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Dynamic disturbance decoupling of nonlinear systems and linearization*

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Abstract

In this paper we investigate the connections between the solvability of the dynamic disturbance decoupling problem with exponential stability (DDDPes) for a nonlinear system and the solvability of the same problem for its linearization around an equilibrium point. It is shown that under generic conditions the nonlinear DDDPes is solvable for a nonlinear system if and only if the static disturbance decoupling problem with stability (DDPs) is solvable for its linearization around an equilibrium point.

Keywords: nonlinear control systems, disturbance decoupling, dynamic feedback, linearization.


1 Introduction

As in linear geometric control theory (see for instance [22]), one of the first structural synthesis problems that has been posed and to some extent (i.e., in a local fashion) has been solved in a nonlinear context is the so called disturbance decoupling problem (DDP). Recall that the DDP may be defined as follows. Consider a nonlinear system \( \Sigma_q \) of the form

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u + p(x)q \\
y &= h(x)
\end{align*}
\]

where \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \) are local coordinates for the state space manifold \( \mathcal{X} \), \( u \in \mathbb{R}^m \) denotes the controls, \( q \in \mathbb{R}^r \) the disturbances and \( y \in \mathbb{R}^p \) the outputs. Also,
let \( g_1, \ldots, g_m \) denote the columns of the matrix \( g \) and similarly \( p_1, \ldots, p_r \) the columns of the matrix \( p \). All data in (1), i.e., the vector fields \( f, g_1, \ldots, g_m, p_1, \ldots, p_r \) as well as the function \( h \) will be assumed to be analytic throughout the paper. In the DDP one is asked to find, if possible, a static state feedback

\[
Q_s : \quad u = \alpha(x) + \beta(x)v
\]

with \( \alpha(x) \) and \( \beta(x) \) respectively an \( m \)-vector and an \((m, m)\)-matrix depending analytically on \( x \) and with \( v \in \mathbb{R}^m \) denoting a new control vector, such that in the closed loop system \( \Sigma_q \circ Q_s \) given by

\[
\left\{
\begin{array}{ll}
\dot{x} &= f(x) + g(x) \alpha(x) + g(x) \beta(x)v + p(x)q \\
y &= h(x)
\end{array}
\right.
\]

the output is not affected by the disturbances \( q \), no matter how the new control \( v \) is chosen. Usually the DDP is considered under the additional assumption that in the static state feedback law (2) the matrix \( \beta(x) \) is nonsingular for all \( x \). This in order to keep as much control to the system as possible, while achieving disturbance decoupling at the same time. The (local) solution of the DDP can be found in [14],[15]. This solution to the DDP very much resembles its linear analogon.

Another disturbance decoupling problem can be posed if one allows for dynamic state feedbacks instead of static state feedbacks (2) (see [10],[11],[16]). This gives rise to the so called dynamic disturbance decoupling problem (DDDP) in which a dynamic state feedback

\[
Q_d \left\{
\begin{array}{ll}
\dot{z} &= \alpha(x, z) + \beta(x, z)v \\
u &= \gamma(x, z) + \delta(x, z)v
\end{array}
\right.
\]

with compensator state \( z \in \mathbb{R}^n \) and control \( v \in \mathbb{R}^m \) is sought, such that in the closed loop system \( \Sigma_q \circ Q_d \) given by

\[
\Sigma_q \circ Q_d \left\{
\begin{array}{ll}
\dot{x} &= f(x) + g(x) \gamma(x, z) + g(x) \delta(x, z)v + p(x)q \\
\dot{z} &= \alpha(x, z) + \beta(x, z)v \\
y &= h(x)
\end{array}
\right.
\]

the disturbances do not influence the outputs, no matter how the new controls \( v \) are chosen. Under the assumption that \( Q_d \) is regular, which roughly says that the relation between the old controls \( u \) and the new controls \( v \) is invertible (the exact definition of a regular dynamic state feedback is postponed till Section 2.1), a local solution of the DDDP has been obtained in [10],[11]. The remarkable observation about the DDDP is that for linear systems the DDDP is solvable if and only if the DDP is, a conclusion which in the nonlinear case no longer turns out to be true.

Nonregular solutions to the DDDP, i.e., solutions in which (4) is not necessarily a regular dynamic state feedback, are described in [8].
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The purpose of this paper is twofold. First we try to relate the solvability of the DDDP for a nonlinear system around an equilibrium point $x_0 \in X$ to the solvability of the (D)DDP for the linearized system (assuming that $h(x_0) = 0$)

$$L \Sigma_q \left\{ \begin{array}{l}
\xi = F\xi + G\eta + Pq \\
\eta = H\xi
\end{array} \right. \quad (6)$$

where, in order not to introduce too many new variables we denote the controls and disturbances by $u$ and $q$ again, and where

$$F = \left( \frac{\partial f}{\partial x} \right)(x_0), \ G = g(x_0), \ P = p(x_0), \ H = \left( \frac{\partial h}{\partial x} \right)(x_0)$$

An engineering paradigm would be to approximately solve the DDDP for the system (1) around $x_0$ by addressing the same problem for its linearization (6) and, once a linear solution of the DDDP has been obtained for (6), to consider this as a first order approximate solution of the DDDP for (1). Recently we have demonstrated that indeed for a large (generic) class of systems this approach makes sense, see [9]. In that paper we have rigorously shown that by allowing for linear regular dynamic state feedbacks (4) in the solution of the disturbance decoupling problem for the linear system (6), we obtain first order approximate solutions for the DDDP for (1), provided the (D)DDP for (6) is solvable. The solution of the DDDP in this case uses particular dynamic state feedbacks, that we call Singh compensators (following the work of [20]). A particular feature of such a Singh compensator is that stepwise some of the input channels are integrated and the other input channels are only changed by a static feedback loop at each step.

So far, in the literature the analysis of the relation between the nonlinear DDDP and the DDP of the linearization has been done without taking into account the important requirement of closed loop stability (if the new controls $v$ and unknown disturbances $q$ are all set equal to zero). For linear systems, this combined problem, the so called disturbance decoupling problem with stability (DDPs) has been solved completely (cf. [22],[1]). In the nonlinear context the combination of the DDP with (local) stabilizability is a more difficult problem that has been studied via different methods. One method is based on the notion of the system being minimum phase (see [3]) which generalizes the linear concept of minimum phase-ness to nonlinear systems. Another method in the analysis of the nonlinear DDPs is essentially based upon deriving sufficient conditions for the solvability of the DDPs for (1) via the solvability of the problem for its linearization (6), see [21] for an extensive discussion. From the above reference it follows also that the DDDP with exponential stability (DDDPes) can be studied in a similar way, based on the extended system (5). Clearly, this analysis has the drawback that it will depend on the choice of the compensator $Q_d$ in (5). For this reason we will develop here, as the second purpose of the paper, results on the solvability of the DDDPes for (1) in relation to the solvability of the (D)DDPes for its linearization (6), herewith avoiding the forementioned drawback in
Roughly stated, our main result says that, for "generic" systems, the DDDPes is locally solvable if and only if the DDPs is solvable for its linearization.

The organization of the paper is as follows. In Section 2 we first introduce the disturbance decoupling problem via regular dynamic state feedback (DDDP) and present a solution to this problem, following [10],[11]. Then we proceed by investigating the connection between the solvability of this problem for a given nonlinear control system and the solvability of the same problem for the linearization of the system around an equilibrium point. The results in this section are from [9]. In Section 3 we investigate the same problems with the extra requirement of exponential stability of the closed loop system. Section 4 contains an example. In Section 5 we draw some conclusions.

2 The dynamic disturbance decoupling problem and linearization

In this section we briefly discuss the dynamic disturbance decoupling problem (DDDP) and its linearization. Throughout we restrict ourselves to square systems of full rank, i.e., systems with as many controls as outputs which are left- and right-invertible, cf. [14],[15]. The non-square and non-invertible case can be treated in the same way, see e.g. [12]. In order to possibly solve the DDDP we introduce in Section 2.1 a particular class of dynamic state feedbacks, the so called Singh compensators. These dynamic state feedbacks have the property that for a full rank system the resulting compensated system will have an invertible decoupling matrix. This means that by applying a Singh compensator such a system can be input-output decoupled. Moreover, among those dynamic state feedbacks that have this property, the Singh compensator is one of minimal dimension (i.e., $\nu$ in (4) is as small as possible, cf. ([12]). In Section 2.2, we formulate the DDDP and present a solution to this problem. In Section 2.3 we make the correspondence with the DDP of the associated linearization of the system.

2.1 Singh compensators

We start with some algebraic concepts that were introduced in [4]. Consider the nonlinear system $\Sigma_0$, which is the system derived from $\Sigma$ by setting $q \equiv 0$. Recall that a meromorphic function $\eta$ is a function of the form $\eta = \pi/\theta$, where $\pi$ and $\theta$ are analytic functions. Assume that the control functions $u(t)$ are $n$ times continuously differentiable. Then define $u^{(0)} := u$, $u^{(i+1)} := (d/dt)u^{(i)}$. View $x, u, \ldots, u^{(n-1)}$ as variables and let $K$ denote the field consisting of the set of rational functions of $(u, \ldots, u^{(n-1)})$ with coefficients that are meromorphic in $x$. Note that $y, \ldots, y^{(n)}$ have components in the field $K$. Let $E$ denote the vector space over $K$ spanned by $\{dx, dy, \ldots, dy^{(n)}\}$. Define subspaces $E_0, \ldots, E_n$ of $E$ by

$$E_k = \text{span}_K \{dx, dy, \ldots, dy^{(k)}\}$$

(7)
Dynamic disturbance decoupling and linearization

Then the rank \( \rho^*(\Sigma_0) \) of \( \Sigma_0 \) is defined by (see [4])

\[
\rho^*(\Sigma_0) = \dim_K \mathcal{E}_n - \dim_K \mathcal{E}_{n-1}
\] (8)

Note that we always have \( \rho^*(\Sigma_0) \leq m \). \( \Sigma_0 \) is said to be of full rank if \( \rho^*(\Sigma_0) = m \).

Now consider a dynamic state feedback \( Q_d \) for \( \Sigma_q \) of the form (4). It is said to be a regular dynamic state feedback for \( \Sigma_q \) if the system

\[
\begin{align*}
x &= f(x) + g(x)\gamma(x, z) + g(x)\delta(x, z)v \\
\dot{z} &= \alpha(x, z) + \beta(x, z)v \\
u &= \gamma(x, z) + \delta(x, z)v
\end{align*}
\] (9)

with controls \( v \) and outputs \( u \) has full rank (see [4]).

The Singh compensator is obtained in the following way. For \( r, s \in \mathbb{N} \), introduce the notation \( I_{r,s} := \{r, \cdots, s\} \). Using e.g. Singh's algorithm ([20],[4]), we can then find a permutation of the outputs and positive integers \( \gamma_1, \cdots, \gamma_m \) satisfying \( \gamma_1 \leq \cdots \leq \gamma_m \leq n \), such that for \( k = 0, \cdots, n \)

\[
\{dx, \{dy^{(j)} \mid \gamma_i \leq k, j \in I_{nk}\}\}
\] (10)

forms a basis for \( \mathcal{E}_k \). Denoting \( \tilde{y}_k = \text{col}(y_i \mid \gamma_i = k) \), \( \hat{y}_k = \text{col}(y_i \mid \gamma_i > k) \), this means that we may write for \( k = 1, \cdots, n \)

\[
\begin{align*}
\tilde{y}_k^{(k)} &= \tilde{a}_k(x, \{\tilde{y}_i^{(j)} \mid i \in I_{1k-1}, j \in I_{ik}\}) + \\
\hat{b}_k(x, \{\hat{y}_i^{(j)} \mid i \in I_{1k-1}, j \in I_{ik-1}\})u
\end{align*}
\] (11)

\[
\begin{align*}
\hat{y}_k^{(k)} &= \hat{y}_k^{(k)}(x, \{\hat{y}_i^{(j)} \mid i \in I_{1k}, j \in I_{ik}\})
\end{align*}
\]

where the matrices \( \tilde{B}_k := (\tilde{b}_k^T \cdots \hat{b}_k^T)^T \) have full row rank over \( K \) (cf. [4],[12]). Define \( \hat{Y}_n := (\hat{y}_1^T \cdots \hat{y}_n^T)^T \) and \( \hat{A} := (\tilde{a}_1^T \cdots \hat{a}_n^T)^T \). Then (11) yields in particular:

\[
\begin{align*}
\hat{Y}_n &= \hat{A}_n(x, \{\hat{y}_i^{(j)} \mid i \in I_{1n-1}, j \in I_{in}\}) + \\
\hat{B}_n(x, \{\hat{y}_i^{(j)} \mid i \in I_{1n-1}, j \in I_{in-1}\})u
\end{align*}
\] (12)

We call a pair \( (x, y) = (x_0, 0) \) a strongly regular point for \( \Sigma_q \) if for every possible permutation of the outputs as described above, the matrix \( \hat{B}_n \) in (12) has full row rank over \( \mathbb{R} \), when evaluated at \( (x_0, 0) \). If the pair \( (x_0, 0) \) is a strongly regular point for \( \Sigma_q \), we know that for (12) there exists a neighborhood \( U \subset X \) of \( x_0 \) and a neighborhood \( Y_0 \subset Y_{nm} \) of \( (\hat{y}_i^{(j)} \mid i \in I_{1n-1}, j \in I_{in-1}) = 0 \) such that \( \hat{B}_n \) is invertible on \( U \times Y_0 \). Then on \( U \times Y_0 \) we obtain from (12):

\[
u = \hat{B}_n^{-1}[\hat{Y}_n - \hat{A}_n]
\] (13)
Clearly, \( \gamma_i \) is the lowest time-derivative of \( y_i \) appearing in the right hand side of (13). Let \( \delta_i \) be the highest time-derivative of \( y_i \) appearing in the right hand side of (13). It can be shown that the integers \( \delta_i \) and \( \sum_{i=1}^{m} \gamma_i \) are intrinsic, i.e., independent of the permutation of the outputs that is chosen (cf. [12]). In fact, the \( \delta_i \) are just the essential orders ([5]) of \( \Sigma_0 \). Hence the integer \( \sigma := \sum_{i=1}^{m} (\delta_i - \gamma_i) \) is an intrinsic number too. Moreover, the right hand side of (13) is affine in \( y_i^{(\delta_i)} \) and we may rewrite it as

\[
u = \phi_1(x, \{y_i^{(j)} : i \in \mathcal{I}_{1m}, j \in \mathcal{I}_{n,\delta_i-1}\}) + \sum_{i=1}^{m} \phi_{2i}(x, \{y_i^{(j)} : i \in \mathcal{I}_{1m}, j \in \mathcal{I}_{n,\delta_i-1}\})y_i^{(\delta_i)}\]

for certain vector-valued functions \( \phi_1, \phi_{2i} \) \( (i = 1, \ldots, m) \). Let \( z_i \) \( (i = 1, \ldots, m) \) be a vector of dimension \( \delta_i - \gamma_i \) and consider the system:

\[
\begin{align*}
\dot{z}_i &= A_iz_i + B_iv_i \quad (i = 1, \ldots, m) \\
u &= \phi_1(x, z_1, \ldots, z_m) + \sum_{i=1}^{m} \phi_{2i}(x, z_1, \ldots, z_m)v_i
\end{align*}
\]

(14)

with \( (A_i, B_i) \) in Brunovsky canonical form. Then (14) is called a Singh compensator for \( \Sigma_q \) around \( x_0 \).

The Singh compensator has the following properties (see [7],[11],[12],[13]).

**Proposition 2.1** Consider the nonlinear system \( \Sigma_0 \) and assume that \( \rho^*(\Sigma_0) = m \). Let \( x_0 \in X \) be an equilibrium point of \( \Sigma_0 \). Assume that \( h(x_0) = 0 \) and that \( (x, y) = (x_0, 0) \) is a strongly regular point for \( \Sigma_0 \). Let \( Q \) be a Singh compensator for \( \Sigma_0 \). Then

(i) \( Q \) is a regular dynamic state feedback for \( \Sigma_0 \).

(ii) \( Q \) is a minimal order input-output decoupling compensator for \( \Sigma_0 \), i.e., the dimension of the compensator state of \( Q \) is minimal among those compensators that achieve input-output decoupling for \( \Sigma_0 \).

(iii) a. The point \( (x, z) = (x_0, 0) \) is an equilibrium point for \( \Sigma_0 \circ Q \).

b. Denote by \( L(\Sigma_0 \circ Q) \) the linearization of \( \Sigma_0 \circ Q \) around \( (x_0, 0) \). Then \( L(\Sigma_0 \circ Q) = L\Sigma_0 \circ LQ \), where \( LQ \), the linearization of \( Q \) around \( (x_0, 0) \), is a Singh compensator for \( L\Sigma_0 \).

c. Conversely, if \( R \) is a Singh compensator for \( L\Sigma_0 \), then there is a Singh compensator \( Q \) for \( \Sigma_0 \) such that \( LQ = R \) and \( L(\Sigma_0 \circ Q) = L\Sigma_0 \circ R \).

In the sequel a dynamic state feedback (4) will be called a Singh compensator for \( \Sigma_q \) if it is a Singh compensator for \( \Sigma_0 \).
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2.2 The dynamic disturbance decoupling problem

The DDDP is defined as:

**Definition 2.2** Disturbance decoupling problem via regular dynamic state feedback (DDDP) Consider the nonlinear system $\Sigma_q$ and let a point $x_0 \in \mathcal{X}$ be given. The DDDP is said to be locally solvable around $x_0$ if there exist a regular dynamic state feedback $Q_d$ for $\Sigma_q$ of the form (4), a neighborhood $U \subset \mathcal{X}$ of $x_0$ and an open subset $Z \subset \mathbb{R}^v$ such that the outputs of the composed system $\Sigma_q \circ Q_d$ restricted to $U \times Z$ are independent of the disturbances.

The following theorem, which can be found in [10],[11], gives a local solution of the DDDP. In the statement of the theorem we use that for $\Sigma_0$, the $y_k^{(k)} (k = 0, \ldots, n; y_0 = y)$ can be viewed as functions on $\mathcal{X}_e := \mathcal{X} \times \mathbb{R}^v$ and so $\text{Ker} y_k^{(k)} (k = 0, \ldots, n)$ defines a distribution on $\mathcal{X}_e$. Let $\mathcal{G}, \mathcal{P}$ denote the distributions spanned by the control vector fields and disturbance vector fields of $\Sigma_q$ respectively. Define the distributions $\mathcal{G}_e, \mathcal{P}_e$ on $\mathcal{X}_e$ by $\mathcal{G}_e := \mathcal{G} \times \{0\}, \mathcal{P}_e := \mathcal{P} \times \{0\}$. For a particular permutation of the outputs of $\Sigma_0$ (as described in Subsection 2.1), define

$$\tilde{\Delta}_e := \bigcap_{k=0}^n \text{Ker} y_k^{(k)}$$

(15)

**Theorem 2.3** Consider the square nonlinear system $\Sigma_q$ and let $x_0 \in \mathcal{X}$ be such that $(x_0, 0)$ is a strongly regular point for $\Sigma_q$. Then the DDDP is locally solvable around $x_0$ if and only if for every permutation of the outputs for $\Sigma_0$ (as described in Subsection 2.1) we have

$$\mathcal{P}_e \subset \tilde{\Delta}_e$$

Moreover, if the DDDP is locally solvable around $x_0$, every Singh compensator for $\Sigma_q$ around $x_0$ solves the DDDP for $\Sigma_q$.

**Remark 2.4** Another way of solving the DDDP can be found in [16].

2.3 The DDDP and its linearization

Next we relate the DDDP for $\Sigma_q$ with the analogous problem for $L\Sigma_q$, the linearization of $\Sigma_q$ around an equilibrium point. Here we follow [9]. So we consider an equilibrium point $x_0 \in \mathcal{X}$ of $\Sigma_q$, satisfying $h(x_0) = 0$, and the linearization $L\Sigma_q$ of $\Sigma_q$ around $x_0$. The following assumption is made.

**Assumption 2.5** (i) $(x, y) = (x_0, 0)$ is a strongly regular point for $\Sigma_q$. 
For every permutation of the outputs of $\Sigma_0$ as described in Subsection 2.1, $P_e$ and $\bar{\Delta} \cap P_e$ have constant dimension on a neighborhood of $(x_0, 0)$ in $X_e$.

Then the following result, relating the DDDP around $x_0$ for $\Sigma_q$ and for $L\Sigma_q$, is proved in [9].

**Theorem 2.6** Consider the square nonlinear system $\Sigma_q$, and assume that it has full rank, i.e., $\rho^*(\Sigma_0) = m$. Let $x_0 \in \mathbb{R}^m$ be an equilibrium point of $\Sigma_q$ with $h(x_0) = 0$ and suppose that Assumption 2.5 is satisfied. Then the DDDP for $\Sigma_q$ is locally solvable around $x_0$ if and only if it is solvable for $L\Sigma_q$.

**Proof** *(necessity)* Assume that the DDDP for $\Sigma_q$ is locally solvable around $x_0$. Let $Q$ be a Singh compensator that solves the problem around $x_0$. It may be checked (see e.g. [12]) that the decoupling matrix ([14],[15]) is the $(m,m)$-identity matrix. This implies (cf. [6])

$$\Delta_e^*(x_0,0) = V_e^*$$

(17)

where $\Delta_e^*$ is the maximal locally controlled invariant distribution in $\text{Ker} dh$ for $\Sigma_q \circ Q$ and $V_e^*$ is the maximal controlled invariant subspace in $\text{Ker} \left( \begin{array}{c} H \end{array} \right)$ for $L(\Sigma_q \circ Q)$. Hence we have, since the DDP is solvable for $\Sigma_q \circ Q$:

$$\text{Im} \left( \begin{array}{c} P \\ 0 \end{array} \right) = P_e(x_0,0) \subset \Delta_e^*(x_0,0) = V_e^*$$

(18)

and thus the DDP is solvable for $L(\Sigma_q \circ Q) = L\Sigma_q \circ LQ$. This implies that the DDP is solvable for $L\Sigma_q$.

*(sufficiency)* Assume that Assumption 2.5 holds and that the DDDP is solvable for $L\Sigma_q$ via a Singh compensator $R$. By Proposition 2.1, there is a Singh compensator $Q$ for $\Sigma_q$ such that $(x_0,0)$ is an equilibrium point of $\Sigma_q \circ Q$ and such that $L(\Sigma_q \circ Q)$, the linearization of $\Sigma_q \circ Q$ around $(x_0,0)$, satisfies $L(\Sigma_q \circ Q) = L\Sigma_q \circ R$. As above, we have that the decoupling matrix of $\Sigma_q \circ Q$ is the $(m,m)$-identity matrix. Then it follows (cf. [14],[15]) that $\Delta_e^*$, the maximal locally controlled invariant distribution in $\text{Ker} dh$ for $\Sigma_q \circ Q$, is given by

$$\Delta_e^* = \bar{\Delta} \cap \text{Ker} dz$$

(19)

and hence by Assumption 2.5, $\Delta_e^*$ has constant dimension around $(x_0,0)$. Let $V_e^*$ be the maximal controlled invariant subspace in $\text{Ker} \left( \begin{array}{c} H \end{array} \right)$ for $L\Sigma_q \circ R$. Then we obtain analogously to (18) that

$$P_e(x_0,0) \subset \Delta_e^*(x_0,0)$$

(20)

Since Assumption 2.5 holds, this implies that in fact we have $P_e \subset \Delta_e^*$ around $(x_0,0)$. Hence the DDP is solvable for $\Sigma_q \circ Q$ and thus the DDDP is solvable for $\Sigma_q$. 

\[\blacksquare\]
The result of Theorem 2.6 has the following practical implications. Besides tackling a synthesis problem via linearization, it is common practice in engineering to add integral actions in order to achieve better disturbance attenuation. From Theorem 2.6 it follows that if Assumption 2.5 holds, the solvability of the DDP for $L \Sigma_q$ implies solvability of the DDDP for $\Sigma_q$, but not necessarily solvability of the DDP for $\Sigma_q$ (for a counter example, see [9] or Section 4). If indeed the DDDP, but not the DDP, is solvable for $\Sigma_q$, no static state feedback that solves the DDP for $L \Sigma_q$ will be a first order approximation of a feedback that solves the DDDP for $\Sigma_q$. As a result of this, such a static state feedback will in general not result in a satisfactory disturbance attenuation when applied to $\Sigma_q$. At the same time the remedy is clear: one should look for a dynamic state feedback that solves the DDDP for $L \Sigma_q$ and that at the same time is the linearization of a dynamic state feedback that solves the DDDP for $\Sigma_q$. By Proposition 2.1 and Theorem 2.6, any Singh compensator for $L \Sigma_q$ will do this job (provided the DDP is solvable for $L \Sigma_q$). In other words, one should incorporate integral action to some of the controls to achieve better disturbance attenuation. In this respect one could view the result of Theorem 2.6 as a partial interpretation of introducing integral action in classical PID-control applied to nonlinear systems. A simulation example that illustrates the difference in using a static versus a dynamic linear feedback as an approximate linear solution for the DDDP is given in [9].

3 The DDDP with (exponential) stability

The DDDP with stability that we will discuss next is defined as:

**Definition 3.1** Disturbance decoupling problem with local stability via regular dynamic state feedback (DDDPs) Consider the nonlinear system $\Sigma_q$ and let a point $x_0 \in \mathcal{X}$ be given. The DDDPs is said to be locally solvable around $x_0$ if there exists a regular dynamic state feedback $Q_d$ that locally solves the DDDP around $x_0$ and in addition the closed loop system $\Sigma_q \circ Q_d$ has $(x, z) = (x_0, 0)$ as a locally asymptotically stable equilibrium point (when setting $v \equiv 0$ and $q \equiv 0$ in (5)).

In the following we will deal with a slightly stronger notion of stability, namely exponential stability. Correspondingly, we will use the abbreviation DDDPes. The reasons to work with exponential stability are in fact twofold. First, in most circumstances the new controls $v$ as well as the unknown disturbances may not equal zero and thus the closed loop asymptotic stability for $v \equiv 0$ and $q \equiv 0$ may not guarantee a nice behavior, as e.g. bounded input-bounded state stability, when $v$ and/or $q$ do not vanish. The second motivation to look at exponential stability is that we want to exploit optimally the relations between the DDDP for (1) around $x_0$ and the DDDP for its linearization (6).

The main result we will establish is that under generic assumptions the DDDPes for (1) is locally solvable around $x_0$ if and only if the (linear) DDPs for its linearization
(6) is solvable. Recall that the (linear) DDPs for (6) requires that there exists a linear feedback

\[ u = M\xi + Nv \]  

(21)

with \( M \) an \((m,n)\)-matrix and \( N \) an \((m,m)\)-matrix, such that in the closed loop system

\[
\begin{aligned}
\dot{\xi} &= (F + GM)\xi + GNv + Pq \\
\eta &= H\xi 
\end{aligned}
\]  

(22)

the disturbances \( q \) do not influence \( \eta \) and, in addition, the matrix \( F + GM \) is a Hurwitz matrix (see [22],[1]). Recall that the DDPs for (6) is solvable if and only if

\[
\text{Im} P \subset \mathcal{V}^*_s 
\]  

(23)

where \( \mathcal{V}^*_s \) is the maximal stabilizability subspace in \( \text{Ker} H \) (cf. [22],[1]). Note further that in the linear context the notions of asymptotic stability and exponential stability coincide, so that the DDPs and the DDPes represent the same problem.

Our main result is the following.

**Theorem 3.2** Consider the square analytic nonlinear system \( \Sigma_q \) and assume that it has full rank, i.e., \( \rho^*(\Sigma_0) = m \). Let \( x_0 \in \mathcal{X} \) be an equilibrium point of \( \Sigma_q \) with \( h(x_0) = 0 \) and suppose that Assumption 2.5 is satisfied. Then the DDDPes is locally solvable around \( x_0 \) if and only if the DDPs is solvable for its linearization (6) around \( x_0 \).

**Proof** (necessity) Assume that under the given conditions the DDDPes is locally solvable for the system (1). Thus, in particular the closed loop system \( \Sigma_q \circ Q_d \) (see (5) has \((x_0,0)\) as an exponentially stable equilibrium point when \( v \equiv 0, q \equiv 0 \). Let us write the linearization of this closed loop system as

\[
\begin{aligned}
\dot{\xi} &= (F + GM_1)\xi + GM_2 z + GNv + Pq \\
\dot{z} &= A_1 \xi + A_2 z + Bv \\
\eta &= H\xi 
\end{aligned}
\]  

(24)

Since the closed loop system (5) is exponentially stable, it follows that (24) is asymptotically stable. Moreover, since the disturbances \( q \) do not influence \( y \) for the system (5), the same holds true for the linearized system (24). Following [2],[18],[19], this implies on its turn that there exists a linear subspace \( \mathcal{V}_e \) in \((\xi,z)\)-space that is invariant for (24), is inner-stable as well as outer-stable ([18],[19]) and satisfies

\[
\text{Im} \left( \begin{array}{c} P \\ 0 \end{array} \right) \subset \mathcal{V}_e \subset \text{Ker}(H \ 0) 
\]  

(25)
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But then we also have that the subspace \( p(\mathcal{V}_e) \subset \mathbb{R}^n \) defined by

\[
p(\mathcal{V}_e) = \{ \xi \in \mathbb{R}^n \mid \exists z \in \mathbb{R}^n : (\xi, z) \in \mathcal{V}_e \}
\]  

(26)

is controlled invariant and inner- as well as outer-stable for the linearization of (1), i.e., for (6). Obviously we also have \( \text{Im}P \subset p(\mathcal{V}_e) \subset \text{Ker}H \). Therefore we may conclude that the DDPs is solvable for (6).

(sufficiency) The proof of the sufficiency-part is based on the following observations:

(i) given the fact that the DDPs is solvable for (6), it follows by direct inspection -for example using the Popov-Belevich-Hautus test- that the system (6), together with any linear Singh compensator \( R \) is stabilizable since the original system (6) is.

(ii) Given the fact that the DDPs is solvable for (6), we have that \( \text{Im}P \subset \mathcal{V}_e^* \). Further, one may check that for any Singh compensator \( R \) for (6) we have that for the closed loop system there exists a stabilizability subspace \( \mathcal{V}_e^* \) that is isomorphic to \( \mathcal{V}_e^* \) and that satisfies \( p(\mathcal{V}_e^*) = \mathcal{V}_e^* \).

Now assume that the DDPs is solvable for \( L\Sigma_q \). Then there exists a Singh compensator \( R \) for \( L\Sigma_q \) that solves the DDDP for \( L\Sigma_q \). By the above observations, we have that for \( L\Sigma_q \circ R \) the DDPs is solvable, say by means of a static state feedback \( \tilde{R}_s \) for \( L\Sigma_q \circ R \). By Proposition 2.1 there exists a Singh compensator \( Q \) for \( \Sigma_q \) that solves the DDDP for \( \Sigma_q \) and has the property that \( L(\Sigma_q \circ Q) = L\Sigma_q \circ R \). Together with the fact that \( \tilde{R}_s \) solves the DDPs for \( L\Sigma_q \circ R \), this implies that \( Q \circ \tilde{R}_s \) solves the DDDP for \( \Sigma_q \).

4 Example

In this section we present an example that illustrates the theory developed in the foregoing sections and the construction of a first order approximate solution to the DDDP for a nonlinear control system \( \Sigma_q \).

Consider the following nonlinear system:

\[
\begin{align*}
\dot{x}_1 &= (x_2 + 1)u_1 \\
\dot{x}_2 &= x_5 \\
\dot{x}_3 &= -x_2 - x_3 + x_4 + (x_4 - 1)u_1 \\
\dot{x}_4 &= u_2 \\
\dot{x}_5 &= -15x_5 + q \\
\dot{x}_6 &= 8x_2 - 8x_4 - 16x_6
\end{align*}
\]  

(27)
around the equilibrium point \( x_0 = 0 \). For this system we find \( \gamma_1 = 1, \gamma_2 = 2 \) and

\[
\begin{align*}
\dot{y}_1 &= (x_2 + 1)u_1 \\
\dot{y}_2 &= -x_2 - x_3 + x_4 + \frac{x_4 - 1}{x_2 + 1} \dot{y}_1 \\
\ddot{y}_2 &= -x_5 + x_2 + x_3 - x_4 - \frac{x_4 - 1}{x_2 + 1} \dot{y}_1 + u_2 + \frac{y_2}{x_2 + 1} \dot{y}_1 - \\
&\quad \frac{x_5(x_4 - 1)}{(x_2 + 1)^2} \dot{y}_1 + \frac{x_4 - 1}{x_2 + 1} \ddot{y}_1
\end{align*}
\]  
(28)

Hence \( \delta_1 = \delta_2 = 2 \), and a Singh compensator for (27) is given by

\[
\begin{align*}
\dot{z} &= \tilde{u}_1 \\
u_1 &= \frac{x}{x_2 + 1} \\
u_2 &= \frac{x_2 + 1}{x_2 + z + 1} \left[-x_2 - x_3 + x_4 + x_5 + \frac{(x_4 - 1)x}{x_2 + 1} + \right. \\
&\quad \left. \frac{x_5(x_4 - 1)}{(x_2 + 1)^2} \dot{y}_1 + \frac{x_4 - 1}{x_2 + 1} \ddot{y}_1 + \dddot{u}_2 \right]
\end{align*}
\]  
(29)

Furthermore, it follows from (28) that

\[
\bar{\Delta}_e = \text{span} \left\{ z(x_2 + 1) \frac{\partial}{\partial x_2} + [(x_2 + 1)^2 + z(x_4 - 1)] \frac{\partial}{\partial x_4}, \right. \\
&\quad \left. (x_4 - 1) \frac{\partial}{\partial x_4} - z \frac{\partial}{\partial z}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6} \right\}
\]

Hence

\[
P_e = \text{span} \left\{ \frac{\partial}{\partial x_5} \right\} \subset \bar{\Delta}_e
\]

and by Theorem 2.3 the DDDP for (27) is solvable around \( x_0 \) via the Singh compensator (29).

The linearization of (27) around \( x_0 = 0 \) is given by

\[
\begin{align*}
\dot{\xi}_1 &= u_1 & \eta_1 &= \xi_1 \\
\dot{\xi}_2 &= \xi_5 & \eta_2 &= \xi_3 \\
\dot{\xi}_3 &= -\xi_2 - \xi_3 + \xi_4 - u_1 \\
\dot{\xi}_4 &= u_2 \\
\dot{\xi}_5 &= -15\xi_5 + q \\
\dot{\xi}_6 &= 8\xi_2 - 8\xi_4 - 16\xi_6
\end{align*}
\]  
(30)
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For (30) we have

\[
\begin{align*}
\dot{\eta}_1 &= u_1 \\
\dot{\eta}_2 &= -\xi_2 - \xi_3 + \xi_4 - \eta_1 \\
\dot{\eta}_2 &= \xi_2 + \xi_3 - \xi_4 - \xi_5 = \eta_1 - \eta_1 + u_2
\end{align*}
\]

(31)

Hence a Singh compensator for (30) is given by

\[
\begin{align*}
\dot{\tilde{z}} &= \tilde{u}_1 \\
u_1 &= \tilde{z} \\
\tilde{u}_2 &= -\xi_2 - \xi_3 + \xi_4 + \xi_5 - \tilde{z} + \tilde{u}_1 + \tilde{u}_2
\end{align*}
\]

(32)

It may be checked that (32) is the linearization of (27) around \((x, z) = (x_0, 0)\). Further, for (30) we have

\[
\mathcal{V}^* = \mathcal{V}_x^* = \operatorname{span}\{e_2 + e_4, e_5, e_6\}
\]

where \(e_i\) denotes the \(i\)-th basis vector of the standard basis of \(\mathbb{R}^6\). Since \(\operatorname{Im}P = \operatorname{span}\{e_5\}\), this implies that the DDPs is solvable for (30) and thus by Theorem 3.2 the DDDPes is solvable for (27). Choosing

\[
\begin{align*}
\tilde{u}_1 &= -4\xi_1 - \tilde{z} + v_1 \\
\tilde{u}_2 &= -\xi_2 - \xi_3 + \xi_4 + \xi_5 - \tilde{z} + v_1 + v_2
\end{align*}
\]

(33)

the feedback (32,33) solves the DDPs for (30). Mutatis mutandis, this implies that the feedback (29,33) is a first order approximate solution of the DDDPes for (27).

5 Concluding remarks

In this paper we have investigated the analogy between the dynamic disturbance decoupling problem with stability around an equilibrium point for a nonlinear system and the problem for its linearization around this equilibrium point. It was shown that under generic conditions the nonlinear problem is solvable if and only if the corresponding linearized problem is. This illustrates an often used engineering paradigm in the way that a nonlinear problem may be approached by tackling the corresponding linearized problem.

At the same time we show that in general only certain dynamic state feedbacks for the linearized system act as first order approximate solutions of the nonlinear dynamic disturbance decoupling problem, although for the solvability of the linear disturbance decoupling problem with stability it is sufficient to limit oneself to static state feedbacks. Of course it is not clear yet to what extent an approximate solution based on the linearization serves as a "reasonably good" solution. For this one needs to develop measures and tools for a robustness analysis of nonlinear controller design. One possible measure, used in [9],[17] is based upon \(L_2\)-norms, like in \(H_\infty\)-control. We leave this issue open for future research.
Another issue that becomes important in the analysis we have developed, is to replace the dynamic state feedbacks in the solution of the (linear or nonlinear) dynamic disturbance decoupling problem by dynamic output feedbacks. Such investigations have been performed successfully in a linear context, see e.g. [18],[19], and it might very well be possible to do a similar research in the nonlinear context.

References


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