_{\alpha} = \{ x \mapsto L(x) = (P \alpha x, N_{-a} g)_Y | g \in N_{\alpha}(Y), \| g \|_Y \leq 1\} \).

Obviously \( \alpha_1 < \alpha_2 \Rightarrow \alpha_1 A_{\alpha_1} \subset A_{\alpha_2} \) since for \( g \in N_{\alpha_1}(Y) \) we have \( N_{-a_1} g = N_{-a_2} (N_{a_2} - a_1) g \) where \( a_2 - a_1 \in N_{a_2}(Y) \) and \( \| N_{-a_1} g \|_Y \leq 1 \). We claim

\[ (*) \quad \forall x \in X \quad \exists \alpha > 0 \quad \exists M > 0 \quad \forall \alpha \in A_{\alpha} \quad |L(x)| \leq M. \]
Indeed, since \( PM_t x \in S_{Y^a} \) for \( \alpha \) sufficiently small \( PM_t x \in Y_\alpha \) and 
\( \gamma \), \( PM_t x \in Y \) so that 
\[
|\ell(x)| = \left| (N_{-\alpha} PM_t x, g) \right| \leq \|N_{-\alpha} PM_t x\|_Y.
\]

Applying a modified version of the Banach-Steinhaus theorem, see \([J]\), see our Appendix C, from (*) it follows 
\[
\exists \alpha > 0 \exists M > 0 \forall x \in X \forall \ell \in A_\alpha \quad |\ell(x)| \leq M\|x\|
\]
But then for all \( x \in X \) and all \( g \in Y_\alpha \)
\[
|PM_t x, N_{-\alpha} g) Y| \leq M\|x\|\|g\|_Y
\]

which means \( PM_t x \in D(N_{-\alpha}) \) and \( N_{-\alpha} PM_t \) bounded.

III \( \Rightarrow \) V. We can take \( \mathbb{E} = Y \). Take \( y \in Y \) fixed. For each \( x \in X \) and each \( t > 0 \) we have
\[
|L_{p_y, Y} (M_t x)| = |(PM_t x, y) Y| \leq C_{t, y}\|x\|_X.
\]
Together with Theorem 4.1 the result follows.

V \( \Rightarrow \) VI. According to Riess' theorem for each \( y \in \mathbb{E} \) and each \( t > 0 \) there exists \( f_t \in X \) such that for each \( h \in X \)
\[
(\forall) \quad L_{p_y, Y} (M_t h) = (PM_t h, y) Y = (h, f_t) X.
\]
Replacing \( h \) by \( M_t x, x \in X \) we observe that \( f_{t+\tau} = M_t f_t \) so that \( f_t \in S_{X,A} \).

From (\forall) we obtain \( f_t = (PM_t)^* y \). So \( D((PM_t)^*) = \mathbb{E} \) which is dense in \( Y \).
Since \( PM_t \) is defined on the whole \( X \) the operator \( (PM_t)^* \) is defined on the whole \( Y \) and bounded. Repeating the argument with arbitrary \( y \in X \) shows \( (PM_t)^* y \in S_{X,A} \).

VI \( \Rightarrow \) III. \( PM_t \) is bounded because \( (PM_t)^* \) is bounded.

IV \( \Rightarrow \) II. Let \( \{u_n\} \) be a null-sequence in \( S_{X,A} \). Then for some \( \alpha > 0 \)
\[
\{M_{-\alpha} u_n\} \text{ is a null-sequence in } X. \quad \text{But then, using the boundedness of}
\]
\[
N_{-\beta} PM_{-\alpha} u_n = \{N_{-\beta} \alpha u_n\} \text{ is a null-sequence in } Y. \quad \text{Hence } \{P_{\alpha} u_n\}
\]
is a null-sequence in \( S_{Y,B} \).
The next corollary is important for applications.

**Corollary 4.3.**

Suppose $Q$ is a densely defined closable operator: $X \to Y$. If

$$D(Q) = S_{X,A}$$

and $Q(S_{X,A}) \subseteq S_{Y,B}$, then $Q$ maps $S_{X,A}$ continuously into $S_{Y,B}$.

**Proof**

$Q$ is closable iff $D(Q^*)$ is dense in $Y$. Since $Q_{M_t} : X \to Y$ is defined on the whole $X$, its adjoint $(Q_{M_t}^*)^*$ is bounded. The adjoint however is densely defined since on $D(Q^*)$ one has $(Q_{M_t}^*)^* = M_tQ^*$. Hence $(Q_{M_t}^*)^*$ is defined on the whole $Y$ and bounded. From this the boundedness of $Q_{M_t}$ follows. Application of Theorem 4.2 III yields the desired result. $\square$

**Theorem 4.4.**

Let $K : S_{X,A} \to S'_{Y,B}$ be a linear mapping. The following four conditions are equivalent.

I. $K$ is continuous with respect to the strong topologies of $S_{X,A}$ and $S'_{Y,B}$.

II. For each $t > 0$, $\alpha > 0$ $N_{t\alpha}KM$ is a bounded linear operator from $X$ into $Y$.

III. For each $t > 0$ $N_{t\alpha}$ is a continuous map from $S_{X,A}$ into $S'_{Y,B}$.

IV. There exists a mapping $Z$ of the interval $(0,\infty)$ into the continuous linear maps from $S_{X,A}$ into $S'_{Y,B}$ such that

(i) For each $t > 0$, $\tau > 0$ $Z(t + \tau) = N_{t\tau}Z(\tau)$.

(ii) For each $u \in S_{X,A}$ $Ku = \lim_{t\to0} \text{emb}(Z(t)u)$, strongly in $S'_{Y,B}$.

**Proof**

I $\Rightarrow$ II. Let $\{x_n\}$ be a null-sequence in $X$. Then $\{M_{\alpha_n}x_n\}$ is a null sequence in $S_{X,A}$. Since $K$ is continuous $\{KM_{\alpha_n}x_n\}$ is a null sequence in $S'_{Y,B}$, which means that for each $t > 0$ $(N_{t\alpha_n}K)\{x_n\}$ is a null sequence in $Y$. Hence $N_{t\alpha}K$ is bounded.
II ⇒ I. Let \( \{u_n\} \) be a null sequence in \( S_{X,A} \). Then for some \( \alpha > 0 \)
\( \{M_{\alpha}u_n\} \) is defined and is a null-sequence in \( X \). But then
\( \{Ku_n\} = \{KM_{\alpha}u_n\} \) is a null-sequence in \( S_{Y,B} \) since for each \( t > 0 \)
\( (KM_{\alpha}u_n)(t) = (N_{\alpha}KM)M_{\alpha}u_n \to 0 \) in \( Y \).

II ⇒ III. Apply theorem 4.2 with \( P = N_{t}K \).

III ⇒ IV. Define \( Z(t) = N_{t}K \). It obviously has the property (i). We
must show that for each \( \tau > 0 \) \( \lim_{t \to 0} N_{t}Z(t)u = (Ku)(\tau) \). But this is also
obvious because \( N_{t}Z(t)u = N_{t+\tau}Ku \) and the semi-group \( N_{t} \) is strongly
continuous.

IV ⇒ III. Trivial because \( N_{t}K = Z(t) \).

Theorem 4.5.:

Let \( \Gamma : S_{X,A}^{t} \to S_{Y,B}^{t} \) be a linear mapping.

Let \( \Gamma_{\tau : X \to Y} \) denote the restriction of \( \Gamma \) to \( X \). The following five
conditions are equivalent.

I. \( \Gamma \) is continuous with respect to the strong topologies of \( S_{X,A}^{t} \) and
\( S_{Y,B}^{t} \).

II. \( \Gamma^{*}_{\tau}(Y) \subset S_{X,A}^{t} \).

III. There exists \( t > 0 \) such that \( \Gamma^{*}_{\tau}(Y) \subset M_{t}(X) \) and \( M_{-t}\Gamma^{*}_{\tau} \)
is a bounded
operator from \( Y \) into \( X \).

IV. There exists \( t > 0 \) such that \( \Gamma_{t}^{*}M_{-t} \) with domain \( X_{t} \subset X \) is bounded as
an operator from \( X \) into \( Y \).

V. There exists \( t > 0 \) and a continuous linear map \( Q : S_{X,A}^{t} \to S_{Y,B}^{t} \) such
that \( \Gamma = QM_{t}^{*} \).

Proof

I ⇒ II. From the continuity of \( \Gamma \) it immediately follows that \( \Gamma_{\tau} \) is
continuous. Its adjoint \( \Gamma^{*}_{\tau} \), defined by \( (x,\Gamma^{*}_{\tau}y)_{X} = (\Gamma_{\tau}x,y)_{Y} \) for all
\( x \in X, y \in Y \), is a bounded operator as well. Now consider the dual
operator \( \Gamma' : S_{Y,B}^{t} \to S_{X,A}^{t} \) defined by
\[ \langle F,\Gamma'G \rangle_{X} = \langle GF,G \rangle_{Y} \] for all \( F \in S_{X,A}^{t} \).
For fixed $G \in S_Y^i$, the righthand-side defines a continuous linear functional on $S_X^i$. $\Gamma^*_r$ is a restriction of $\Gamma'$ and therefore maps $Y$ into the set $S_X^i \subset X$.

II $\Rightarrow$ III. For $\beta > 0$ consider the following class of continuous linear functionals on $Y$

$$B_\beta = \{ y \mapsto \ell(y) = (\Gamma_r M_{-\beta r}^* y) y | y \in M_{\beta}(X), \| y \|_X \leq 1 \}.$$  

As in the proof of Theorem 4.2 one shows that

$$\beta_1 < \beta_2 \Rightarrow B_{\beta_1} \subset B_{\beta_2}.$$  

We claim

$$\forall y \in Y \exists \beta > 0 \exists M > 0 \forall \ell \in B_\beta \ | \ell(y) | \leq M.$$  

Indeed, since $\Gamma^*_r y \in S_X^i$, one has for $\beta$ sufficiently small

$$| \ell(y) | = | (\ell, M_{-\beta r}^* y) y | \leq \| M_{-\beta r}^* y \|_X.$$  

Applying the modified Banach-Steinhaus theorem, see Appendix C, it follows

$$\exists \beta > 0 \exists M > 0 \forall y \in Y \forall \ell \in B_\beta \ | \ell(y) | \leq M \| y \|_Y.$$  

But then for all $\ell \in M_{\beta}(X)$ and all $y \in Y$

$$| (M_{-\beta r}^* \ell, \Gamma^*_r y) y | \leq M \| \ell \|_X \| y \|_Y$$

which implies $\Gamma^*_r y \in D(M_{-\beta r}^*) = X^i$ and $M_{-\beta r}^*$ is bounded.

III $\Rightarrow$ IV. $M_{-\beta r}^*$ has a bounded adjoint $(M_{-\beta r}^*)^*$ but $(M_{-\beta r}^*)^* = \Gamma_r M_{-\beta r}^*$ with domain $X^i \subset X$.

IV $\Rightarrow$ V. Take $Q = (M_{-\beta r}^*)^*$. For each $\alpha > 0$ $QM_{\alpha}$ is the extension of $\Gamma_{M_{-\beta r}^*}$ which is bounded. So $Q$ satisfies condition III of Theorem 4.2. We have $\Gamma = Q M_{\beta}$.

V $\Rightarrow$ I. If $\Gamma = Q M_{\beta}$ the mapping $\Gamma$ is obviously continuous.
Theorem 4.6.

Let $\phi : S_{X,A}' \rightarrow S_{Y,B}'$ be a linear mapping.
Let $\phi_r : X \rightarrow S_{Y,B}'$ denote the restriction of $\phi$ to $X$.
The following six conditions are equivalent.

I. $\phi$ is continuous with respect to the strong topologies of $S_{X,A}'$ and $S_{X,A}$

II. For each $g \in S_{Y,B}$ the expression $\langle g, \phi_F \rangle_Y$, $F \in S_{X,A}'$ is a continuous linear functional on $S_{X,A}$

III. For each $t > 0$ the operator $N_t \phi$ is a continuous map from $S_{X,A}'$ into $S_{Y,B}'$

IV. For each $t > 0$ $(N_t \phi_r)^*(Y) \subset S_{X,A}'$

V. For each $t > 0$ there exists $\beta > 0$ such that $(N_t \phi_r)^*(Y) \subset M_\beta(X)$

VI. For each $t > 0$ there exists $\beta > 0$ such that $N_t \phi M_\beta = N_t \phi M_{t\beta}$ on the domain $X_\beta \subset X$ is bounded as an operator from $X$ into $Y$.

Proof.

We proceed according to the scheme

\[\begin{array}{cccccc}
I & \Rightarrow & II & \Rightarrow & III & \Rightarrow \\
& & IV & \Rightarrow & V & \Rightarrow \\
& & & & VI & \Rightarrow \\
& & & & & I
\end{array}\]

I $\Rightarrow$ II. Trivial.

I $\Rightarrow$ III. Trivial because $N_t : S_{Y,B}' \rightarrow S_{Y,B}$ is continuous.

III $\Rightarrow$ IV. Theorem 5.4, condition II.

IV $\Rightarrow$ V. Apply Theorem 4.5 to $N_t \phi_r$.

V $\Rightarrow$ VI. The adjoint of the bounded operator $M_\beta (N_t \phi_r)^*$ is an extension of $(N_t \phi_r) M_{t\beta}$

Therefore the latter is bounded.

VI $\Rightarrow$ I. Let $\{F_n\}$ be a null sequence in $S_{X,A}'$. Then for each $t > 0$

there exists $\beta > 0$ such that we can write $(\phi F_n)(t) = N_t \phi M_{t\beta} F_n(\beta)$.

This converges to zero as $n \rightarrow \infty$ because of the boundedness of $N_t \phi M_{t\beta}$.
II \rightarrow V. \quad \langle \varphi, \Phi F \rangle_Y has the representation \langle \varphi, F \rangle_X', see Theorem 3.2.IV, here \( f = \varphi' g \in S_{X', A'} \). Taking \( F = u \in S_{X, A} \) we observe that
\[
(u, \varphi' g)'_X = \langle u, \varphi' g \rangle_X = \langle \varphi u, g \rangle_Y \]
is, as a function of \( g \), a continuous linear functional on \( S_{Y, B} \). Then with Theorem 4.2.V it follows that
\( \varphi' \) maps \( S_{Y, B} \) continuously into \( S_{X, A} \). Then, by Theorem 4.2.IV, for each \( t > 0 \) there exists \( \beta > 0 \) such that \( M_{-\beta} \varphi' N_t \) is a bounded operator. But \( M_{-\beta} \varphi' N_t = M_{-\beta} (N_t \varphi')^* \) because for all \( x \in X, y \in Y \)
\[
(\varphi' N_t y, x)'_X = \langle N_t y, \varphi x \rangle_Y = \langle y, N_t \varphi x \rangle_Y = \langle (N_t \varphi)' y, x \rangle_X.
\]
The interesting question arises which densely defined (possibly unbounded) operators from \( X \) into \( Y \) can be extended to a continuous mapping from \( S_{X, A} \) into \( S_{Y, B} \).

**Theorem 4.7.**

Let \( E \) be a linear map \( X \supset D(E) \rightarrow Y \) with \( D(E) = X \). \( E \) can be extended to a continuous linear map \( E : S_{X, A} \rightarrow S_{Y, B} \) iff \( E \) has a densely defined adjoint \( E^* : Y \supset S_{Y, B} \rightarrow D(E^*) \rightarrow X \) with \( E^*(S_{Y, B}) \subset S_{X, A'} \).

**Proof**

\( \Rightarrow \) For each \( \xi \in D(E) \) and \( g \in S_{Y, B} \) one has \( \langle \xi, g \rangle_Y = \langle \xi, E^* g \rangle_X \).

It follows from Theorem 4.2.V that \( E^* \) maps \( S_{Y, B} \) continuously into \( S_{X, A'} \).

We define \( \tilde{E}_F, F \in S_{X, A} \), by
\[
\langle v, \tilde{E}_F \rangle_Y = \langle E^* v, F \rangle_X, \quad \forall v \in S_{Y, B}.
\]
This functional is continuous as a function of \( v \), so \( \tilde{E}_F \in S_{Y, B}' \).

It is obviously continuous as a function of \( F \). Then by Theorem 4.6.II the continuity of \( \tilde{E} \) follows.

\( \Leftarrow \) If \( \tilde{E} \) exists as a continuous map, its dual operator \( \tilde{E}' \) maps \( S_{Y, B} \)
into \( S_{X, A'} \).

For each \( x \in D(E) \) and \( g \in S_{Y, B} \) one has \( \langle g, E x \rangle_Y = \langle g, E x \rangle_Y = \langle \tilde{E}' g, x \rangle_X = \langle \tilde{E}' g, x \rangle_X 
\]
It follows that \( E^* = \tilde{E}' \) and \( E^*(S_{Y, B}) \subset S_{X, A'} \).
Corollary 4.8.

A continuous linear map \( Q : S_{X,A} \to S_{Y,B} \) can be extended to a continuous linear map \( \overline{Q} : S_{X,A}^1 \to S_{Y,B}^1 \) iff \( Q \) has a Hilbert space adjoint \( Q^* \) with \( D(Q^*) = S_{Y,B} \) and \( Q^*(S_{Y,B}) \subseteq S_{X,A}^1 \).
CHAPTER 5. Topological tensor products of spaces of type \( S^1 \), \( S^1 \)

For two separable Hilbert spaces \( X \) and \( Y \) we consider the complex vector space consisting of all Hilbert-Smidt operators \( Z \) from \( X \) into \( Y \). We shall denote this vector space by \( X \otimes Y \). For any \( Z \in X \otimes Y \) and any orthonormal basis \( \{ e_i \} \subset X \) we have

\[
\| Z \|_2^2 = \sum_{i=1}^{\infty} \| Ze_i \|_Y^2 < \infty.
\]

The double norm \( \| \cdot \| \) does not depend on the choice of the orthonormal basis \( \{ e_i \} \). We introduce an inner product in \( X \otimes Y \) by

\[
\langle Z, K \rangle_{X \otimes Y} = \sum_{i=1}^{\infty} \langle Ze_i, Ke_i \rangle_Y.
\]

Endowed with this inner product \( X \otimes Y \) is a Hilbert space. See [RS] Ch. VIII.10. Examples of elements in \( X \otimes Y \) are \( \xi \otimes \eta \), \( \xi \in X \), \( \eta \in Y \) defined by \( (\xi \otimes \eta)f = (f, \xi)_X \eta \), for all \( f \in X \), and finite linear combinations of these: \( \sum_{j=1}^{N} (\xi_j \otimes \eta_j) \) with \( \xi_j \in X \), \( \eta_j \in Y \). The linear subspace of \( X \otimes Y \) which consists of all H.S. operators of the type just mentioned will be denoted by \( X \otimes_a Y \), i.e. the (sesquilinear) algebraic tensor product of \( X \) and \( Y \). \( X \otimes Y \) may be regarded as the completion of \( X \otimes_a Y \) with respect to the double norm \( \| \cdot \| \). Therefore \( X \otimes Y \) is called the completed (sesquilinear) topological tensor product of \( X \) and \( Y \). For later reference we mention the following properties taken from [RS] Ch. VIII.

Properties 5.1.

(a) \( \forall x, \xi \in X \forall y, \eta \in Y \quad (\xi \otimes \eta, x \otimes y)_{X \otimes Y} = (x, \xi)_X (\eta, y)_Y \)

(b) \( \forall \lambda \in \mathbb{C} \forall \xi \in X \forall \eta \in Y \quad \lambda (\xi \otimes \eta) = (\lambda \xi) \otimes \eta = \xi \otimes (\lambda \eta) \).

Thus the canonical mapping \( X \times Y \to X \otimes Y \), defined by \( [xy] \to x \otimes y \) is anti-linear in \( x \) and linear in \( y \).
(c) \( \forall z \in X \otimes Y \ \forall x \in X \ \forall y \in Y \ \ (z, x \otimes y)_{X \otimes Y} = (zx, y)_Y \)

Let \( H \) resp. \( J \) denote bounded linear operators on \( X \), resp. \( Y \) into themselves. \( H \otimes J \) denotes the linear mapping of \( X \otimes Y \) into itself defined by \( (H \otimes J) (x \otimes y) = (Hx) \otimes (Jy) \) and linear extension, followed by continuous extension.

(d) The uniform operator norms of \( H, J \) and \( H \otimes J \) are related by
\[
\| H \otimes J \| = \| H \| \| J \| .
\]

(e) \( \forall z \in X \otimes Y \ \ (H \otimes J)z = JzH^* \)

(f) \( H \) and \( J \) injective \( \Rightarrow H \otimes J \) is injective.

The theory of closable tensor products of unbounded closable operators and the description of their properties in terms of corresponding properties of their factors presents greater difficulties. Only rather recently significant general results have been attained [T].

**Definition 5.2.**

Let \( A \) with domain \( D(A) \) be a densely defined closed linear operator in \( X \). Let \( B \) with domain \( D(B) \) be the same in \( Y \). On \( D(A) \otimes a D(B) \subset X \otimes Y \) we introduce the operator \( A \otimes a I + I \otimes a B \) by \( (A \otimes a I + I \otimes a B)(x \otimes y) = (Ax) \otimes y + x \otimes (By) \) and linear extension. This extension is well defined and closable, [RS] p. 298.

**Lemma 5.3.**

Let \( A \) resp. \( B \) be self-adjoint operators in \( X \) resp. \( Y \).

I. \( A \otimes a I + I \otimes a B \) is essentially self-adjoint in \( X \otimes Y \). We denote the unique self-adjoint extension by \( A \otimes a I + I \otimes a B \) or, briefly, \( A \oplus B \).

II. \( A \geq 0 \) and \( B \geq 0 \) implies \( A \oplus B \geq 0 \)

**Proof**

I. We follow Nelson's approach [S]. The operator \( C = A \otimes a I + I \otimes a B \) is obviously symmetric. \( C \) is essentially self-adjoint iff \( C^*Z = \pm iZ \) implies \( Z = 0 \).
On $X \otimes Y$ we introduce the bounded linear operator $e^{-iAt} \otimes e^{iBt}$, $t \in \mathbb{R}$. This operator is isometric and

$$\langle e^{-iAt} \otimes e^{iBt} \rangle (D(A) \otimes_{a} D(B)) \subset D(A) \otimes_{a} D(B).$$

For $Z \in D(A) \otimes_{a} D(B)$ we have

$$\frac{d}{dt}(e^{-iAt} \otimes e^{iBt})Z = iC(e^{-iAt} \otimes e^{iBt})Z.$$

Now let $K \in D(C^*)$ and $C^*K = \pm iK$

Then for each $Z \in D(A) \otimes I + I \otimes B$,

$$\frac{d}{dt}(Z, (e^{-iAt} \otimes e^{iBt})K)_{X \otimes Y} = \tau Z, (e^{-iAt} \otimes e^{iBt})K)_{X \otimes Y}.$$

Thus

$$\langle Z, (e^{-iAt} \otimes e^{iBt})K \rangle_{X \otimes Y} = e^{\tau t} \langle Z, K \rangle_{X \otimes Y}.$$

But

$$|\langle Z, (e^{-iAt} \otimes e^{iBt})K \rangle_{X \otimes Y}| \leq \|Z\|_{X \otimes Y} \|K\|_{X \otimes Y}.$$

This contradicts (*) except when $(Z, K)_{X \otimes Y} = 0$ for all $Z \in D(A) \otimes_{a} D(B)$, which is a dense set. Therefore $K$ must be zero.

II. We must show that sums of type

$$\sum_{i=1}^{N} \sum_{j=1}^{N} ((A \otimes I)x_i \otimes y_i, x_j \otimes y_j)_{X \otimes Y} = \sum_{i=1}^{N} \sum_{j=1}^{N} (Ax_i, x_j) \langle y_i, y_j \rangle_{Y}$$

are non-negative. This follows by taking orthonormal bases in the span of $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ and straightforward calculation. Since the graph of $A \otimes I + I \otimes B$ is dense in the graph of $A \otimes B$ it follows that also the latter must be non-negative.

**Theorem 5.4.**

On $X \otimes Y$ we have for $t \geq 0$

$$e^{-(A \otimes B)t} = e^{-At} \otimes e^{-Bt} = M_t \otimes N_t.$$
Proof

\( e^{-At} \otimes e^{-Bt} \) is a strongly continuous one-parameter semi-group of bounded non-negative hermitean operators. Obviously
\[ (e^{-At} \otimes e^{-Bt}) (D(A) \otimes D(B)) \subseteq D(A) \otimes D(B). \]

The infinitesimal generator of \( M_t \otimes N_t \) restricted to \( D(A) \otimes D(B) \) is \(-C\). The infinitesimal generator must therefore contain the closure of \(-C\) which is the self-adjoint operator \( -(A \oplus B) \). The infinitesimal generator however has to be self-adjoint. Hence it must be equal to \(-(A \oplus B)\). \( M_t \otimes N_t \) is, of course, holomorphic.

Applying the results of the preceding chapters we can introduce the spaces \( S_{X \otimes Y, A \oplus B}, S'_{X \otimes Y, A \oplus B} \) and, by taking \( A = 0 \) or \( B = 0 \), the spaces \( S_{X \otimes Y, A \oplus I}, S'_{X \otimes Y, I \oplus B} \), etc.

Definition 5.5.

The canonical sesquilinear map \( \otimes : S_{X,A} \times S_{Y,B} \rightarrow S_{X\otimes Y, A\oplus B} \) is defined by \([u,v] \rightarrow u \otimes v\). Here the symbol \( \otimes \) is the same as in Properties 5.1. This definition is consistent because for \( u \in S_{X,A}, v \in S_{Y,B} \) there exist \( x \in X, y \in Y \) and \( t > 0 \) such that \( u = M_t x, v = N_t y \). Further \( u \otimes v = (M_t \otimes N_t)(x \otimes y) = (M_t x) \otimes (N_t y) \). So that \( u \otimes v \in S_{X\otimes Y, A\oplus B} \).

Theorem 5.6.

\( S_{X\otimes Y, A\oplus B} \) is a complete topological tensor product of \( S_{X,A} \) and \( S_{Y,B} \).

By this we mean:

I. \( S_{X\otimes Y, A\oplus B} \) is complete.

II. The canonical sesquilinear map \( \otimes : S_{X,A} \times S_{Y,B} \rightarrow S_{X\otimes Y, A\oplus B} \) is continuous.

III. The span of the image of \( \otimes \), i.e. the algebraic tensor product \( S_{X,A} \otimes a S_{Y,B} \), is dense in \( S_{X\otimes Y, A\oplus B} \).
Proof

I. The completeness follows from Theorem 1.11.

II. It is enough to check the continuity of \( \otimes \) at \([0;0]\). Let \( \mathcal{W} \) be a convex open neighbourhood of 0 in \( S_{X \otimes Y, A \oplus B} \). Then for each \( t > 0 \)
\[ \mathcal{W} \cap (X \otimes Y)_t \] is an open 0-neighbourhood in \( (X \otimes Y)_t \) and it contains an open ball centered at 0 and radius \( r_t, \ 0 < r_t < 1 \). In \( X_t \) resp. \( Y_t \) we introduce open balls \( A_t \) resp. \( B_t \), centered at 0 and with radius both \( r_t \). Let \( A = \bigcup_{t>0} A_t \subset S_{X, A} \) and \( B = \bigcup_{t>0} B_t \subset S_{Y, B} \).
Then \( \otimes \) maps \( A \times B \) in \( \mathcal{W} \) since \( \|x \otimes y\|_{_{\min(t,\tau)}} \leq \|x\|_t \|y\|_t \leq r_{\min(t,\tau)} \) whenever \( x \in A, y \in B \). Let \( \hat{A} \) resp. \( \hat{B} \) denote the convex hulls of \( A \) resp. \( B \). Then \( \otimes \) maps \( \hat{A} \times \hat{B} \) in \( \mathcal{W} \). The set \( \hat{A} \) is convex and \( \hat{A} \cap X_t \) contains an open 0-neighbourhood in \( X_t \). From Theorem 1.4.11 it follows that \( A \) contains an open set \( U_{\psi, \epsilon} \).
Similarly \( \hat{B} \) contains an open set \( V_{\chi, \delta} \). We conclude that \( \otimes \) maps \( U_{\psi, \epsilon} \times V_{\chi, \delta} \) into \( \mathcal{W} \).

III. For each \( t > 0 \), \( X_t \otimes Y_t \) is dense in \( (X \otimes Y)_t \). From this the desired result follows.

Remark

Our strong topology on \( S_{X \otimes Y, A \oplus B} \) is, generally speaking, not the projective tensor product topology. Cf. [SCH] p. 93. Therefore the universal factorization property for continuous sesquilinear maps on this space does not hold in general.

Definition 5.7.

The canonical sesquilinear map \( \otimes : S_{X, A}' \times S_{Y, B}' \rightarrow S_{X \otimes Y, A \oplus B}' \),
\[ [F,G] \rightarrow F \otimes G, \] is defined by \( (F \otimes G)(t) = F(t) \otimes G(t) \).
Here \( \otimes \) is the same as in Properties 5.1. The definition is consistent because \( (F \otimes G)(t + \tau) = M_{t+\tau} F(t) \otimes N_{t+\tau} G(t) = (M_{t} \otimes N_{t}) (F(t) \otimes G(t)) = (M_{t} \otimes N_{t}) (F \otimes G)(t) \).
Theorem 5.8.

$S'_{X\otimes Y, A \boxplus B}$ is a complete topological tensor product of $S'_{X, A}$ and $S'_{Y, B}$.

By this we mean:

I. $S'_{X\otimes Y, A \boxplus B}$ is complete.

II. The canonical sesquilinear map $\otimes : S'_{X, A} \times S'_{Y, B} \to S'_{X\otimes Y, A \boxplus B}$ is continuous.

III. The span of the image of $\otimes$, i.e. the algebraic tensor product $S'_{X, A} \otimes S'_{Y, B}$, is dense in $S'_{X\otimes Y, A \boxplus B}$.

Proof

I. The completeness follows from Theorem 2.5.

II. For each $t > 0$ we have $\|F(t) \otimes G(t)\|_{X\otimes Y} = \|F(t)\|_X \|G(t)\|_Y$. From this the continuity at $[0; 0)$ follows.

III. $X \otimes_a Y$ is dense in $X \otimes Y$ which is dense in $S'_{X\otimes Y, A \boxplus B}$.

We now want to introduce mixed sesquilinear topological tensor products of type $S'_{X, A} \otimes S'_{Y, B}$, etc. The reader who is already content with the Kernel Theorems case a and case b of the next chapter may skip the remainder of this chapter.

Definition 5.9.

We introduce the following linear subspace of $S'_{X\otimes Y, A \boxplus B}$

$$\Sigma'_{X\otimes Y, I \boxplus B, A \boxplus B} = \{ \phi \mid \phi \in S'_{X\otimes Y, I \boxplus B}, \forall t > 0 \phi(t) \in S'_{X\otimes Y, A \boxplus B} \}$$

If no confusion is likely to arise we use the shorter notation $\Sigma'_{B}$ for this space. For $\phi \in \Sigma'_{B}$ we have for each $t, \tau > 0$ $\phi(t + \tau) = (I \otimes N)\phi(t)$. On $\Sigma'_{B}$ we introduce semi-norms $\{\rho_{t, \psi}\}$, $t > 0$, $\psi \in B_+$, by

$$\rho_{t, \psi}(\phi) = \|\psi(A \boxplus B)\phi(t)\|_{X\otimes Y}$$

and the corresponding locally convex topology.
Definition 5.10.

The canonical sesquilinear map \( \Phi : S_{X,A} \times S_{Y,B} \rightarrow \Sigma^I_B \) is defined by \( (\Phi \otimes G)(t) = \Phi \otimes G(t) \). Here the symbol \( \Phi \) is the same as in Properties 5.1. The span of the image of \( \Phi \) is denoted by \( S_{X,A} \otimes a S_{Y,B} \).

Theorem 5.11.

\( \Sigma^I_B \) is a complete topological tensor product of \( S_{X,A} \) and \( S_{Y,B} \). By this we mean

I. \( \Sigma^I_B \) is complete.

II. The canonical sesquilinear map \( \Phi : S_{X,A} \times S_{Y,B} \rightarrow \Sigma^I_B \) is continuous.

III. \( S_{X,A} \otimes a S_{Y,B} \) is dense in \( \Sigma^I_B \).

Proof

I. Let \( \{ \Phi_\alpha \} \) be a Cauchy net. The \( \alpha \)'s belong to a directed set \( D \). First take \( \psi = 1 \). \( \Phi_\alpha \) tends to a limit point \( \Phi \in S_{X \otimes Y, A \oplus B} \) because the latter is complete. It remains to show that for each \( t > 0 \) \( \Phi(t) \in S_{X \otimes Y, A \oplus B} \). For each \( \psi \in B^+ \) and each \( t > 0 \) \( \psi(A \oplus B) \Phi_\alpha \) converges in \( X \otimes Y \).

From the closedness of \( \psi(A \oplus B) \) it follows that \( \Phi(t) \in D(\psi(A \oplus B)) \). This is true for each \( \psi \in B^+ \) and therefore by Lemma 1.10, for each \( t > 0 \), \( \Phi(t) \in S_{X \otimes Y, A \oplus B} \).

II. It is enough to check the continuity at \([0;0]\). Let \( \mathcal{U}_{t,\psi,\varepsilon} \) be the set of elements in \( \Sigma^I_B \) for which the semi-norm \( \rho_{t,\psi} \) is smaller than \( \varepsilon \); it is an open 0-neighbourhood. Let \( \mathcal{V}_{\psi,\varepsilon} \) denote the convex open 0-neighbourhood in \( S_{X \otimes Y, A \oplus B} \) consisting of elements with semi-norm \( \rho_{\psi} \) smaller than \( \varepsilon \). Then \( \mathcal{U}_{t,\psi,\varepsilon} \) consists of those trajectories in \( \Sigma^I_B \) which pass at \( t \) through \( \mathcal{V}_{\psi,\varepsilon} \). According to the proof of part II of Theorem 5.6 there exist open 0-neighbourhoods \( U \subset S_{X,A} \), \( V \subset S_{Y,B} \) such that \( U \otimes V \subset \mathcal{V}_{\psi,\varepsilon} \). \( V \cap Y_{ht} \) contains an open ball, centered at 0, of radius \( \delta \), say. Now let \( \Omega = \{ G \mid G \in S_{Y,B} , \| G(ht) \|_Y < \delta \} \}. \) \( \Omega \) is open in \( S_{Y,B} \) and the set \( \{ G(t) \mid G \in \Omega \} \subset V \). It follows that \( \Phi \otimes G \in \mathcal{U}_{t,\psi,\varepsilon} \) whenever \( \Phi \in U \) and \( G \in \Omega \). This proves the continuity of \( \Phi \).
III. $S_{X, A} \otimes_{a} S_{Y, B}$ can be considered as a subspace of $S_{X, A} \otimes S_{Y, B}$.

$S_{X, A} \otimes_{a} S_{Y, B}$ is a dense subspace of $S_{X \otimes Y, A \oplus B}$. See Theorem 5.6. So we are ready if we show that $S_{X \otimes Y, A \oplus B}$ is densely embedded in $\mathfrak{L}'_B$. Let $\phi \in \Sigma'_B$ then for $t > 0$ $\phi(t) \in S_{X \otimes Y, A \oplus B}$. We claim that $\phi(t) \to \phi$ in $\mathfrak{L}'_B$-sense as $t \to 0$. Indeed

$$
\rho_t, \psi(\phi(t) - \phi) = \| \psi(A \oplus B)(\{ \phi(t + \tau) - \phi(t) \}) \|_{X \otimes Y} = $$

$$
= (\{ I \otimes N_t - I \otimes I \} \phi(t), \psi^2(A \oplus B)(I \otimes N_t - I \otimes I) \phi(t) )_{X \otimes Y} .
$$

The first factor in this inner product tends to 0 as $t \to 0$ because $I \otimes N_t$ is a strongly continuous semi-group. The second factor remains bounded as $t \to 0$. This is so because it can be written.

$$
\{ \psi^2(A \oplus B)(M_{-\epsilon} \otimes N_{-\epsilon}) \} \{ I \otimes N_t - I \otimes I \} (M_{-\epsilon} \otimes N_{-\epsilon}) \phi(t)
$$

with $\epsilon$ sufficiently small. The operators between $\{ \}$ are (uniformly) bounded. This finishes the proof.

**Remark.**

In the proofs of parts III of Theorems 5.6 and 5.11 we circumvented the problem whether the strong topology on $S_{X \otimes Y, A \oplus B}$ is induced by seminorms of the form $\psi(A) \otimes \chi(B)$ with $\psi, \chi \in B^*_+$.

**Definition 5.12.**

We introduce the following subspace of $S'_{X \otimes Y, A \oplus I}$

$$
\Sigma'_{X \otimes Y, A \oplus I, A \oplus B} = \{ P | P \in S'_{X \otimes Y, A \oplus I}, \quad \forall t > 0 \quad P(t) \in S_{X \otimes Y, A \oplus B} \}
$$

If no confusion is likely to arise we use the shorter notation $\Sigma'_A$ for this space. For $P \in \Sigma'_A$ we have for each $t, \tau > 0$ $P(t + \tau) = (M_t \otimes I)P(t)$. On $\Sigma'_A$ we introduce the seminorms $\rho_{t, \psi}$ of Definition 5.9 and the corresponding locally convex topology.
Definition 5.13.

The canonical sesquilinear map \( \mathcal{O} : S^t_{X,A} \times S_{Y,B} \to \Sigma_A' \) is defined by
\[(F \mathcal{O} \psi)(t) = F(t) \mathcal{O} \psi.\]
The span of the image of \( \mathcal{O} \) is denoted by \( S^t_{X,A} \mathcal{O} S_{Y,B} \).

The proof of the following theorem runs the same as the proof of Theorem 5.11.

Theorem 5.14.

\( \Sigma_A' \) is a complete topological tensor product of \( S^t_{X,A} \) and \( S_{Y,B} \). By this we mean

I. \( \Sigma_A' \) is complete.

II. The canonical sesquilinear map \( \mathcal{O} : S^t_{X,A} \times S_{Y,B} \to \Sigma_A' \) is continuous.

III. \( S^t_{X,A} \mathcal{O} S_{Y,B} \) is dense in \( \Sigma_A' \).

We conclude this chapter by introducing another candidate for a sesquilinear tensor product of mixed type. For \( \alpha > 0 \) the map \( M_\alpha \mathcal{O} I \) which carries
\( S_{X \mathcal{O} Y, A \mathcal{O} B} \) into itself can be extended to a continuous linear map from
\( S^t_{X \mathcal{O} Y, A \mathcal{O} B} \) into itself. This follows from Corollary 4.8. The extension is realized by \( ((M_\alpha \mathcal{O} I)K)(t) = (M_\alpha \mathcal{O} I)K(t). \) The map \( M_\alpha \mathcal{O} I \) is injective, see Properties 5.1.f, we denote its inverse by \( M_{-\alpha} \mathcal{O} I. \)

Definitions 5.15.

We introduce the locally convex topological vector space
\[
\Sigma_{X \mathcal{O} Y, A \mathcal{O} I, A \mathcal{O} B} = \bigcup_{\alpha > 0} (M_\alpha \mathcal{O} I) (S^t_{X \mathcal{O} Y, A \mathcal{O} B}).
\]
The topology on \( (M_\alpha \mathcal{O} I) (S_{X \mathcal{O} Y, A \mathcal{O} B}) \) is given by the metric \( d_\alpha \). Here
\( d_\alpha(T) = d((M_{-\alpha} \mathcal{O} I)T) \) and \( d(\cdot) \) denotes the metric on \( S_{X \mathcal{O} Y, A \mathcal{O} B}. \) See Theorem 2.5. The topology on \( \Sigma_{X \mathcal{O} Y, A \mathcal{O} I, A \mathcal{O} B} \) or, briefly, \( \Sigma_A' \) is the inductive limit topology with respect to the metrics \( d_\alpha, \alpha > 0. \) We will not look for an explicit system of semi-norms for \( \Sigma_A' \), nor investigate its completeness here.
On $\Sigma_A^I \times \Sigma_A$ we introduce the sesquilinear form

$$<p, t>_A = <p(e), (M_{-e} \circ I)T>_X \otimes Y$$

for $e > 0$ sufficiently small. The definition does not depend on the choice of $e$. Further we define the embedding of $\Sigma_A$ into $\Sigma_B$ by

$$(*) \quad (\text{emb } T)(t) = (I \times N_t)T.$$

**Theorem 5.16.**

I. $S_X, A \otimes_{a} S'_Y, B$ is dense in $\Sigma_A$.

II. For each fixed $P \in \Sigma'_A$ the linear functional $<p, t>_A$, as a function of $T$, is continuous on $\Sigma_A$.

III. The embedding $(*)$ of Definitions 5.15 is continuous.

**Proof**

I. $X \otimes_a Y$ is dense in $S_X \otimes Y, A \otimes B$. Therefore for $a > 0$ $X \otimes_a Y$ is dense in $\Sigma'_A \otimes_{a} S'_Y, B$. We conclude that $U \otimes_a Y$ which is a subset of $\Sigma'_A \otimes_{a} S'_Y, B$ is dense in $\Sigma_A$.

II. It is sufficient to prove the continuity for restrictions to each $(M_{-a} \circ I)(S'_X \otimes Y, A \otimes B), a > 0$.

We have $|<p, t>_A| \leq c d(M_{-a} \circ I)T = c d_a(T)$. Here $c$ is a constant which depends only on $P$ and $a$.

III. $\rho_{t, \psi}(\text{emb } T) = \|\{p(A \otimes B)(M_{-a} \circ I)(I \otimes N_t)\}(M_{-a} \circ I)\|_X \otimes Y$ for $a$ sufficiently small. The operator between $\{\}$ is bounded. Therefore $\rho_{t, \psi}(\text{emb } T) \leq c d_a(T)$ where $c$ only depends on $t$, $\psi$ and $a$.

**Definitions 5.17.**

Similar to Definitions 5.15 we introduce the space

$$\Sigma_X \otimes Y, J \otimes B, A \otimes B = \bigcup_{\beta > 0} (I \otimes N_{\beta})(S'_X \otimes Y, A \otimes B),$$

briefly denoted by $\Sigma_B$, with the appropriate inductive limit topology.

For $[\phi, F] \in \Sigma'_B \times \Sigma_B$ we introduce the pairing $<\phi, F>_B = <\phi(e), (I \times N_{-e})F>_X \otimes Y$

Further we define the embedding of $\Sigma_B$ into $\Sigma_A$ by $(\text{emb } F)(t) = (M_t \circ I)F.$
Analogous to Theorem 5.16 we have

**Theorem 5.18.**

I. $S_{X,A}^\prime \subseteq S_{Y,B}^\prime$ is dense in $E_B$.

II. For each fixed $\phi \in S_{X,A}^\prime$ the linear functional $\langle \phi, F \rangle_{E_B}$, as a function of $F$, is continuous on $E_B$.

III. The embedding described in Def. 5.17 is continuous.
CHAPTER 6. Kernel Theorems

In this final chapter we show that the elements of the completed sesquilinear topological tensor products of the preceding chapter can, in a very natural way, be interpreted as linear maps of the types we discussed in Chapter 4. We give necessary and sufficient conditions on the semi-groups $M_t = e^{-tA}$ and $N_t = e^{-tB}$ which ensure that the topological tensor products comprise all continuous linear maps. In this case we say that a Kernel Theorem holds.

CASE a: Continuous linear maps $S^t_{X,A} + S^t_{Y,B}$. We consider an element $\theta \in S_{X\otimes Y, A \oplus B}$ as a linear operator $S^t_{X,A} + S^t_{Y,B}$ in the following way. Let $F \in S^t_{X,A}$, we define $\theta F$ by

\[ \theta F = \sum_{e_1} (N_{e_1} \theta M_{e_1}) F(e_1) \cdot e_1 \cdot e_1 \cdot e_1 \cdot e_1 \cdot e_1 \]

For $\epsilon > 0$ and sufficiently small this definition makes sense and does not depend on $\epsilon$.

Theorem 6.1.

I. For each $\theta \in S_{X\otimes Y, A \oplus B}$ the linear operator $\theta : S^t_{X,A} + S^t_{Y,B}$ as defined by (a), is continuous.

II. For each $\theta \in S_{X\otimes Y, A \oplus B}$, $F \in S^t_{X,A}$, $G \in S^t_{Y,B}$

\[ \langle \theta F, G \rangle_Y = \langle \theta, F \circ G \rangle_{X\otimes Y} \]

III. If for each $t > 0$ at least one of the operators $M_t$, $N_t$ is H.S. then $S_{X\otimes Y, A \oplus B}$ comprises all continuous linear operators from $S^t_{X,A}$ into $S^t_{Y,B}$.

IV. $S_{X\otimes X, A \oplus A}$ comprises all continuous linear operators from $S^t_{X,A}$ into $S^t_{X,A}$ iff for each $t > 0$ the operator $M_t$ is H.S.

Proof

I. We shall prove that $\theta$ satisfies condition IV of Theorem 4.5. Since
\( \theta \in X \otimes Y \) we have \( \theta_{t} = \theta \). Since \( \theta \in S_{X \otimes Y, A \oplus B} \) we have for \( t \) sufficiently small \( N_{-\epsilon} \theta M_{-t} \in X \otimes Y \). Therefore \( \theta M_{-t} = N_{t} (N_{-t} \theta M_{-t}) \) is bounded.

II. For \( \epsilon \) sufficiently small \( N_{-\epsilon} \theta M_{-\epsilon} \in X \otimes Y \), therefore by Properties 5.1.c.

\[
\langle \Theta F, G \rangle_{Y} = \langle N_{\epsilon} (N_{-\epsilon} \theta M_{-\epsilon}) F(\epsilon), G \rangle_{Y} = \langle (N_{-\epsilon} \theta M_{-\epsilon}) F(\epsilon), G(\epsilon) \rangle_{Y} = (N_{-\epsilon} \theta M_{-\epsilon}, F(\epsilon) \otimes G(\epsilon))_{X \otimes Y} = \langle \Theta, F \otimes G \rangle_{X \otimes Y}.
\]

III. Let \( \Gamma : S_{X,A}^{t} \to S_{Y,B}^{t} \) be continuous. By Theorem 4.5.V there exists \( \tau > 0 \) and continuous \( O : S_{X,A} \to S_{Y,B}^{t} \) such that \( \Gamma = OM_{t} \). By Theorem 4.2.IV there exists \( \beta > 0 \) such that \( N_{-\beta} O M_{t} \) is a bounded operator. Put \( a = \frac{1}{2} \min (b, \frac{1}{2}) \), then

\[ \Gamma = N_{a} \{ N_{-\beta} (N_{-\epsilon} O M_{t}) M_{\frac{1}{2} - \alpha} \} M_{a} \]

The operator between ( ) is bounded. Further the operator between \{ \} is H.S. since \( N_{-\beta} \) or \( M_{\frac{1}{2} - \alpha} \) is H.S.

It follows that \( \Gamma \in S_{X \otimes Y, A \oplus B}^{t} \).

IV. The if-part is a special case of III. For the only if-part consider the special map \( \Gamma = M_{a} : S_{X,A}^{t} \to S_{X,A}^{t} \) for some \( a > 0 \). In order that \( \Gamma \in S_{X \otimes Y, A \oplus B}^{t} \) it has to be H.S.

\textit{Case b.} Continuous linear maps \( S_{X,A}^{t} \to S_{Y,B}^{t} \).

Let \( K \in S_{X \otimes Y, A \oplus B}^{t} \). For \( f \in S_{X,A}^{t} \) we define \( Kf \in S_{Y,B}^{t} \) by

\[ (Kf)(t) = N_{e} K(e) M_{-e} f \]

For any \( f \in S_{X,A}^{t} \) and \( t > 0 \) this definition makes sense for \( \epsilon > 0 \) sufficiently small. Moreover \( (Kf)(t) \) does not depend on \( \epsilon \).

Theorem 6.2.

I. For each \( K \in S_{X \otimes Y, A \oplus B}^{t} \) the linear operator \( K : S_{X,A} \to S_{Y,B}^{t} \) defined by (b) is continuous.

II. For each \( K \in S_{X \otimes Y, A \oplus B}^{t}, f \in S_{X,A}^{t}, g \in S_{Y,B}^{t} \)

\[ \langle g, Kf \rangle_{Y} = \langle f \otimes g, K \rangle_{X \otimes Y} \]


III. If for each $t > 0$ at least one of the operators $M_t, N_t$ is H.S.
then $S'_{X\otimes Y, A \otimes B}$ comprises all continuous linear operators from $S_{X, A}$ into $S_{Y, B}$.

IV. $S'_{X, A \otimes A}$ comprises all continuous linear operators from $S_{X, A}$ into $S'_{X, A}$ iff for each $t > 0$ the operator $M_t$ is H.S.

Proof

I. We use condition II of Theorem 4.4.
For each $t > 0$, $a > 0$ $N_{t/a}K$ is a bounded operator from $X$ into $Y$ because for $\varepsilon$ sufficiently small

$$N_{t/a}K = N_{t-\varepsilon}K(\varepsilon)M_{a-\varepsilon}$$

All operators in the last expression are bounded.

II. For each $r > 0$ $K(t)$ is a H.S.-map. For $\varepsilon$ sufficiently small, with Properties 5.1.c.

$$<g, Kf>Y = (N_{-\varepsilon}g, (N_{-\varepsilon}K(\varepsilon))(M_{-\varepsilon}f))Y = (M_{-\varepsilon}f \otimes N_{-\varepsilon}g, K(\varepsilon))_{X \otimes Y} = <\varepsilon \otimes g, K>_{X \otimes Y}$$

III. Let $L : S'_{X, A} \rightarrow S_{Y, B}$ be continuous. According to Theorem 4.4.II the operators $N_{tLM_t}$ are bounded for each $t > 0$. However if $N_t$ or $M_t$ is H.S. for each $t > 0$ then $N_{tLM_t}$ is H.S. for each $t > 0$ and it defines an element in $S'_{X \otimes Y, A \otimes B}$. A simple verification shows that this element reproduces $L$ by recipe (b).

IV. The if-part is a special case of III. For the only-if-part consider the special map $L = \text{emb} = I$. Here $I$ is the identity map. $M_tIM_t = M_{2t}$ can be considered as an element of $S'_{X \otimes Y, A \otimes A}$ iff $M_t$ is H.S. for all $t > 0$.

CASE c: continuous linear maps: $S'_{X, A} \rightarrow S_{Y, B}$
Let $P \in S'_{X, A}$. For $f \in S_{X, A}$ we define $Pf \in S_{Y, B}$ by

$$(c) \quad Pf = P(\varepsilon)M_{-\varepsilon}f$$
Pf \in S_{X,A} Y,B \because P(\varepsilon) \in S_{X,Y,A+B}. The definition makes sense for \varepsilon sufficiently small and does not depend on the choice of \varepsilon.

\section*{Theorem 6.3.}

I. For each \( P \in \Sigma' \), the linear operator \( P : S_{X,A} \to S_{Y,B} \) defined by (c) is continuous.

II. For each \( P \in \Sigma' \), each \( f \in S_{X,A} \), each \( G \in S_{Y,B} \)

\[ \langle Pf, G \rangle_X = \langle P, f \odot G \rangle_A. \]

III. If for each \( t > 0 \) at least one of the operators \( M_t, N_t \) is H.S. then \( \Sigma' \) comprises all continuous linear operators from \( S_{X,A} \) into \( S_{Y,B} \).

IV. Consider the special case \( Y = X \) and \( B = A \).

\( \Sigma' \) comprises all continuous linear operators from \( S_{X,A} \) into itself iff for each \( t > 0 \) the operator \( M_t \) is H.S.

\section*{Proof}

I. We use condition II of Theorem 4.2.

Let \( f_n \to 0 \) strongly in \( S_{X,A} \). For some \( \varepsilon > 0 \) \( M_{-\varepsilon} f_n \to 0 \) in \( X \).

\( P(\varepsilon) \in S_{X,Y,A+B} \), therefore there exists \( \delta > 0 \) such that \( N_{-\delta} P(\varepsilon) \) is a bounded operator. But then \( N_{-\delta} Pf_n = (N_{-\delta} P(\varepsilon)) M_{-\varepsilon} f_n \to 0 \) in \( Y \), which shows that \( Pf_n \to 0 \) strongly in \( S_{Y,B} \).

II. For \( \beta \) and \( \delta \) sufficiently small and positive we have

\[ \langle P, f \times G \rangle_A = \langle P(\beta), (M_{-\beta} \odot I)(f \odot G) \rangle_{X \odot Y} \]

\[ = \langle P(\beta), M_{-\beta} f \odot G \rangle_{X \odot Y} = (N_{-\delta} P(\beta) M_{-\delta}, (M_{-\beta+\delta} f) \odot G(\delta))_{X \odot Y} \]

\[ = (N_{-\delta} P(\beta) M_{-\delta} M_{-\beta+\delta} f, G(\delta))_Y = \]

\[ \langle P(\beta) M_{-\beta} f, G \rangle_Y = \langle Pf, G \rangle_Y. \]

III. Let \( Q : S_{X,A} \to S_{Y,B} \) be continuous. According to Theorem 4.2.IV for each \( t > 0 \) there exist \( \beta(t) > 0 \) such that \( N_{-\beta(t)} Q M_t \) is a bounded map from
X into Y. Now because of the assumption on \( M \), \( N \) we find that
\[
QM_t = N\beta(t)(N^{-\beta}(t)QM_t)
\]
is an element of \( \Sigma' \). It reproduces the operator \( Q \) if the recipe (c) is applied.

IV. The if-part is a special case of III. For the only-if-part consider the identity map \( I : S_{X,A} \rightarrow S_{X,A} \). In order that \( IM_t \) as a function of \( t \) is an element of \( \Sigma' \), the operator \( M_t \) has to H.S. for all \( t > 0 \).

CASE d: Continuous linear maps: \( S_{X,A}^t \rightarrow S_{Y,B}^t \). Let \( \phi \in \Sigma' \). For \( F \in S_{X,A} \) we define \( \phi F \in S_{Y,B}^t \) by
\[
(\phi F)(t) = \phi(t)M_{-\epsilon(t)}F(\epsilon(t))
\]
This definition makes sense for \( t > 0 \) and \( \epsilon(t) > 0 \) sufficiently small.
\[
(\phi F)(t) \in S_{Y,B} \quad \text{since} \quad \phi(t) \in S_{X,Y,A \oplus B}. \quad \text{Moreover}
\]
\[
N(t)(\phi F)(t) = N(t)\phi(t)M_{-\epsilon(t)}F(\epsilon(t)) = \phi(t + t)M_{-\epsilon(t)}F(\epsilon(t)) = (\phi F)(t + t).
\]

Theorem 6.4.

I. For each \( \phi \in \Sigma' \) the linear operator \( \phi : S_{X,A}^t \rightarrow S_{Y,B}^t \) defined by (d) is continuous.

II. For each \( \phi \in \Sigma' \), \( F \in S_{X,A}^t \), \( g \in S_{Y,B}^t \)
\[
< \phi, F \ast g >_Y = < \phi, F \ast g >_B
\]

III. If for each \( t > 0 \) at least one of the operators \( M_t \), \( N_t \) is H.S. then
\( \Sigma' \) comprises all continuous linear operators from \( S_{X,A}^t \) into \( S_{Y,B}^t \).

IV. Consider the special case \( Y = X \) and \( B = A \).
\( \Sigma' \) comprises all continuous linear operators from \( S_{X,A}^t \) into itself iff for each \( t > 0 \) the operator \( M_t \) is H.S.

Proof

I. We use Theorem 4.6.III. For each \( t > 0 \) \( N_t \phi \in S_{X,Y,A \oplus B}^t \). Then according to case a, \( N_t \phi \) is a continuous linear map from \( S_{X,A}^t \) into \( S_{Y,B}^t \).
II. For $a$ and $\delta$ sufficiently small and positive

$$\langle \phi, F \ast g \rangle = \langle \phi(a), (I \ast N_{-a})(F \ast g) \rangle_{X \otimes Y} =$$

$$\langle \phi(a), F \ast N_{-a}g \rangle_{X \otimes Y} = \langle (N_{-\delta} \phi(a) M_{-\delta}, F(\delta) \ast N_{-a+\delta}g) \rangle_{X \otimes Y} =$$

$$= \langle (N_{-\delta} \phi(a) M_{-\delta} F(\delta), N_{-a+\delta}g) \rangle_Y = \langle (\phi(a) M_{-\delta} F(\delta), N_{-a}g) \rangle_Y$$

III. Let $\Psi : S'_{X,A} \to S'_{Y,B}$ be continuous, According to Theorem 4.6.VI for each $t > 0$ there exists $\beta(t) > 0$ such that $N_t \Psi M_{t, \beta(t)}$ is a densely defined and bounded operator from $X$ into $Y$. If one of the operators $N_t, M$ is H.S. for arbitrary small positive $a$ it follows that $N_t \Psi$ is H.S. for $t > 0$, because $N_t \Psi = N_t \Psi M_{t, \beta(t)}$. Here $\Psi M_{t, \beta(t)}$ denotes the extension of $N_t \Psi$ to the whole of $X$.

Since $N_t \Psi = N_t \Psi M_{t, \beta(t)}$ it follows $N_t \Psi \in S'_{X \otimes Y, A \otimes B}$.

Hence $N_t \Psi$, as a function of $t$, belongs to $S'_{X \otimes Y}$. By recipe (d) the operator $\Psi$ is reproduced.

IV. The if-part is a special case of III. For the only-if-part consider the identity map $I$. In order that $M_t I$, as a function of $t$, can be considered as an element in $S'_{Y, B}$ the operator $M_t$ should be H.S. for all $t > 0$.
APPENDIX A. Functions of self-adjoint operators.

A self-adjoint operator $A$ in a Hilbert space $H$ admits a spectral decomposition

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda.$$  

Here $E_\lambda$ is the so-called resolution of the identity, i.e. $E_\lambda$ is a monotonically increasing function with values in the projection operators, $E_\lambda$ is strongly continuous from the right, $E(-\infty) = 0$, $E(\infty) = I$, $E_\lambda E_\mu = E_{\min(\lambda, \mu)}$. Let $\psi$ be a complex valued and everywhere finite Borel function on $\mathbb{R}$. On the domain

$$D(\psi(A)) = \{h \in H| \int_{-\infty}^{\infty} |\psi(\lambda)|^2 d(\lambda, h, h) < \infty\}$$

the operator $\psi(A)$ is defined by

$$\psi(A)h, x = \int_{-\infty}^{\infty} \psi(\lambda) d(\lambda, h, x) \quad \forall x \in H$$

where the integration is carried out with respect to the Borel measure determined by $m((\lambda_1, \lambda_2]) = (E_{\lambda_2}^\lambda h, x) - (E_{\lambda_1}^\lambda h, x)$. We write formally

$$\psi(A) = \int_{-\infty}^{\infty} \psi(\lambda) dE_\lambda$$

It can be proved, see [Y] Ch. XI.12, that $\psi(A)$ is a normal operator. $\psi(A)$ is self-adjoint if $\psi$ is real valued. Further $(\psi\chi)(A) = \psi(A)\chi(A)$ where the usual precautions concerning the domains must be regarded.
In our case $A \geq 0$ and the functions $\psi$ are always piece-wise continuous and piece-wise of bounded variation. We frequently employ the notations
\[
\int_{a}^{b} \psi(\lambda) dE_{\lambda} , \quad 0 < a < b \leq \infty .
\]
By this we mean
\[
\int_{-\infty}^{\infty} \chi_{a,b}(\lambda) \psi(\lambda) dE_{\lambda}
\]
with
\[
\chi_{a,b}(\lambda) = \begin{cases} 
1 & \text{on } (a,b] \\
0 & \text{elsewhere .} 
\end{cases}
\]
By the notation
\[
\int_{0}^{b} \psi(\lambda) dE_{\lambda} , \quad 0 < b \leq \infty
\]
we mean
\[
\int_{-\infty}^{\infty} \chi_{-1,b}(\lambda) \psi(\lambda) dE_{\lambda} .
\]

The elementary and elegant presentation of spectral theory in [LS] is quite sufficient for our work. We referred to [Y] because the theory in [LS] is not completely dressed for our purposes.
APPENDIX B Holomorphic semi-groups

On a Banach space $X$ we consider a one-parameter strongly continuous semi-group of bounded operators $P_t$, $t \geq 0$, with the properties

$$P_0 = I, \quad P_{t+\tau} = P_t P_{\tau}, \quad t \geq 0, \quad \tau \geq 0.$$  

Such a semi-group describes the solutions of the convolution problem in $X$:

$$\frac{du(t)}{dt} = -A u(t), \quad u(0) = u_0 \text{ by } u(t) = P_t u_0.$$  

Here $-A$ is the so-called infinitesimal generator of the semi-group $P_t$. $A$ is a densely defined closed linear operator which enjoys the following property (cf. [Y] Ch. IX):

There exist constants $M$ and $a$ such that for every $\lambda > 0$ the resolvent $R_\lambda = (A + \lambda I)^{-1}$ exists and $\|R_\lambda^n\| \leq M(\lambda - a)^{-n}$. Then

$$P_t u_0 = \lim_{n \to \infty} (I + \frac{t}{n} A)^{-n} u$$  

and

$$u_0 \in D(A^m) \Rightarrow \forall t > 0 \quad P_t u_0 \in D(A^m).$$

Our interest is centered upon holomorphic semi-groups then $\forall t > 0 \forall x \in H$ $P_t x \in D(A)$, $P_t x$ is an infinite differentiable function of $t$ and has the local representation

$$\forall x \in X \quad P_t x = \sum_{n=0}^{\infty} \frac{(\lambda - t)^n}{n!} A^n x = \sum_{n=0}^{\infty} \frac{(\lambda - t)^n}{n!} P_t^{(n)} x.$$  

In this case $P_t$ has a holomorphic extension into a sector $|\arg t| < \phi$ in the complex $t$-plane.

A necessary and sufficient condition on $P_t$ in order to be holomorphic is

$$\forall x \in H \quad \forall t > 0 \quad P_t x \in D(A)$$  

and

$$\exists \alpha > 0 \quad \forall t, 0 < t \leq 1 \quad \|tP_t\| \leq \alpha.$$  

Then $\phi = \arctan (ae)^{-1}$. See [Y] Ch. IX. A necessary and sufficient
condition on the infinitesimal generator \(-A\) to generate a holomorphic semi-group is:

\[ \exists \beta > 0 \; \exists \varepsilon > 0 \; \forall \lambda, \; \Re \lambda \geq 1 + \varepsilon \; \| \lambda(\lambda I + A)^{-1} \| \leq \beta \]

See [Y] Ch. IX.

An interesting subclass of holomorphic semi-groups on a Hilbert space \(H\) is generated by operators of the form \(A = Q + B\) where \(Q\) is a positive self-adjoint operator and \(B\) is subjugated to \(Q\) in a specified sense. See [G].

In case \(B = 0\) the operators \(P_t, \; t \geq 0\) are all selfadjoint and \(P_t\) has a holomorphic extension into the whole open right half plane and a strongly continuous extension into the closed right half plane. Whenever \(t > 0\) the operator \(P_t\) has a self-adjoint inverse which is unbounded iff \(A\) is unbounded. In this paper we consider only the case that \(A\) is unbounded, self adjoint and non-negative.
APPENDIX C. A Banach-Steinhaus Theorem.

The following is taken from [J].

Theorem. Let $(B, \| \cdot \|)$ be a Banach space. Suppose that for every $a > 0$ there is given a collection $A_a$ of bounded linear functionals of $B$ such that $a_1 < a_2 \Rightarrow A_a_1 \subset A_a_2$ ($a_1 > 0, a_2 > 0$). Then the following statements are equivalent:

(i) $\forall f \in B \exists a > 0 \exists M > 0 \forall L \in A_a \left[ |Lf| \leq M \right]$

(ii) $\exists a > 0 \exists M > 0 \forall f \in B \forall L \in A_a \left[ |Lf| \leq M\|f\| \right]$

Proof

It is easily seen that (ii) $\Rightarrow$ (i): if $N > 0$ and $B > 0$ are such that $|Lf| \leq N f$ for every $f \in B$, $L \in A_B$, then we can take $a = B$ and $M = N\|f\|$ in (i).

We now suppose (i), and we shall show that

(*) $\forall a > 0 \forall M > 0 \exists f \in B \exists L \in A_a \left[ |Lf| > M\|f\| \right]$

yields a contradiction. To every $n \in \mathbb{N}$ we can then find an $f_n \in B$ and an $L_n \in A_a$ such that $|L_n f_n| > n\|f_n\|$. Note that $L_n$ is a bounded linear functional of $B$ for every $n \in \mathbb{N}$, and that $(L_n f_n)_{n \in \mathbb{N}}$ is a bounded sequence for every $f \in B$. It follows therefore from the Banach-Steinhaus theorem that there is an $M > 0$ such that $|L_n f| \leq M\|f\|$ for every $n \in \mathbb{N}$ and every $f \in B$. Contradiction.
Acknowledgements

During the research I have profited from very stimulating discussions with A.J.E.M. Janssen.
I wish to thank J.W. Nienhuys and J.J. Seidel for making valuable suggestions on the presentation of the introduction.
Finally I wish to thank J. Boersma for showing me the way in Whittaker and Watson.
References


