On the degree of approximation of functions in $C^1 [0,1]$ by Bernstein polynomials

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On the degree of approximation of functions in $C^1[0,1]$ by Bernstein polynomials

by

F. Schurer and F.W. Steutel

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Abstract
Let $f$ be a real function defined on the interval $[0,1]$ and let $B_n(f;x)$ denote its $n$-th order Bernstein polynomial. The object of this paper is to study the exact degree of approximation with Bernstein polynomials for functions in $C^1[0,1]$. We estimate the difference $|B_n(f;x) - f(x)|$ in terms of $\omega(f';\delta)$, the modulus of continuity of $f'$, with $\delta = \frac{1}{\sqrt{n}}$. Starting-point of our considerations is a theorem of Lorentz ([5], p. 21). Similar work on the degree of approximation with Bernstein polynomials for functions in $C[0,1]$ has been done by Sikkema ([10],[11]) and Esseen [1]. Results for functions in $C^1[0,1]$ and $\delta = \frac{1}{n}$ may be found in [8].

AMS Subject Classification: 41A25
1. Introduction and summary

Let $C[0,1]$ be the set of real continuous functions defined on $[0,1]$. The expression

$$B_n(f;x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)p_{n,k}(x),$$

where $f \in C[0,1]$ and

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad (x \in [0,1], \; n = 1, 2, \ldots; \; k = 0, 1, \ldots, n),$$

is called the Bernstein polynomial of order $n$ of the function $f$. Bernstein proved as early as 1912 that

$$(1.1) \quad B_n(f;x) \to f(x) \quad (n \to \infty),$$

uniformly on $[0,1]$. For a proof of this result the reader is referred to [5], pp. 5-6. We note that $B_n$ is a positive linear operator, i.e. $f \geq 0$ on $[0,1]$ implies $B_n f \geq 0$ on $[0,1]$. This property can be used to give an elegant proof of (1.1) (cf. [3], pp. 28-30). There is an extensive literature on the rapidity with which $B_n(f;x)$ tends to $f(x)$ as $n \to \infty$. As an illustration we cite here a result of Popoviciu [6], who proved that

$$(1.2) \quad \max_{0 \leq x \leq 1} |B_n(f;x) - f(x)| \leq \frac{3}{2} \omega(f; \frac{1}{\sqrt{n}})$$

for all $f \in C[0,1]$ and all $n \in \mathbb{N}$. Here $\omega(f; \delta)$ denotes the modulus of continuity of $f$, i.e.

$$\omega(f; \delta) = \max_{|x-y| \leq \delta} |f(x) - f(y)| \quad (x, y \in [0,1], \; \delta > 0).$$

A refinement of (1.2) can be found in [5], p. 20. There also the problem was raised of determining the best constant in the right-hand side of (1.2). This problem was solved by Sikkema in a couple of papers ([10], [11]). He proved that for all $f \in C[0,1]$ and all $n \in \mathbb{N}$

$$(1.3) \quad \max_{0 \leq x \leq 1} |B_n(f;x) - f(x)| \leq \kappa \omega(f; \frac{1}{\sqrt{n}}),$$

where

$$(1.4) \quad \kappa = \frac{4306 + 837\sqrt{6}}{5832} \approx 1.089887,$$

*) Here and elsewhere the numbers are rounded to the last digit shown.
and that $k$ in (1.3) cannot be replaced by any number smaller than the one
given in (1.4) without invalidating the inequality.
Esseen [1] proved that for all $f \in C[0,1]$

$$\max_{0 \leq x \leq 1} |B_n(f;x) - f(x)| \leq A \omega(f;\frac{1}{\sqrt{n}})$$

(1.5) with

$$A = 2 \sum_{j=0}^{\infty} (j+1)(\phi(2j+2) - \phi(2j)) = 1.045564,$$

(1.6) where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt,$$

(1.7) and he showed that the number $A$ in (1.5) cannot be replaced by any number
smaller than the one given in (1.6).

This paper deals with similar problems. Here the setting is the space $C^1[0,1]$
of real functions that have a continuous derivative on $[0,1]$. Starting-point
of our considerations is a result of Lorentz ([5], p. 21) concerning the de­
gree of approximation with Bernstein polynomials for this class of functions.
His theorem reads as follows.

**Theorem 1.1 (Lorentz).** Let $f \in C^1[0,1]$ and let $\omega_1(f;\delta) := \omega(f';\delta)$ be the mo­
dulus of continuity of $f'$, then for $n \in \mathbb{N}$ one has

$$\max_{0 \leq x \leq 1} |B_n(f;x) - f(x)| \leq C \frac{1}{\sqrt{n}} \omega_1(f;\frac{1}{\sqrt{n}}),$$

(1.8) with $C = 3/4$.

As in the case of $f \in C[0,1]$, one may ask for the best constant in (1.8). To
be more precise, for each fixed $n \in \mathbb{N}$ let (cf. remark 1.2 on p. 4)

$$c_n := \sup_{f \in C^1[0,1]} \frac{\max_{0 \leq x \leq 1} |B_n(f;x) - f(x)|}{\omega_1(f;\frac{1}{\sqrt{n}})}.$$

(1.9)
The problem then is to determine

\[(1.10) \quad c^{(1)} := \sup_{n \geq 1} c_n.\]

To this end we introduce the functions \(c_n(x)\) defined by

\[(1.11) \quad c_n(x) = \sup_{f \in C^1[0,1]} \frac{\sqrt{n} |B_n(f;x) - f(x)|}{\omega_1(f; \frac{1}{\sqrt{n}})}.
\]

We shall derive explicit expressions for the \(c_n(x)\), and obtain \(c^{(1)}\), making use of the obvious equality

\[(1.12) \quad \sup_{f \in C^1[0,1]} \sup_{0 \leq x \leq 1} \frac{|B_n(f;x) - f(x)|}{\omega_1(f; \frac{1}{\sqrt{n}})} \leq \max_{0 \leq x \leq 1} \sup_{f \in C^1[0,1]} \frac{|B_n(f;x) - f(x)|}{\omega_1(f; \frac{1}{\sqrt{n}})} \]

where, in fact, on both sides of (1.12) \(\sup\) may be replaced by \(\max\).

We now give a sketch of the contents of the various sections of this report. Section 2 contains some preliminary results that will be needed later. In order to make the paper reasonably self-contained, we start section 3 with Lorentz's proof of theorem 1.1. By a slight modification of this proof we obtain a small improvement of the estimate (1.8). Then it is shown by elementary means that \(c^{(1)} < \frac{1}{2}\). In section 4 the so-called extremal functions are introduced; these play a fundamental role in determining \(c_n(x)\) as defined in (1.11), and hence in determining \(c_n\) and \(c^{(1)}\). In section 5 we calculate \(c_n\) for \(n = 1, 2, \ldots, 5\). A simple proof of the fact that \(c^{(1)} = c_1 = \frac{1}{2}\) is given in section 6, using the positivity of the operators \(B_n\). In section 7 we obtain

\[(1.13) \quad c^{(2)} := \sup_{n \geq 2} c_n,
\]

and, finally, in section 8 we derive \(\lim_{n \to \infty} c_n(x)\) and \(\lim_{n \to \infty} c_n\), and we give some numerical information concerning the numbers \(c_n\).

Remark 1.1. In [8] similar problems are treated for functions \(f \in C^1[0,1]\) normalized by \(\omega_1(f; \frac{1}{n})\) instead of \(\omega_1(f; \frac{1}{\sqrt{n}})\). There it is proved that for \(n \in \mathbb{N}\) the smallest constant \(d_n\) satisfying the inequality

\[(1.14) \quad \max_{0 \leq x \leq 1} |B_n(f;x) - f(x)| \leq d_n \omega_1(f; \frac{1}{n})
\]
for all \( f \in C^1[0,1] \), is given by

\[
d_n = \begin{cases} 
\frac{1}{8} + \frac{1}{8(n+1)} & \text{if } n \text{ is even}, \\
\frac{1}{8} + \frac{1}{8n} & \text{if } n \text{ is odd}.
\end{cases}
\]

It is, of course, possible to consider norming by \( \omega_1(f,n^{-\alpha}) \) for, say, \( 0 < \alpha \leq 1 \). It seems, however, that the case \( \alpha = \frac{1}{2} \) is the most interesting, and the most natural from an asymptotic point of view. The case \( \alpha = 1 \) is by far the most tractable.

Remark 1.2. Inequalities of the type (1.8) and (1.14) are satisfied by linear functions (which are left unchanged by \( B_n \)) for all positive values of the constant \( C \) or \( d_n \). It follows that the linear functions are of no interest for the problem we are concerned with. As the right-hand side of (1.9), and similar expressions elsewhere, are undefined for linear functions, in the remaining part of this report we shall often disregard these functions, without explicitly indicating this in our notation.

2. Preliminary results

This section contains three lemmas, the contents of which will be needed later. We start out with a well-known result that may be found in [4], p. 122 or [5], p. 14.

Lemma 2.1. Let

\[
T_n,s(x) := \sum_{k=0}^{n} (k - nx)^s p_{n,k}(x) \quad (n=1,2,...; s=0,1,2,...).
\]

Then one has the following recursion formula

\[
T_{n,s+1}(x) = x(1-x)\{T_{n,s}(x) + nsT_{n,s-1}(x)\},
\]

where

\[
T_{n,0}(x) = 1, \quad T_{n,1}(x) = 0.
\]
Corollary 2.1. If \( x(1-x) \) is denoted by \( X \), then in particular

\[
(2.4) \quad T_{n,2}(x) = nX, \quad T_{n,4}(x) = 3n^2x^2 + nX(1 - 6X),
\]

\[
(2.5) \quad T_{n,6}(x) = 15n^3x^3 + 5n^2x^2(5 - 26X) + nX(1 - 30X + 120X^2).
\]

The proof of lemma 2.1 is omitted. Corollary 2.1 is a straightforward consequence of (2.2), using (2.3).

The next lemma deals with a particular sum that plays a prominent role in the calculation of the functions \( c_n(x) \) as defined in (1.11); we list some of its properties.

**Lemma 2.2.** Defining

\[
(2.6) \quad S_n(x) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left| \frac{k}{n} - x \right| p_{n,k}(x) \quad (x \in [0,1]; \quad n = 1,2,\ldots),
\]

one has

\[
S_n(x) = S_n(1-x).
\]

If \([a]\) denotes the largest integer not exceeding \( a \) and if \( \|S_n\| := \max_{x \in [0,1]} |S_n(x)| \), then

\[
(2.7) \quad S_n(x) = \frac{1}{\sqrt{n}} (n - x) \left( n \right)^{r+1} (1-x)^{n-r} \quad (r = [nx]),
\]

\[
(2.8) \quad \max_{x \in [0,\frac{1}{n}]} S_n(x) < \max_{x \in [\frac{1}{n},\frac{2}{n}]} S_n(x) < \ldots < \max_{x \in [\frac{r}{n},\frac{r+1}{n}]} S_n(x) = \max_{x \in [\frac{r}{n},\frac{r}{n} + \frac{1}{2}]} S_n(x) = \|S_n\| \quad (r = \left[ \frac{n-1}{2} \right]),
\]

\[
\begin{align*}
\frac{1}{4} &= \|S_1\| > \|S_3\| > \|S_5\| > \ldots, \\
\frac{4}{27\sqrt{2}} &= \|S_2\| > \|S_4\| > \|S_6\| > \ldots,
\end{align*}
\]

\[
(2.9) \quad \lim_{n \to \infty} S_n(x) = \sqrt{\frac{x(1-x)}{2\pi}} =: S(x),
\]

\[
(2.10) \quad \lim_{n \to \infty} S_n(x) = \sqrt{\frac{x(1-x)}{2\pi}} =: S(x).
\]
\begin{align*}
(2.11) \quad \|S_n\| \to S(\frac{1}{2}) &= \frac{1}{2\sqrt{2\pi}} = 0.19947114 \quad (n \to \infty) .
\end{align*}

**Proof.** We shall first establish formula (2.7). Let \( x \in [0,1] \) and let \( r = [nx] \). Taking into account the second part of (2.3) we have

\[
\frac{1}{n} \sum_{k=0}^{n} \frac{|k-x|}{n} p_{n,k}(x) = \frac{1}{2} \sum_{k=0}^{r} (x - \frac{k}{n}) p_{n,k}(x) + \frac{1}{2} \sum_{k=r+1}^{n} \left( \frac{k}{n} - x \right) p_{n,k}(x) =
\]

\[
= \frac{r}{n} \sum_{k=0}^{r} (\frac{k}{n} - x) p_{n,k}(x) =: f_r(n,x) .
\]

In order to evaluate the \( f_r(n,x) \) we consider \( r \) to be independent of \( n \) and \( x \) for the moment, and we take generating functions. Changing the order of summation and using (2.3) again it is easily verified that one has

\[
\sum_{r=0}^{n} f_r(n,x) z^r = \sum_{k=0}^{n} \left( \frac{k}{n} - x \right) p_{n,k}(x) z^r = \sum_{k=0}^{n} \left( \frac{k}{n} - x \right) p_{n,k}(x) \frac{z^r - \frac{z^{k+1}}{1-z}}{1-z} =
\]

\[
= \frac{1}{1-z} (x(z+1-x))^n \sum_{k=0}^{n} \frac{n}{k} (xz)^k (1-x)^{n-k} =
\]

\[
= \frac{1}{1-z} (x(z+1-x))^n \sum_{k=0}^{n} \frac{n-1}{k-1} (xz)^k (1-x)^{n-k} =
\]

\[
= \frac{1}{1-z} (x(z+1-x))^n \sum_{k=0}^{n} (n-1)(xz)^k (1-x)^{n-r} = x(1-x)(xz+1-x)^{n-1} .
\]

Expanding the last expression in powers of \( z \) we obtain

\[
x(1-x)(xz+1-x)^{n-1} = x(1-x) \sum_{r=0}^{n-1} \binom{n-1}{r} (xz)^r (1-x)^{n-1-r} =
\]

\[
= \sum_{r=0}^{n} \binom{n-1}{r} x^{r+1} (1-x)^{n-r} z^r .
\]

Equating the coefficients of \( z^r \), and taking into account the definitions of \( f_r(n,x) \) and \( S_n(x) \), it follows that (with \( r = [nx] \) again)

\[
S_n(x) = \sqrt{n} \binom{n-1}{r} x^{r+1} (1-x)^{n-r} = \frac{1}{\sqrt{n}} (n-r) \binom{n}{r} x^{r+1} (1-x)^{n-r} .
\]
This proves (2.7). We omit verification of \( S_n(x) = S_n(1-x) \); it is an easy consequence of (2.7). We note that a different proof of (2.7) can be given by making use of Hilfssatz I in [10].

The monotonicity of the various maxima of \( S_n(x) \) on the interval \([0,1]\) can be shown as follows. Obviously, for fixed \( r \) the maximum of \( S_n(x) \) on the interval \( \left[ \frac{r}{n}, \frac{r+1}{n} \right] \) is attained at \( x = \frac{r + 1}{n + 1} = \left[ \frac{nx}{n+1} \right] + 1 \). In order to prove (2.8) it is therefore sufficient to show that this maximum, i.e.

\[
(2.12) \quad \frac{(n+1)}{\sqrt{n}} \frac{r+1}{n+1} \frac{n-r}{n+1} r+1 \frac{n-r}{n+1}
\]

is an increasing function of \( r \) on \( \{0,1,2,\ldots,\left[\frac{n-1}{2}\right]\} \).

The quotient of two successive maxima is equal to

\[
\frac{(n)(n + 1) r+1 (n-r) n-r+1}{(r-1)(n + 1) r (n-r) n-r+2} = \frac{(n+1)(n-r+1) n-r+1}{(1 - \frac{1}{r+1}) r+1}.
\]

As \((1-x)^x\) is an increasing function for \( x > 1 \), this ratio is at least one as long as \( n-r+1 \geq r+1 \), i.e. \( r \leq \frac{n}{2} \). Taking into account the range of \( r \), it thus follows that for \( n \) even the largest maximum of \( S_n(x) \) is attained when \( r = \frac{n}{2} - 1 \). In case \( n \) is odd \( S_n(x) \) attains its largest maximum when \( r = \frac{n-1}{2} \).

This proves (2.8). As a consequence we have

\[
(2.13) \quad \|S_n\| = \begin{cases} S_n \left( \frac{n}{2m+2} \right) & \text{if } n \text{ is even}, \\ S_n \left( \frac{1}{2} \right) & \text{if } n \text{ is odd}. \end{cases}
\]

We proceed with the proof of (2.9). As for the first part of it, this amounts to showing (cf. (2.12) and (2.13)) that for \( n = 2m + 1 \) we have

\[
\frac{2m + 2}{\sqrt{2m + 1}} \left( \frac{2m+1}{m} \right)^{\frac{1}{2}} > \frac{2m + 4}{\sqrt{2m + 3}} \left( \frac{2m+3}{m+1} \right)^{\frac{1}{2}} (m = 0,1,2,\ldots) \hfill (2m+5)
\]

This inequality is equivalent to

\[
2(m + 1) > \sqrt{(2m + 1)(2m + 3)} (m = 0,1,2,\ldots),
\]

which is apparently true.
The verification of the second part of (2.9) is more tedious. Assuming \( n = 2m \) and taking into account formulae (2.12) and (2.13), it is easily verified that we have to show that

\[
\sqrt{m + 1} \cdot m^{1\over 2} (2m + 3)^{2m + 3} > 2(2m + 1)^{2m + 2}(m + 2)^{m + 2} \quad (m = 1, 2, \ldots)
\]

Taking logarithms of both sides we have to establish that for \( m = 1, 2, \ldots \)

\[
(2.14) \quad \frac{1}{2} \log m(m + 1) + m \log m + (2m + 3) \log(2m + 3) - \log 2 - (2m + 2) \log(2m + 1) +
\]

\[- (m + 2) \log(m + 2) > 0,
\]

which is easily seen to be true for \( m = 1 \) and \( m = 2 \). It can be shown that the derivative of the left-hand side of (2.14) is negative for \( m \geq 2 \). This observation, together with the fact that (2.14) holds for large \( m \) (as can be seen from its expansion in powers of \( 1/m \)), assures that the second assertion of (2.9) holds. We omit all computational details. Finally, an application of the central limit theorem easily yields (2.10). Assertion (2.11) then is an immediate consequence. For details we refer to section 6, where similar limits are computed. This completely proves lemma 2.2.

In table 2.1 we show the numerical values of \( \|S_n\| \), \( n = 1, 2, \ldots, 30 \), together with the corresponding values of \( x \), where the maxima are attained.

We proceed with a simple lemma that will be used in sections 3 and 6.

**Lemma 2.3.** If \( c_n \) is defined as in (1.9), then

\[
c_1 = 1/4.
\]

**Proof.** Using the mean value theorem and the definition of the modulus of continuity we have

\[
|B_1(f; x) - f(x)| = |x(1 - x)f'(\xi_0) - x(1 - x)f'(\xi_1)| =
\]

\[
x(1 - x)|f'(\xi_0) - f'(\xi_1)| \leq \frac{1}{3} \omega_1(f; 1)
\]

Taking \( f(x) = \frac{1}{2}|x - \frac{1}{2}|, 0 \leq x \leq 1 \), it follows that \( c_1 = \frac{1}{4} \). The fact that \( f \) is not differentiable at \( x = \frac{1}{2} \) does not, of course, affect the argument.
Table 2.1

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3. An upper bound for \(c(1)\)

In the introductory section we have formulated theorem 1.1 of Lorentz. As theorems of this type are the central theme of this report, for the sake of completeness, we here reproduce the proof of Lorentz' theorem as given by him in [5], p. 21.

Proof of theorem 1.1. We have

\[
(3.1) \quad f(x_1) - f(x_2) = (x_1 - x_2)f'(\xi) = \\
= (x_1 - x_2)f'(x_1) + (x_1 - x_2)(f'(\xi) - f'(x_1)) \quad (x_1 < \xi < x_2) .
\]

Let \(x \in [0,1]\) be arbitrary and fixed, and let \(\delta\) be an arbitrary positive number. In view of (3.1) and the second part of (2.3), we deduce, using a well-known property of the modulus of continuity, that we have
\[ |B_n(f;x) - f(x)| = | \sum_{k=0}^{n} (f(x) - f(x \cdot k_n))p_{n,k}(x)| \leq \]

\[ \leq \sum_{k=0}^{n} n^{k} \cdot |x - k/n| f'(x)p_{n,k}(x)| + \omega_1(f;\delta) \sum_{k=0}^{n} \frac{k}{n} - x |p_{n,k}(x) + \frac{1}{\delta} \sum_{k=0}^{n} \frac{k}{n} - x |^2 p_{n,k}(x) \leq \]

\[ \leq \omega_1(f;\delta) \sum_{k=0}^{n} \frac{k}{n} - x |p_{n,k}(x) + \frac{1}{\delta} \sum_{k=0}^{n} \frac{k}{n} - x |^2 p_{n,k}(x) \leq \]

by Schwarz' inequality. By the first part of (2.4) we have

\[ \frac{n}{\delta} \sum_{k=0}^{n} \frac{k}{n} - x |^2 p_{n,k}(x) = x(1 - x) \quad (x \in [0,1]), \]

and hence

\[ |B_n(f;x) - f(x)| \leq \omega_1(f;\delta) \left( \frac{1}{2\sqrt{n}} + \frac{1}{4n\delta} \right). \]

Putting \( \delta = \frac{1}{\sqrt{n}} \) here, we obtain theorem 1.1.

We next show that by a slight modification of the above proof it is possible to improve on the constant \( \frac{1}{4} \).

**Theorem 3.1.**

\[ c(1) := \sup_{n \geq 1} \sup_{f \in C^1[0,1]} \frac{\sqrt{n} \cdot \max_{0 \leq x \leq 1} |B_n(f;x) - f(x)|}{\omega_1(f;\frac{1}{\sqrt{n}})} < \frac{11}{16}. \]

**Proof.** Proceeding as in the proof of theorem 1.1 one has

\[ |B_n(f;x) - f(x)| \leq \omega_1(f;\delta) \sum_{k=0}^{n} \frac{k}{n} - x |p_{n,k}(x) + \frac{1}{\delta} \sum_{k=0}^{n} \frac{k}{n} - x |^2 p_{n,k}(x) \leq \]

\[ \leq \omega_1(f;\delta) \sum_{k=0}^{n} \frac{k}{n} - x |p_{n,k}(x) + \frac{1}{\delta} \sum_{k=0}^{n} \frac{k}{n} - x |^2 p_{n,k}(x) \leq \]

\[ \leq \omega_1(f;\delta) \sum_{k=0}^{n} \frac{k}{n} - x |p_{n,k}(x) + \frac{1}{\delta} \sum_{k=0}^{n} \frac{k}{n} - x |^4 p_{n,k}(x) \leq \]

(3.3)
Using (2.4) and taking into account that \( T_{n,4}(x) \) is maximal at \( x = \frac{1}{4} \) for all \( n \geq 2 \), it follows that

\[
\sum_{k=0}^{n} \left( \frac{k}{n} - x \right)^{4} p_{n,k}(x) \leq \frac{3}{16n^{2}} - \frac{1}{8} < \frac{3}{16n^{2}} \quad (n = 1, 2, \ldots),
\]

where the case \( n = 1 \) has to be verified separately. Using this and (3.2) we obtain

\[
|B_{n}(f;x) - f(x)| \leq \omega_{1}(f;\delta) \left( \frac{1}{\sqrt{n}} + \frac{3}{16n^{2/3}} \right).
\]

Taking \( \delta = \frac{1}{\sqrt{n}} \) it follows that for all \( x \in [0,1] \)

\[
|B_{n}(f;x) - f(x)| \leq \frac{11}{16} \frac{1}{\sqrt{n}} \omega_{1}(f;\frac{1}{\sqrt{n}}) \quad (n = 1, 2, \ldots).
\]

This proves theorem 3.1.

Remark 3.1. As is obvious from the considerations above, we also have

\[
|B_{n}(f;x) - f(x)| \leq \omega_{1}(f;\delta) \left( \sum_{k=0}^{n} \left| \frac{k}{n} - x \right| p_{n,k}(x) + \frac{1}{\delta} \sum_{k=0}^{n} \left( \frac{k}{n} - x \right)^{6} p_{n,k}(x) \right).
\]

The second sum in the right-hand side can be evaluated by using (2.5). However, it turns out that this yields a constant that is worse than the constant of theorem 3.1.

Remark 3.2. Instead of applying Schwarz' inequality to the first sum in the right-hand side of (3.3) one can use the estimates (2.9) of lemma 2.2. In this way, treating the case \( n = 1 \) separately (cf. lemma 2.3), one can improve slightly further on the upper bound for \( c^{(1)} \). We shall not pursue this, but instead improve on this upper bound by a more effective method.

We have
Theorem 3.2.
\[
\sup_{n \geq 1} \sup_{f \in C[0,1]} \frac{\sqrt{n} \max_{0 \leq x \leq 1} |B_n(f;x) - f(x)|}{\omega^{1/2}(f;\frac{1}{\sqrt{n}})} < \frac{1}{2}.
\]

Proof. Let \( n \geq 2 \), let \( x \in [0,1] \) and let \( \delta \) be positive. In view of (2.3) and using a well-known property of the modulus of continuity it is easily verified that one has

\[
|B_n(f;x) - f(x)| = \left| \sum_{k=0}^{n-1} \left( f \left( \frac{k}{n} \right) \right) - f(x) \right| \leq \sum_{k=0}^{n-1} \left| f \left( \frac{k}{n} \right) - f(x) \right| \leq \sum_{k=0}^{n-1} \left( f'(t) - f'(x) \right) dt \leq \omega^{1/2}(f;\frac{1}{\sqrt{n}}) \sum_{k=0}^{n-1} \left| f \left( \frac{k}{n} \right) - f(x) \right|
\]

Putting \( \delta = \frac{1}{\sqrt{n}} \) and taking into account definitions (2.1) and (2.6), we obtain

\[
|B_n(f;x) - f(x)| \leq \frac{1}{\sqrt{n}} \omega^{1/2}(f;\frac{1}{\sqrt{n}}) \left\{ 2S_n(x) + \frac{1}{4n^2} T_n(x) \right\}.
\]

The expression between brackets in (3.6) can be evaluated by means of the second part of (2.4) and formulae (2.7), (2.9) of lemma 2.2. Using these results and observing (3.4), by straightforward calculation one has
Consequently, in view of (3.6) and lemma 2.3 for the case \( n = 1 \), it follows that \( c^{(1)} < \frac{1}{4\sqrt{3}} + \frac{3}{64} < \frac{1}{2} \).

**Remark 3.3.** Considering the proof of theorem 3.2, the following inequality apparently also holds:

\[
|B_n(f;x) - f(x)| \leq \omega_1(f;\delta) \left\{ \sum_{k=0}^{n} \left| \frac{k}{n} - x \right| p_{n,k}(x) + \frac{1}{\delta^{2s-1}} \sum_{\frac{k}{n} - x > \delta x} \int \left| t - x \right|^{2s-1} dt |p_{n,k}(x)| \right\},
\]

where \( s \) is an arbitrary positive number. It turns out that \( s = 2 \) is a suitable choice when one sets out to prove that \( c^{(1)} < \frac{1}{4} \). Taking \( s = 1 \) gives rise to simpler calculations, but then a few cases corresponding to small values of \( n \) have to be treated separately. Choosing \( s = 3 \), one can use (2.5), but the calculations become somewhat more intricate.

4. **The extremal functions**

Up to now we have not made use of the functions \( c_n(x) \) defined in section 1, formula (1.11), but instead we have obtained a (rather crude) upper bound for \( c^{(1)} \). In this section we derive an explicit expression for \( c_n(x) \), which will be used in the following sections to determine the quantities \( c_n \) \((n = 1, 2, \ldots, 5)\), \( c^{(1)} \) and \( c^{(2)} \) as defined in (1.9), (1.10) and (1.13).

We first slightly simplify the notation and define

\[
\Delta_n(f;x) = B_n(f;x) - f(x).
\]

We shall make use of the representation (cf. (3.5))

\[
\Delta_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) \int_{x}^{k/n} f'(t) dt ,
\]

\[
\parallel S_n \parallel \leq \parallel S_3 \parallel = \frac{1}{6} \sqrt{3} \quad (n = 2, 3, \ldots),
\]

\[
\frac{1}{4n^2} \parallel T_{n,4} \parallel = \frac{1}{2} \max_{0 \leq x \leq 1} \sum_{k=0}^{n} (k - nx)^4 p_{n,k}(x) < \frac{3}{64} \quad (n = 2, 3, \ldots).
\]
and of the fact that for every linear function \( \ell \) we have
\[
(4.3) \quad \Delta_n(f + \ell, x) = \Delta_n(f; x).
\]

The main object of this section is to prove the following theorem.

**Theorem 4.1.** For each \( n \in \mathbb{N} \), for each \( x_0 \in [0,1] \) and each \( \delta > 0 \)
\[
(4.4) \quad \sup_{f \in C^1[0,1]} \frac{|\Delta_n(f; x_0)|}{\omega_1(f; \delta)} = \Delta_n(\tilde{f}; x_0),
\]
where \( \tilde{f} \), which depends on \( x_0 \) and \( \delta \), is defined for all real \( x \) by the conditions
\[
(4.5) \quad \begin{cases} 
\tilde{f}(x_0) = 0, \\
\tilde{f}(x) = j + \frac{1}{\delta} \quad (j \delta < x - x_0 \leq (j + 1) \delta; \ j = 0, \pm 1, \pm 2, \ldots).
\end{cases}
\]

The functions \( \tilde{f} \) will be called *extremal functions*. We shall prove theorem 4.1 in a number of small steps, stated as lemmas, which gradually narrow down the class of functions to be considered. We first slightly widen the class \( C^1[0,1] \) to the class \( K_\delta \) of functions on \((-\infty, \infty)\) defined as follows:
\[
(4.6) \quad K_\delta = \left\{ f : f \text{ is continuous, } f' \text{ is bounded, } f' \text{ has finitely many jump discontinuities on finite intervals and no other discontinuities, } 0 < \omega_1(f; \delta) \leq 1 \right\}.
\]

The restriction \( \omega_1(f; \delta) > 0 \) excludes the linear functions (cf. remark 1.2 on p. 4), and the restriction \( \omega_1(f; \delta) \leq 1 \) is simply a matter of scaling. We might, in fact, restrict ourselves to functions with \( \omega_1(f; \delta) = 1 \), but this is not practical for our purposes.

In order to avoid trivial, but troublesome, difficulties at the boundary points 0 and 1, we continue all functions to the interval \((-\infty, \infty)\), in such a way that their essential properties, e.g. convexity, extend to this interval.

We now state and prove our lemmas.

**Lemma 4.1.**
\[
(4.7) \quad \sup_{f \in C^1[0,1]} \frac{|\Delta_n(f; x_0)|}{\omega_1(f; \delta)} = \sup_{f \in K_\delta} \frac{|\Delta_n(f; x_0)|}{\omega_1(f; \delta)}.
\]
Proof. On \([0,1]\) every \(f \in K_{\delta}\) is the pointwise limit of functions in \(C^1[0,1]\) with the same value of \(\omega_1(f;\delta)\), as is easily seen by approximating \(f'\) by continuous functions. The result then follows from the continuity of \(B_n\) with respect to pointwise convergence.

Lemma 4.2.
\[
\sup_{f \in K_{\delta}} \frac{|\Delta_n(f;x_0)|}{\omega_1(f;\delta)} = \sup_{f \in K_{\delta}} \frac{\Delta_n(f;x_0)}{\omega_1(f;\delta)}.
\]

Proof. Without loss of generality we take, here and in the sequel, \(f \in K_{\delta}\) such that \(\Delta_n(f;x_0) \geq 0\). We define a function \(\bar{f}\) by (see figure 4.1)
\[
\bar{f}(x_0) = f(x_0), \quad \bar{f}'(x) = \begin{cases} \inf f'(u) & \text{if } x \leq x_0, \\ \sup f'(u) & \text{if } x \geq x_0. \end{cases}
\]

Clearly \(\bar{f}'\) is nondecreasing, i.e. \(\bar{f}\) is convex. As \(\bar{f}' \leq f'\) on \((-\infty, x_0]\) and \(\bar{f}' \geq f'\) on \([x_0, \infty)\), it follows from (4.2) that \(\Delta_n(\bar{f};x_0) \geq \Delta_n(f;x_0)\). Moreover, \(\omega_1(\bar{f};\delta) \leq \omega_1(f;\delta)\). This can be seen as follows: if on \([x,x+\delta]\) the derivative \(\bar{f}'\) varies by \(\varepsilon\), i.e. if \(\bar{f}'(x + \delta) - \bar{f}'(x) = \varepsilon\), then by the definition of \(\bar{f}'\), for each \(n > 0\), there exist \(y_1\) and \(y_2\) with \(x \leq y_1 < y_2 \leq x + \delta\) such that \(f'(y_2) - f'(y_1) > \varepsilon - n\). This proves that \(\omega_1(\bar{f};\delta) \leq \omega_1(f;\delta)\). It is easily verified that \(\bar{f}\) satisfies the remaining conditions for \(K_{\delta}\), and the lemma is proved.
For arbitrary \( f \) on \( (-\infty, \infty) \) we define \( f^* \) by

\[
\begin{align*}
\text{\( f^* \) is continuous} \\
f^*(x_0 + j\delta) = f(x_0 + j\delta) \\
f^* \text{ is linear on each interval } (x_0 + j\delta, x_0 + j\delta + \delta).
\end{align*}
\]

Lemma 4.3. If \( f \) is convex and \( f \in K_\delta \), then \( f^* \) is convex and \( f^* \in K_\delta \).

Proof. The function \( f^* \) is trivially convex: its graph is a polygon inscribed in the graph of \( f \). In order to prove that \( f^* \in K_\delta \), we show that \( \omega_1(f^*;\delta) \leq \omega_1(f;\delta) \) and hence \( \omega_1(f^*;\delta) \leq 1 \); the other conditions are easily checked. We proceed as follows. If \( t \) is not of the form \( x_0 + j\delta \), then \( f^*(t) \) is well defined. For \( t = x_0 + j\delta \) we define \( f^*(t) \) by continuity from the left. Now, for any two points \( t_1 \) and \( t_2 \) with \( t_1 < t_2 \leq t_1 + \delta \) we have for some integer \( j \)

\[
0 \leq f^*(t_2) - f^*(t_1) \leq f^*(t_1 + \delta) - f^*(t_1) = \frac{f(x_0 + j\delta + \delta) - f(x_0 + j\delta)}{\delta} - \frac{f(x_0 + j\delta) - f(x_0 + j\delta - \delta)}{\delta} = \frac{1}{\delta} \int_{j\delta}^{(j+1)\delta} f'(x_0 + t) dt - \frac{1}{\delta} \int_{(j-1)\delta}^{j\delta} f'(x_0 + t) dt = \frac{1}{\delta} \int_{j\delta}^{(j+1)\delta} \{f'(x_0 + t) - f'(x_0 + t - \delta)\} dt.
\]

From this inequality it follows that \( \omega_1(f^*;\delta) \leq \omega_1(f;\delta) \leq 1 \), and the lemma is proved.

Lemma 4.4. If \( f \in K_\delta \) is convex, then

\[
\frac{\Delta_n(f^*;x_0)}{\omega_1(f^*;\delta)} \geq \frac{\Delta_n(f;x_0)}{\omega_1(f;\delta)}.
\]

Proof. As \( f^*(x) \geq f(x) \) for all \( x \), by the positivity of the operator \( B_n \) and the fact that \( f^*(x_0) = f(x_0) \), we have \( \Delta_n(f^*;x_0) \geq \Delta_n(f;x_0) \). From the proof of the preceding lemma we conclude that \( \omega_1(f^*;\delta) \leq \omega_1(f;\delta) \), and the lemma follows.
We now define a class $K^*_0$ of piecewise linear functions by

$$K^*_0 = \{ f; f \in K_0, f \text{ convex}, f \equiv f^*, f(x_0) = 0, f'(x) = \frac{1}{2} \text{ for } x_0 < x \leq x_0 + \delta \},$$

where the restrictions on $f(x_0)$ and $f'(x)$ are inessential because of (4.3).

From the preceding four lemmas we now obtain

**Lemma 4.5.**

$$\sup_{f \in C_1([0,1])} \frac{|\Delta_n(f;x_0)|}{\omega_1(f;\delta)} = \sup_{f \in K^*_0} \frac{\Delta_n(f;x_0)}{\omega_1(f;\delta)}.$$

We are now ready for the proof of the main result of this section.

**Proof of theorem 4.1.** For $f \in K^*_0$ we have in view of (4.2)

$$\frac{\Delta_n(f;x_0)}{\omega_1(f;\delta)} = \sum_{k=0}^{n} p_{n,k}(x_0) \int_{x_0}^{x_0 + k/n} \frac{f'(t)}{\omega_1(f;\delta)} \, dt,$$

where $f'$ is a nondecreasing stepfunction with largest step equal to $\omega_1(f;\delta)$. It follows that $f'/\omega_1(f;\delta)$ is a nondecreasing stepfunction with largest step equal to 1, i.e. with modulus of continuity equal to 1. As is obvious from (4.9), $\Delta_n(f;x_0)/\omega_1(f;\delta)$ is maximal if all jumps of $f'/\omega_1(f;\delta)$ are equal to 1, i.e. if $f/\omega_1(f;\delta) = \tilde{f}$ as defined in (4.5). This proves the theorem.

We conclude this section by giving explicit expressions for $\tilde{f}$ and $\Delta_n(\tilde{f};x_0)$. From (4.5) we have for $x > x_0$

$$\tilde{f}'(x) = \frac{1}{2} + \sum_{j=1}^{\infty} H(x - x_0 + j\delta),$$

where $H$ denotes the unit stepfunction, taken to be continuous from the left. Hence, because $\tilde{f}$ is symmetric with respect to $x_0$,

$$\tilde{f}(x) = \frac{1}{2} |x - x_0| + \sum_{j=1}^{\infty} (|x - x_0| - j\delta)_+,$$

where $a_+ := \max(a, 0)$. As $\tilde{f}(x_0) = 0$ we have

$$\Delta_n(\tilde{f};x_0) = B_n(\tilde{f};x_0).$$
and therefore

\[ \Delta_n(f;x_0) = \frac{1}{n} \sum_{k=0}^{n} \left| \frac{k}{n} - x_0 \right| p_{n,k}(x_0) + \frac{1}{n} \sum_{j=1}^{\infty} \sum_{k=0}^{n} \left( \left| \frac{k}{n} - x_0 \right| - j\delta \right) p_{n,k}(x_0), \]

or

\[
(4.11) \quad \Delta_n(f;x_0) = \frac{1}{n} \sum_{k=0}^{n} \left| \frac{k}{n} - x_0 \right| p_{n,k}(x_0) + \sum_{j=1}^{\infty} \sum_{k=0}^{n} \left( \left| \frac{k}{n} - x_0 \right| - j\delta \right) p_{n,k}(x_0).
\]

From a graph of \( \tilde{f} \) (see figure 4.2) one easily obtains

\[
\tilde{f}(x) = (\bar{\varepsilon} + \frac{1}{\delta}) |x - x_0| - \frac{1}{\delta}(\bar{\varepsilon} + 1) \delta,
\]

where \( \bar{\varepsilon} = \left[ \frac{|x - x_0|}{\delta} \right] \). Hence we have

\[
\Delta_n(\tilde{f};x_0) = B_n(\tilde{f};x_0) = \frac{1}{n} \sum_{k=0}^{n} \left| \frac{k}{n} - x_0 \right| p_{n,k}(x_0) \left( \bar{\varepsilon} + \frac{1}{\delta} \right) \frac{k}{n} - x_0 - \frac{1}{\delta} (\bar{\varepsilon} + 1) \delta,
\]

with \( \bar{\varepsilon} = \left[ \frac{|k/n - x_0|}{\delta} \right] \). This can be rewritten as

\[
(4.12) \quad \Delta_n(\tilde{f};x_0) = \frac{1}{n} \sum_{k=0}^{n} \left| \frac{k}{n} - x_0 \right| p_{n,k}(x_0) \left( \bar{\varepsilon} \frac{k}{n} - x_0 - \frac{1}{\delta} (\bar{\varepsilon} - 1) \delta \right),
\]

with \( \bar{\varepsilon} = \left[ \frac{|k/n - x_0|}{\delta} \right] + 1 \). Formula (4.12), with \( \delta = \frac{1}{\sqrt{n}} \), has been used for the computer calculations (cf. table 8.1, p. 37).

![Figure 4.2.](image)
5. **Calculation of \( c_n \) for some small values of \( n \)**

The object of this section is to determine the first few constants \( c_n \) by using the results of the preceding section. For that purpose we take \( \delta = \frac{1}{\sqrt{n}} \) and we write \( \tilde{f}_n \) instead of \( f \). Furthermore, we shall restrict ourselves here to the cases \( n = 1, 2, 3, 4, 5 \). It turns out that for these small values of \( n \) the calculations involved to determine \( c_n \) are still manageable; for \( n = 5 \), however, the computational effort is already considerable. As will be clear from theorem 7.1 of section 7, the constant \( c_5 \) is the one we are particularly interested in. The exact determination of the constants \( c_n \) for \( n \geq 6 \) does not seem to be easy, in particular when \( n \) is even. In principle, it can be done in the same way as we are proceeding in this section. Ultimately, it amounts to determining the absolute maximum of a piecewise polynomial function on \([0, \frac{1}{2}]\), but for \( n \geq 6 \) the calculations involved become rather intricate. Therefore, in section 7 we use a method that yields estimates for the constants \( c_n \) (\( n \geq 6 \)), that are sufficiently sharp for our purposes. The values of \( c_n \) can also be obtained numerically; for these results we refer to table 8.1.

In order to determine \( c_1, \ldots, c_5 \) we recall that in section 4 we proved that

\[
(5.1) \quad \Delta_n (\tilde{f}_n; x_0) = B_n (\tilde{f}_n; x_0)
\]

where, according to formula (4.10),

\[
\tilde{f}_n (x) = \frac{1}{2} |x - x_0| + Q_n (x),
\]

with

\[
Q_n (x) = \sum_{j=1}^{\infty} \left( |x - x_0| - \frac{j}{\sqrt{n}} \right) + .
\]

As \( \omega_1 (\tilde{f}_n; \frac{1}{\sqrt{n}}) = 1 \), we have (cf. (1.11), (4.1) and (4.5))

\[
(5.2) \quad c_n (x_0) = \sqrt{n} \Delta_n (\tilde{f}_n; x_0) = \sqrt{n} B_n (\tilde{f}_n; x_0),
\]

and hence by (4.11), writing \( x \) instead of \( x_0 \)

\[
(5.3) \quad c_n (x) = \frac{1}{\sqrt{n}} \sum_{k=0}^{\infty} \frac{1}{n} \left| \frac{k}{n} - x \right| p_{n,k}(x) + \sqrt{n} \sum_{j=1}^{\infty} \left( \sum_{k=0}^{\infty} \left| \frac{k}{n} - x \right| - \frac{j}{\sqrt{n}} \right) p_{n,k}(x),
\]

or
(5.4) \[ c_n(x) = S_n(x) + R_n(x) , \]
with
\[ R_n(x) = \sqrt{n} R_n(Q_n;x) \]
and \( S_n(x) \) as defined in (2.6).
A precise evaluation of \( R_n(x) \) is only feasible for small values of \( n \). Together with lemma 2.2, formula (5.4) then allows one to determine the maximum \( c_n \) of \( c_n(x) \) without lengthy calculations, say for \( n \leq 5 \). In section 7 we shall obtain upper bounds for \( R_n(x) \).
The calculation of the constants \( c_n \) rests completely upon the representation for \( c_n(x) \) as given in (5.3). (We recall that \( c_1 \) was already determined in lemma 2.3.) In what follows, we shall consider the cases \( n = 1, \ldots, 5 \). Because of symmetry we restrict ourselves to \( 0 \leq x \leq \frac{1}{2} \).

\( n = 1 \). In this case the second contribution of (5.3) to \( c_1(x) \) is zero, and in view of formula (2.7) we have
\[ c_1(x) = x(1 - x) \quad (0 \leq x \leq \frac{1}{2}) . \]
Hence
\[ c_1 = \max_{0 \leq x \leq \frac{1}{2}} c_1(x) = c_1\left(\frac{1}{2}\right) = \frac{1}{4} . \]

\( n = 2 \). There are two cases to be considered, viz. \( 0 \leq x \leq 1 - \frac{1}{\sqrt{2}} \) and \( 1 - \frac{1}{\sqrt{2}} \leq x \leq \frac{1}{2} \). According to (5.3) and using (2.7) we have
\[ c_2(x) = \sqrt{2} x(1 - x)^2 + \sqrt{2}(1 - x - \frac{1}{\sqrt{2}})x^2 \quad (0 \leq x \leq 1 - \frac{1}{\sqrt{2}}) , \]
\[ c_2(x) = \sqrt{2} x(1 - x)^2 \quad (1 - \frac{1}{\sqrt{2}} \leq x \leq \frac{1}{2}) . \]
One easily verifies that
\[ \max_{0 \leq x \leq 1 - \frac{1}{\sqrt{2}}} c_2(x) = \frac{\sqrt{2} - 1}{2} = 0.207107 , \]
\[ \max_{1 - \frac{1}{\sqrt{2}} \leq x \leq \frac{1}{2}} c_2(x) = c_2\left(\frac{1}{3}\right) = \frac{4}{27\sqrt{2}} = 0.209513 . \]
Hence
\[ c_2 = \frac{4}{27\sqrt{2}} . \]
\(n = 3\). In view of (5.3) and (2.7) one has
\[c_3(x) = \sqrt{3} x (1 - x)^3 + \sqrt{3} (1 - x - \frac{1}{3}) x^3 + (2 - 3x - \sqrt{3}) x^2 (1 - x) \quad (0 \leq x \leq \frac{2}{3} - \frac{1}{\sqrt{3}}), \]
\[c_3(x) = \sqrt{3} x (1 - x)^3 + \sqrt{3} (1 - x - \frac{1}{3}) x^3 \quad \left(\frac{2}{3} - \frac{1}{\sqrt{3}} \leq x \leq \frac{1}{3}\right), \]
\[c_3(x) = 2\sqrt{3} x^2 (1 - x) + \sqrt{3} (1 - x - \frac{1}{3}) x^3 \quad \left(\frac{1}{3} \leq x \leq 1 - \frac{1}{\sqrt{3}}\right), \]
\[c_3(x) = 2\sqrt{3} x^2 (1 - x)^2 \quad \left(1 - \frac{1}{\sqrt{3}} \leq x \leq \frac{1}{2}\right). \]
Again, one easily verifies that
\[\max_{0 \leq x \leq \frac{2}{3} - \frac{1}{\sqrt{3}}} c_3(x) < 0.120955, \quad \max_{\frac{2}{3} - \frac{1}{\sqrt{3}} \leq x \leq \frac{1}{3}} c_3(x) < 0.213834, \]
\[\max_{\frac{1}{3} \leq x \leq 1 - \frac{1}{\sqrt{3}}} c_3(x) = 0.206267, \quad \max_{1 - \frac{1}{\sqrt{3}} \leq x \leq \frac{1}{2}} c_3(x) = c_3(\frac{1}{2}) = \frac{1}{8\sqrt{3}} = 0.216506, \]
and thus
\[c_3 = \frac{1}{8\sqrt{3}}. \]

\(n = 4\). Obviously, there are two cases to be considered, viz. \(0 \leq x \leq \frac{1}{4}\) and \(\frac{1}{4} \leq x \leq \frac{1}{2}\). Taking into account (5.3) and (2.7) we find
\[c_4(x) = 2x (1 - x)^4 + (1 - 2x) x^4 + 2 (1 - 4x) x^3 (1 - x) \quad (0 \leq x \leq \frac{1}{4}), \]
\[c_4(x) = 6x^2 (1 - x)^3 + (1 - 2x) x^4 \quad \left(\frac{1}{4} \leq x \leq \frac{1}{2}\right). \]
Elementary calculations show that
\[\max_{0 \leq x \leq \frac{1}{4}} c_4(x) = c_4\left(\frac{1}{4}\right) = \frac{523}{3125} = 0.16736, \]
\[\max_{\frac{1}{4} \leq x \leq \frac{1}{2}} c_4(x) = c_4\left(\frac{3}{4}\right) = \frac{664}{3125} = 0.21248. \]
Consequently,
\[c_4 = \frac{664}{3125}. \]
\( n = 5 \). A close examination of (5.3) (cf. figure 5.1) shows that one has to deal with the following expressions for \( c_5(x) \).

\[
c_5(x) = \sqrt{5} x(1-x)^5 + \sqrt{5}((1-x-\frac{1}{5}\sqrt{5})x^5 + (4-5x-\sqrt{5})x^4(1-x) + \\
+ (6-10x-2\sqrt{5})x^3(1-x)^2 + (1-x-\frac{2}{5}\sqrt{5})x^5 \) \quad (0 \leq x \leq 1 - \frac{2}{5}\sqrt{5}),
\]

\[
c_5(x) = \sqrt{5} x(1-x)^5 + \sqrt{5}((1-x-\frac{1}{5}\sqrt{5})x^5 + (4-5x-\sqrt{5})x^4(1-x) + \\
+ (6-10x-2\sqrt{5})x^3(1-x)^2 \) \quad (1 - \frac{2}{5}\sqrt{5} \leq x \leq \frac{3}{5} - \frac{1}{5}\sqrt{5}),
\]

\[
c_5(x) = \sqrt{5} x(1-x)^5 + \sqrt{5}((1-x-\frac{1}{5}\sqrt{5})x^5 + (4-5x-\sqrt{5})x^4(1-x) \) \quad (\frac{3}{5} - \frac{1}{5}\sqrt{5} \leq x \leq \frac{1}{3}),
\]

\[
c_5(x) = 4\sqrt{5} x^2(1-x)^4 + \sqrt{5}((1-x-\frac{1}{5}\sqrt{5})x^5 + (4-5x-\sqrt{5})x^4(1-x) \) \quad (\frac{1}{3} \leq x \leq \frac{4}{5} - \frac{1}{5}\sqrt{5}),
\]

\[
c_5(x) = 4\sqrt{5} x^2(1-x)^4 + \sqrt{5}((1-x-\frac{1}{5}\sqrt{5})x^5 \) \quad (\frac{4}{5} - \frac{1}{5}\sqrt{5} \leq x \leq \frac{2}{3}),
\]

\[
c_5(x) = 6\sqrt{5} x^3(1-x)^3 + \sqrt{5}((1-x-\frac{1}{5}\sqrt{5})x^5 \) \quad (\frac{2}{3} \leq x \leq \frac{1}{5}\sqrt{5}),
\]

\[
c_5(x) = 6\sqrt{5} x^3(1-x)^3 + \sqrt{5}((1-x-\frac{1}{5}\sqrt{5})x^5 + (x-\frac{1}{5}\sqrt{5})(1-x)^5) \) \quad (\frac{1}{5}\sqrt{5} \leq x \leq \frac{1}{2}).
\]

\[
\begin{array}{cccccccc}
\frac{3}{5} - \frac{1}{5}\sqrt{5} & \frac{2}{5} & \frac{1}{5} & \frac{4}{5} - \frac{1}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} & \frac{3}{5} & \frac{4}{5} & 1 \\
0 & 1 - \frac{2}{5}\sqrt{5} & \frac{1}{5} & \frac{2}{5} & \frac{1}{2}
\end{array}
\]

**Figure 5.1.**

For our purposes it is not necessary to determine the maxima of \( c_5(x) \) on all the respective intervals; estimates will be sufficient. Elementary calculations show that the maximum of \( c_5(x) \) on the interval \( [\frac{1}{5}\sqrt{5}, \frac{1}{2}] \) is attained at \( x = 1 \) and \( c_5(1) = \frac{2\sqrt{5} - 1}{16} = .217008 \). Once this number is available we can compare it with (upper bounds for) the maxima of \( c_5(x) \) on the remaining intervals. Proceeding in this way we arrive at the following results.
In view of these results we conclude that

\[ c_5 = \frac{2\sqrt{5} - 1}{16}. \]  

The graph of the extremal function corresponding to the constant \( c_5 \) is shown in figure 5.2.

---

Remark 5.1. As will be clear from the example treated above, the method with which the constants \( c_n \) can be determined, is straightforward and simple in principle. However, it is also obvious that the amount of computational work involved grows quite rapidly. Furthermore, certain numerical complications arise when determining the absolute maximum of the piecewise polynomial function \( c_n(x) \) for large values of \( n \). Most of these complications can be avoided, however, by using suitable estimates for \( c_n(x) \) and \( R_n(x) \) (cf. remark 6.3 and the contents of section 7).
The results of this section are collected in the following theorem.

**Theorem 5.1.** For $c_n$ defined as in (1.9), we have

$$
\begin{align*}
c_1 &= c_1 \left( \frac{1}{2} \right) = \frac{1}{4} = 0.250000, \\
c_2 &= c_2 \left( \frac{1}{3} \right) = \frac{4}{27} = 0.209513, \\
c_3 &= c_3 \left( \frac{1}{2} \right) = \frac{1}{8} = 0.125000, \\
c_4 &= c_4 \left( \frac{2}{5} \right) = \frac{664}{3125} = 0.212480, \\
c_5 &= c_5 \left( \frac{1}{2} \right) = \frac{2\sqrt{3} - 1}{16} = 0.217008.
\end{align*}
$$

6. A simple proof of $c^{(1)} = 1/4$

In theorem 4.1 we obtained the extremal function $\tilde{f}$, depending on an arbitrary positive number $\delta$. Since we wish to sharpen Lorentz' theorem 1.1, we take $\delta = \frac{1}{\sqrt{n}}$ and again write $\tilde{f}_n$ instead of $\tilde{f}$. In view of (4.1) one has

$$
(6.1) \quad \tilde{f}_n(x) = \frac{1}{4} |x - x_0| + \sum_{j=1}^{\infty} \left( |x - x_0| - \frac{j}{\sqrt{n}} \right)_+, 
$$

where $a_+ = \max(a, 0)$.

Using the functions $\tilde{f}_n$ we shall prove in an elementary way (cf. [9]) that $c_n \leq \frac{1}{4}$ for all $n \in \mathbb{N}$. To this end we introduce a quadratic function $q_n$ defined by

$$
(6.2) \quad q_n(x) = \frac{1}{8\sqrt{n}} + \frac{1}{\sqrt{n}} (x - x_0)^2.
$$

The graph of $q_n$ is a parabola that is tangent to the graph of $\tilde{f}_n$ in the mid-points of each of the linear pieces of that graph (cf. figure 6.1).

The properties of the function $q_n$ are formally stated in the following lemma.
Lemma 6.1. Let $q_n$ be defined by (6.2) and let $f_n$ be the extremal function defined by (6.1), then we have

i) $q_n(x_0 + \frac{2k + 1}{2\sqrt{n}}) = f_n(x_0 + \frac{2k + 1}{2\sqrt{n}}) = k + \frac{1}{4} (k = 0, \pm 1, \pm 2, \ldots)$,

ii) $q_n'(x_0 + \frac{2k + 1}{2\sqrt{n}}) = f_n'(x_0 + \frac{2k + 1}{2\sqrt{n}}) = k + \frac{1}{4} (k = 0, \pm 1, \pm 2, \ldots)$,

iii) $q_n(x) \geq f_n(x) \quad (x \in [0, 1])$,

iv) $\sqrt{n} B_n(q_n; x_0) = \frac{1}{8} + \frac{1}{2} x_0 (1 - x_0)$.

Proof. In view of the second part of (4.5) it follows by integration from $x_0$ to $x_0 + \frac{2k + 1}{2\sqrt{n}}$ that for $k \geq 0$ we have

$$
\tilde{f}_n(x_0 + \frac{2k + 1}{2\sqrt{n}}) = \frac{1}{\sqrt{n}} \left( \frac{1}{2} + \frac{3}{2} + \ldots + \frac{2k - 1}{2} + \frac{2k + 1}{4} \right) = \frac{2k^2 + 2k + 1}{4\sqrt{n}} = q_n(x_0 + \frac{2k + 1}{2\sqrt{n}}).
$$

By symmetry we obtain i) also for $k < 0$. From (6.1) and (6.2) we immediately have ii). Taking into account that $q_n(x_0) > \tilde{f}_n(x_0)$ and the fact that $q_n$ is a convex function, property iii) now follows from i) and ii). Finally, iv) is an easy consequence of the first part of (2.4). This proves the lemma.
We are now in a position to prove one of the main results of this report (cf. remark 1.2).

**Theorem 6.1.**

\[
\begin{align*}
  c^{(1)} := \sup_{n \geq 1} \sup_{f \in C^1[0,1]} \max_{0 \leq x \leq 1} \frac{B_n(f; x) - f(x)}{\omega_1(f; \frac{1}{\sqrt{n}})} &= \frac{1}{4}.
\end{align*}
\]

**Proof.** Noting that $B_n$ is a positive operator, it follows from properties iii) and iv) of lemma 6.1 that for all $x_0 \in [0,1]$ one has in view of (5.2)

\[
(6.3) \quad c_n(x_0) = \sqrt{n} B_n(f; x_0) \leq \sqrt{n} B_n(q_n; x_0) = \frac{1}{8} + \frac{1}{2} x_0(1 - x_0).
\]

Hence, $c_n \leq \frac{1}{4}$ for $n = 1, 2, 3, \ldots$. Taking into account lemma 2.3 and observing definition (1.10) of $c^{(1)}$ we obtain $c^{(1)} = c_1 = \frac{1}{4}$.

**Remark 6.1.** In order to get a lower bound for $c_n(x_0)$ we consider the function $\tilde{q}_n$ defined by

\[
\tilde{q}_n(x) = \frac{\sqrt{n}}{2} (x - x_0)^2.
\]

It is easily verified that one has (cf. figure 6.1)

\[
\tilde{q}_n(x_0 + \frac{k}{\sqrt{n}}) = \tilde{q}_n(x_0 + \frac{k}{\sqrt{n}}) = \frac{k^2}{2\sqrt{n}} \quad (k = 0, \pm 1, \pm 2, \ldots),
\]

\[
\tilde{q}_n(x) \leq \tilde{q}_n(x) \quad (x \in [0,1]).
\]

Proceeding in the same way as in the proof of theorem 6.1 we deduce

\[
(6.4) \quad c_n(x_0) = \sqrt{n} B_n(\tilde{f}_n; x_0) \geq \sqrt{n} B_n(\tilde{q}_n; x_0) = \frac{1}{4} x_0(1 - x_0).
\]

**Remark 6.2.** The estimate (6.3) can be improved by using a function $\tilde{q}_n$, that differs slightly from the function $q_n$ appearing in lemma 6.1. This function is chosen to be of the form

\[
\tilde{q}_n(x) = \frac{a}{\sqrt{n}} + \frac{b\sqrt{n}}{2} (x - x_0)^2,
\]

where the parameters $a$ and $b$ are chosen such that the graph of $\tilde{q}_n$ is tangent
to the two linear pieces of \( \tilde{f}_n \) adjacent to \( x = x_0 \), and such that \( \sqrt{n} B_n(\tilde{q}_n; x_0) \) is minimal.

One finds

\[
\tilde{q}_n(x) = \frac{1}{2} \left\{ \frac{\sqrt{x_0(1-x_0)}}{n} \right\} + \frac{\sqrt{x_0(1-x_0)}}{x_0(1-x_0)} (x - x_0)^2 \right\} ;
\]

the graph of this function is slightly steeper than that of \( q_n \). As \( \tilde{q}_n(x) \geq \tilde{f}_n(x) \) for all \( x \in [0,1] \), one derives in a similar way as in the proof of theorem 6.1

\[
(6.5) \quad c_n(x_0) = \sqrt{n} B_n(\tilde{f}_n; x_0) \leq \sqrt{n} B_n(\tilde{q}_n; x_0) = \frac{\sqrt{x_0(1-x_0)}}{2} \left\{ \frac{1}{2} x_0(1-x_0) \right\} (x_0 \neq \frac{1}{2}).
\]

We note that the functions \( q_n \) and \( \tilde{q}_n \) are identical if \( x_0 = \frac{1}{2} \).

**Corollary 6.1.** For all \( x \in [0,1] \) and all \( n \in \mathbb{N} \)

\[
\frac{1}{4} x (1-x) \leq c_n(x) \leq \frac{1}{4} x (1-x).
\]

**Proof.** This is an immediate consequence of (6.4) and (6.5).

**Corollary 6.2.** If \( 0 \leq x \leq 0.2517 \) or \( 0.7483 \leq x \leq 1 \), then

\[
(6.6) \quad c_n(x) < c_5 = \frac{2\sqrt{5} - 1}{16} = 0.217008.
\]

**Proof.** Using (5.5) the inequality in (6.6) easily follows from (6.5).

**Remark 6.3.** We note that corollaries 6.1 and 6.2 are of some relevance for the numerical investigation of \( \max_{x \in [0,1]} c_n(x) \): small values of \( x \) need not be taken into consideration. For instance, when \( n = 5 \) the first three cases of p. 22 can be disposed of immediately.

7. **Determination of \( c(2) \)**

Having the extremal functions available, it is a comparatively simple matter to obtain the best constant in Lorentz' theorem 1.1, when \( n \) runs through the set of all positive integers. This was done in the preceding section.

The simplicity of this problem is mainly due to the fact that \( \sup_{n \geq 1} c_n = c_1 \).
and also to the fact that estimate (6.3) becomes an equality if \( n = 1 \).

Thus, case \( n = 1 \) can be regarded as rather special, and it seems natural to ask for \( c^{(2)} = \sup_{n \geq 2} c_n \), \( c_n \) being defined as in (1.9). This question will be answered in the present section. We recall that in section 5, formula (5.2), we established that

\[
(7.1) \quad c_n(x_0) = \sqrt{n} B_n(\tilde{f}_n;x_0),
\]

where, according to theorem 4.1, for all \( x \)

\[
\tilde{f}_n(x) = \frac{1}{n} |x - x_0| + q_n(x),
\]

with

\[
q_n(x) = \sum_{j=1}^{\infty} \left( |x - x_0| - \frac{j}{\sqrt{n}} \right).\]

In what follows we shall obtain upper bounds for \( R_n(x_0) = \sqrt{n} B_n(q_n;x_0) \), which, together with lemma 2.2 and some numerical results, yield one of the main theorems of this report. We have

**Theorem 7.1.**

\[
c^{(2)} := \sup_{n \geq 2} \sup_{f \in C[0,1]} \frac{\sqrt{n} \max_{0 \leq x \leq 1} |B_n(f;x) - f(x)|}{\omega_1(f;\frac{1}{\sqrt{n}})} = c_5 = \frac{2\sqrt{5} - 1}{16} = 0.217008497.
\]

**Proof.** In order to prove this theorem we use (7.1) and we write, replacing \( x \) by \( x_0 \) in (5.4),

\[
(7.2) \quad c_n(x_0) = S_n(x_0) + R_n(x_0).
\]

In lemma 2.2 it was proved that \( S_n(x_0) \) has on \([0, \frac{1}{2}]\) a unique absolute maximum, denoted by \( \|S_n\| \). For \( 1 \leq n \leq 30 \), the values of \( \|S_n\| \) are given in table 2.1, p. 9. We now proceed to obtain upper bounds \( R_n^* \) for \( R_n(x_0) \). To this end we approximate \( q_n \) by polynomials, \( P_{n,s} \), of the form

\[
P_{n,s}(x) = a_{n,s}(x - x_0)^{2s} \quad (s = 1, 2, 3, \ldots).
\]

These polynomials are chosen in such a way that the graph of \( P_{n,s} \) touches the (non-horizontal) linear pieces of the graph of \( q_n \), nearest to \( x_0 \) (see figure 7.1).
One finds

\begin{equation}
\phi_{n,s}(x) = \frac{(2s - 1)^{2s-1}}{(2s)^{2s}} n^{s-\frac{1}{2}} (x - x_0)^{2s}.
\end{equation}

It is easily verified that \( \phi_{n,s}(x) \geq Q_n(x) \) for all \( x \). Taking into account the positivity of the Bernstein operator \( B_n \), we have the following upper bounds for \( R_n(x_0) \).

\begin{equation}
R_n(x_0) \leq \sqrt{n} B_n (\phi_{n,s}; x_0) = \frac{(2s - 1)^{2s-1}}{(2s)^{2s}} n^{\frac{s-1}{2}} B_n ((x - x_0)^{2s}; x_0) \quad (s = 1, 2, \ldots).
\end{equation}

The best bound is obtained for \( s = 3 \), and using formulae (2.1) and (2.5) we get

\begin{equation}
R_n(x_0) \leq \frac{5}{6} \frac{5}{n^3} T_n(x_0) = \frac{5}{6^2} \left\{ 5X_0^3 + \frac{5}{n} X_0^2 (5 - 26X_0) + \frac{1}{n^2} X_0 (1 - 30X_0 + 120X_0^2) \right\},
\end{equation}

where \( X_0 = x_0 (1 - x_0) \).

As the last expression in (7.5) is increasing in \( X_0 \) for all \( n \geq 4 \), its maximum is attained at \( X_0 = \frac{1}{2} \), i.e. at \( x_0 = \frac{1}{2} \). It follows that

\begin{equation}
R_n(x_0) \leq R_n^* := \frac{5}{2^3} \frac{5}{3^2} \left\{ 1 - \frac{2}{n} + \frac{16}{15n^2} \right\} < 0.015699 \quad (n \geq 4).
\end{equation}

Taking into account formulae (7.1), (7.2), (7.4), to prove theorem 7.1 it is sufficient to show that for all \( n \neq 5 \) we have \( \| S_n \| + R_n^* \leq 0.217008 \), or, equivalently, that

![Figure 7.1.](image)
Theorem 5.1 takes care of the cases $n = 2, 3, 4$. In Table 7.1 the values of $\|S_n\|$ and $\alpha_n$ are given for $6 \leq n \leq 29$, and it turns out that inequality (7.7) does indeed hold for all these values, with the exception of $n = 7, 9, 11$.

$$\|S_n\| \leq 0.217008 - \frac{6}{2^m} \left( 1 - \frac{2}{n} + \frac{16}{15n^2} \right) =: \alpha_n.$$  

(7.7)

As $\|S_{28}\| < 0.217008 - 0.015699$ and $\|S_{29}\| < 0.217008 - 0.015699$, the values of $n \geq 30$ are taken care of by the monotonicity of $\|S_{2m}\|$ and $\|S_{2m+1}\|$, cf. (2.9). So, what remains to be done is a separate treatment of the cases $n = 7, 9, 11$.

$n = 7$. In order to show that $c_7(x) < c_5$ for all $x \in [0, 1]$, it is sufficient to restrict $x$ to the interval $[0.48, 0.50]$. This can be seen as follows. From (7.6) and Table 7.1 it follows that

$$R_7^x = 0.011555.$$  

(7.8)
The behaviour of the sum $S_7(x)$ can be dealt with by noting that it has a maximum $0.199588$ at $x = \frac{3}{8}$ and, moreover, that $S_7(x)$ is decreasing on $[\frac{3}{8}, \frac{3}{7}]$ and increasing on $[\frac{3}{7}, \frac{1}{2}]$ (cf. figure 7.2 and also (2.8)).

As $S_7(0.48) = 0.205380$, one thus has in view of lemma 2.2 and (7.8) that for all $x \in [0, 0.48]\n (7.9) \quad S_7(x) + R_7^* \leq 0.216935 < c_5 = 0.217008$.

To evaluate $c_7(x)$, $0.48 \leq x \leq 0.50$, we use formula (5.3). It is easily verified that for this range of $x$ one has

$$c_7(x) = 20\sqrt{7} x^4 (1-x)^4 + \sqrt{7}(1-x)^7(x-\frac{1}{\sqrt{7}}) + x^7(1-x-\frac{1}{\sqrt{7}}) \),
$$

which is maximal for $x = \frac{1}{4}$, with $c_7(\frac{1}{4}) = \frac{11\sqrt{7} - 2}{128} = 0.211744 < c_5$. This, together with (7.9), proves that $c_7 < c_5$.

$n = 9$. Similarly, restricting $x$ to $[\frac{4}{7}, \frac{1}{2}]$, we have in view of (5.3)

$$c_9(x) = 210x^5(1-x)^5 +$$

$$+ 3((1-x)^9(x-\frac{1}{3})+(1-x)^8(x-9/4)+x^8(1-x)(5-9x)+x^9(\frac{2}{3} - x)),
$$

which is again maximal for $x = \frac{1}{4}$, with $c_9(\frac{1}{4}) = \frac{109}{512} = 0.212891 < c_5$. This establishes that $c_9 < c_5$. 
n = 11. This case can be covered in an analogous way as n = 7, n = 9. However, the expression for \( c_{11}(x) \) as given by formula (5.3) becomes somewhat awkward to deal with, as there are contributions for \( k = 0,1,2,9,10,11 \). Case \( n = 11 \) can also be handled by improving slightly on the estimate (7.4).

Considering the difference \( P_{11,3}(x) - Q_{11}(x) \), we find, now restricting \( x \) to the interval \([0.49,0.50]\), that \( P_{11,3}(0) - Q_{11}(0) > 0.17 \), and \( P_{11,3}(1) - Q_{11}(1) > 0.20 \). It follows that the estimates (7.5) and (7.6) can be improved by

\[
\sqrt{11}(0.17(1-x)^{11} + 0.20x^{11}) > 0.000550 \quad (0.49 \leq x \leq 0.50).
\]

As \( S_{11}(\frac{1}{2}) = 0.204050 \) and \( R_{11}^* = 0.012982 \) (cf. table 7.1), this suffices to prove that \( c_{11} < c_{5} \). This concludes the proof of theorem 7.1.

Remark 7.1. We recall that by a considerable amount of computation we proved in section 5 that \( c_{5} = c_{5}(\frac{1}{2}) = \frac{2\sqrt{5} - 1}{16} \). Using the methods of this section this result can be deduced in a much easier way. In fact, in examining \( c_{5}(x) \) it is sufficient to restrict \( x \) to the interval \([0.46,0.50]\), as it is easily verified that one has

\[
S_{5}(x) \leq 0.205632 \quad (0 \leq x \leq 0.46),
\]

\[
R_{5}^* = 0.010089.
\]

Consequently,

(7.10) \( c_{5}(x) < 0.215721 \quad (0 \leq x \leq 0.46) \).

Using (5.3) and (2.7) we have on \([0.46,0.50]\)

\[
c_{5}(x) = 6\sqrt{5} x^3(1-x)^3 + \sqrt{5}\{(1-x-\frac{x}{5})x^5 + (x-\frac{x}{5})(1-x)^5\},
\]

which attains its maximum at \( x = \frac{1}{2} \), with \( c_{5}(\frac{1}{2}) = \frac{2\sqrt{5} - 1}{16} = 0.217008 \). Because of (7.10) it then follows that \( c_{5} = c_{5}(\frac{1}{2}) \).

Remark 7.2. It is perhaps appropriate to note that in dealing with the cases \( n = 7,9,11 \) as above, we have not shown that \( c_{7} = c_{7}(\frac{1}{2}), c_{9} = c_{9}(\frac{1}{2}), c_{11} = c_{11}(\frac{1}{2}) \), though this can be proved by carefully applying the method of section 5.
The limiting behaviour of $c_n(x)$

As we remarked in the introductory section 1, Esseen [1], complementing part of the work of Sikkema [10], determined the constant

$$
\lim_{n \to \infty} \sup_{f \in C[0,1]} \frac{\max_{0 \leq x \leq 1} |B_n(f;x) - f(x)|}{\omega(f; \frac{1}{\sqrt{n}})}
$$

whereas earlier Popoviciu [6] and then Sikkema [10] had given estimates for this quantity. In view of this it seems natural to put the analogous problem here, i.e. to ask for

$$
\lim_{n \to \infty} \sup_{f \in C[0,1]} \frac{\sqrt{n} \max_{0 \leq x \leq 1} |B_n(f;x) - f(x)|}{\omega(f; \frac{1}{\sqrt{n}})}
$$

This section will be concerned with this kind of problem. In fact, using the central limit theorem we shall prove the following result, which is of a more detailed character.

**Theorem 8.1.** For $c_n(x)$ defined as in (1.11), we have

$$
c(x) := \lim_{n \to \infty} c_n(x) = \sqrt{\frac{X}{2\pi}} + 2\sqrt{X} \sum_{j=1}^{\infty} \int_{j/\sqrt{X}}^{\infty} (u - j)^2 \varphi(u) du \quad (0 < x < 1),
$$

$$
\lim_{n \to \infty} \max_{x \in [0,1]} c_n(x) = c(\frac{1}{2}) = \frac{1}{2\sqrt{2\pi}} + \sum_{j=1}^{\infty} \int_{2j}^{\infty} (u - 2j) \varphi(u) du = 0.20796899,
$$

where

$$
\varphi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}, \quad X = x(1-x).
$$

In order to prove this theorem we need two lemmas.
Lemma 8.1. If \( U \) is a nonnegative random variable with distribution function \( F \), then for \( a \geq 0 \)
\[
E(U - a)_+ = \int_a^\infty (1 - F(u)) \, du ,
\]
where \( E \) denotes expectation.

**Proof.**
\[
E(U - a)_+ = \int_a^\infty (u - a) dF(u) ,
\]
and the assertion of the lemma follows on integration by parts. \( \blacksquare \)

Lemma 8.2. If \( V_n \) is a binomial random variable with expectation \( nx \) and variance \( nx \), and if we put \( U_n = \frac{V_n - nx}{\sqrt{nx}} \), then for the distribution function \( F_n \) of \( |U_n| \) one has for all \( u \geq 0 \) and all \( x \in (0,1) \)
\[
1 - F_n(u) \leq 2e^{-u^2 x(1-x)} .
\]

**Proof.** Following Lorentz ([5], pp. 18-19) and Rathore ([7], p. 123) one has
\[
\phi_n(v,x) := \sum_{k=0}^{n} \frac{p_{n,k}(x)e^{v(k-nx)}}{\sqrt{n!}} = (xe^{v(1-x)} + (1-x)e^{-vx})^n .
\]
Expanding \( xe^{v(1-x)} + (1-x)e^{-vx} \) in powers of \( v \), one obtains for \( |v| \leq \frac{3}{2} \) the inequality
\[
xe^{v(1-x)} + (1-x)e^{-vx} \leq 1 + \frac{v^2}{8}(1 + \frac{|v|}{3} + \frac{v^2}{12(1 - \frac{|v|}{3})}) \leq 1 + 0.221v^2 \leq 0.221v^2 .
\]
Defining
\[
\psi_n(v,x) = \sum_{k=0}^{n} p_{n,k}(x)e^{v|k-nx|} ,
\]
we therefore have
\[
\psi_n(v,x) \leq \phi_n(v,x) + \phi_n(-v,x) \leq 2e^{0.221n|v|^2} \quad (|v| \leq \frac{3}{2}) .
\]
Now, by a Chebyshev-type argument, one has
\[
\exp(v|k-nx|) \geq c_\psi_n(x) p_{n,k}(x) \leq \frac{1}{c} \quad (c > 0),
\]
and therefore, by inequality (8.5)
\[
\exp(v|k-nx|) \geq 2c \exp(0.221nv^2) \quad p_{n,k}(x) \leq \frac{1}{c}.
\]
Putting \( c = \frac{1}{e} \) and \( v = \frac{3}{2} \) we have, as \( \frac{9}{4} \cdot 0.221 = 0.49725 < \frac{1}{2} \),
\[
\sum_{|k-nx| \geq \delta n} p_{n,k}(x) \leq 2e^{-\frac{\delta^2}{2}},
\]
which yields (8.4) if we take \( \delta = \sqrt{n(1-x)} \). This proves the lemma. \( \blacksquare \)

Remark 8.1. Inequality (8.6) is contained in the "Stellingen" section of Van de Ven's dissertation ([12], stelling X).

Proof of theorem 8.1. Using the notation of lemma 8.1, we have in view of (5.3)
\[
c_n(x) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n} \frac{|k-nx| p_{n,k}(x) + \sqrt{n} \sum_{j=1}^{\infty} \sum_{k=0}^{n} (|k-nx| - \frac{j}{\sqrt{n}}) p_{n,k}(x)}{\sqrt{n}} =
\]
\[
(8.7) \quad \sqrt{n} \{ E(|U_n|) + \sum_{j=1}^{\infty} E(U_n - \frac{j}{\sqrt{n}}) \}.
\]
An application of lemma 8.1 yields
\[
E(|U_n|) = \int_0^\infty (1 - F_n(u)) du,
\]
\[
(8.8) \left\{ \begin{align*}
E(|U_n| - \frac{j}{\sqrt{n}}) + \int_{j/\sqrt{n}}^{\infty} (1 - F_n(u)) du
\end{align*} \right. \]
Introduce
\( (8.9) \quad \phi(u) = \int_{-\infty}^{u} \varphi(v) dv, \)

where \( \varphi(v) \) is given by (8.3). Then by the Berry-Esseen theorem (cf. [2], p. 542) \( 1 - F_{n}(u) \) tends to \( 2(1 - \phi(u)) \) as \( n \to \infty \), uniformly in \( u \) and uniformly in \( x \), with \( x \in [\delta, 1 - \delta] \), for any \( \delta > 0 \). By lemma 8.2 the integrals in (8.8) converge uniformly in \( n \), \( j \) and \( x \in [\delta, 1 - \delta] \). It follows that

\[
E(|U_n|) = 2 \int_0^\infty (1 - \phi(u)) du = 2 \int_0^\infty u \varphi(u) du = \frac{2}{\sqrt{2\pi}},
\]

uniformly in \( x \in [\delta, 1 - \delta] \), and that

\[
E\left(|U_n| - \frac{1}{\sqrt{x}}\right) = 2 \int_j^\infty (1 - \phi(u)) du = 2 \int_j^\infty \frac{(u - \frac{1}{\sqrt{x}}) \varphi(u) du}{\sqrt{x}},
\]

uniformly in \( j \) and \( x \in [\delta, 1 - \delta] \).

As, also by lemma 8.2, the sums in (8.7) converge uniformly in \( n \) and
\( x \in [\delta, 1 - \delta] \), it follows that \( c_n(x) \to c(x) \) for all \( x \), and uniformly for \( x \in [\delta, 1 - \delta] \) for any \( \delta > 0 \). This proves (8.1).

In order to prove (8.2), we note that from \( c_n(x) \leq \frac{1}{\sqrt{x(1-x)}} \) (cf. (6.5)) it easily follows that \( \max_{x} c_n(x) = c_n(x_n) \), with \( x_n \) bounded away from 0 and 1.

As \( c_n(x) \to c(x) \), uniformly in \( x \), and as \( \max_{x} c(x) = c(\frac{1}{2}) \), it follows that

\[
\lim_{n \to \infty} \max_{x} c_n(x) = c(\frac{1}{2}),
\]

because for large \( n \) and arbitrary \( \varepsilon > 0 \) we have

\[
c_n(x_n) \geq c_n(\frac{1}{2}) \geq c(\frac{1}{2}) - \varepsilon,
\]

whereas on the other hand

\[
c_n(x_n) = c_n(x_n) - c(x_n) + c(x_n) - c(\frac{1}{2}) + c(\frac{1}{2}) \leq \varepsilon + c(\frac{1}{2}).
\]

Remark 8.2. The expression for \( c(\frac{1}{2}) \) occurring in (8.2) can be rewritten as

\[
c(\frac{1}{2}) = \frac{1}{2\sqrt{2\pi}} + \sum_{j=1}^{\infty} e^{-2j^2} - 2 \sum_{j=1}^{\infty} j(1 - \phi(2j)),
\]

where \( \phi \) is defined by (8.9). This formula has been used to compute \( c(\frac{1}{2}) \).
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Table 8.1.
We conclude this section with table 8.1, containing the numerical values of
the coefficients \( c_n = \max_{x \in [0,1]} c_n(x) \), and the points where these maxima are
attained, for \( n = 1, 2, \ldots, 100, 1000, 1001 \). These data were computed on the
Burroughs 6700 of the Computing Centre of the Eindhoven University of Tech­
nology. In computing these numbers use was made of formulae (4.12) and (5.3).
Taking into account theorem 5.1 (where it was proved that \( c_1 = c_1(\frac{1}{2}) \),
\( c_3 = c_3(\frac{1}{3}) \), \( c_5 = c_5(\frac{1}{5}) \)), remark 7.2 (containing the assertion that \( c_7 = c_7(\frac{1}{7}) \),
\( c_9 = c_9(\frac{1}{9}) \), \( c_{11} = c_{11}(\frac{1}{11}) \)), and examining the first part of the table, one is
led to the conjecture that if \( n \) is odd, \( c_n = c_n(\frac{1}{n}) \). For \( n = 57 \), however, the
computer indicates that \( c_{57} > c_{57}(\frac{1}{3}) \). A similar phenomenon takes place for
\( n = 73 \) and \( n = 91 \).

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The authors are indebted for encouragement to Prof.dr. P.C. Sikkema, Delft
University of Technology, whose papers [10], [11] stimulated the research
carried out in this report. We also acknowledge that he, as early as 1960,
conjectured that \( \sup c_n = c_1 = \frac{1}{3} \) (unpublished note). In that year he commu­
nicated this conjecture at a meeting of the GAMM-Tagung in Hannover.
We are particularly grateful to L.G.F.C. van Bree in the mathematics depart­
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sults. Finally, we wish to thank Drs. H.G. ter Morsche, also in the mathema­
tics department, for reading part of the manuscript and for useful comments.

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