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A relaxation of Lyapunov conditions and controller synthesis for discrete-time periodic systems

Christoph Böhm, Mircea Lazar, Frank Allgöwer

Abstract—This paper proposes a novel approach to stability analysis and controller synthesis for discrete-time periodically time-varying systems. Firstly, a relaxation of standard Lyapunov conditions is derived. This leads to a less conservative Lyapunov function that is required to decrease at every period, rather than at each time instant. Secondly, several solutions for synthesizing such periodic control Lyapunov functions are presented. These solutions make use of on-line optimization and can be formulated as a semi-definite program for constrained linear periodic systems. An example illustrates the effectiveness of the developed method.

I. INTRODUCTION

Periodically time-varying systems represent an important system class for both control theory and applications. Some of the most relevant real-life problems that involve control of periodic systems are magnetic satellite control problems, see, for example, [1], [2], and control of helicopter rotors [3]. Furthermore, as it was recently pointed out in [4], time-invariant systems that are controlled by asynchronous inputs, in particular multirate and multiplexed inputs, can be modeled by periodically time-varying systems.

The recent monograph [5] provides an excellent overview on existing results in the field of periodically time-varying systems (or shortly, *periodic systems*) for both stability analysis and controller synthesis. Most of the existing methods [6]–[8] make use of standard Lyapunov theory to solve the analysis and synthesis problem for periodic systems. In [6], it is shown that every periodically time-varying nonlinear system described by difference equations with continuous right hand side is uniformly globally asymptotically stable if and only if it admits a periodically time-varying Lyapunov function (LF). In the linear case, the well-known periodic Lyapunov lemma (PLL), see, for example, [5], establishes existence of a quadratic periodically time-varying LF for asymptotically stable periodic systems.

In particular, in this work we focus on receding horizon control (RHC) of discrete-time periodic systems. Recently, there has been an increasing interest towards designing stabilizing RHC schemes for periodic systems. For example, the linear matrix inequalities (LMIs) based RHC approach of [9] for uncertain linear systems was extended to periodic systems in [10] and periodic systems with uncertainty in [11]. Further extensions were discussed in [12]. The solution of the periodic discrete-time algebraic Riccati equation was used

in [4] to derive a suitable terminal cost and terminal set for a RHC scheme that uses a periodic linear model. Further results for calculating a terminal set for RHC of nonlinear periodic systems were presented in [13]. Notice that all of the above-mentioned RHC schemes rely on the PLL for discrete-time linear periodic systems.

The first contribution of this paper is to indicate that the standard Lyapunov conditions can be relaxed in the case of periodic systems. This relaxation yields a less conservative version of a standard LF, which is called a periodic Lyapunov function (PLF), see also [14]. It is important to point out that the proposed concept of a PLF is different from the definition of a periodically time-varying LF given in [5]–[8]. The difference consists in the fact that a PLF, which may also be periodically time-varying, is only required to decrease at each period, rather than at each sampling instant. Although under typical continuity assumptions, existence of a standard periodically time-varying LF is a necessary condition in stability analysis of periodic systems [5]–[7], it was shown in [14] that in the linear case the concept of PLFs can yield a larger region of attraction. Furthermore, as it will be shown in this paper, PLFs are particularly useful for controller synthesis. As such, after introducing a corresponding definition of a periodic control Lyapunov function (PCLF), several optimization based stabilizing control schemes are proposed. These algorithms rely on the on-line synthesis, in a receding horizon manner, of a PCLF with a corresponding control law. As recently pointed out in [15], on-line calculation of a CLF yields a trajectory-dependent CLF, rather than a “global” CLF. However, the main advantage of such an approach is the significant enlargement of the domain of attraction, due to less conservative Lyapunov conditions. For linear periodic systems, two alternative solutions to calculate quadratic trajectory-dependent PCLFs are presented. Both algorithms require a prediction of the future state and PCLF for one period and they can be formulated as a single semi-definite programming (SDP) problem.

II. PRELIMINARIES

A. Notation and basic definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$ we define $\Pi_{\geq c} := \{k \in \Pi \mid k \geq c\}$, and similarly $\Pi_{\leq c}$, $\mathbb{R}_{\Pi} := \Pi$ and $\mathbb{Z}_{\Pi} := \{k \in \mathbb{Z} \mid k \in \Pi\}$. For $N \in \mathbb{Z}_{\geq 1}$, $\Pi^N := \Pi \times \dots \times \Pi$. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, we denote by $\text{int}(\mathcal{S})$ the interior of $\mathcal{S}$. For a vector $x \in \mathbb{R}^n$,

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$[x]_i$ denotes the i -th element of x and $\|\cdot\|$ denotes the 2-norm, i.e., $\|x\| := \sqrt{\sum_{i=1}^n |[x]_i|^2}$. $I_n \in \mathbb{R}^{n \times n}$ denotes the n -th dimensional identity matrix. For a symmetric matrix $Z \in \mathbb{R}^{n \times n}$ let $Z \succ 0$ ($\succeq 0$) denote that Z is positive definite (semi-definite). For a positive definite matrix $Z \in \mathbb{R}^{n \times n}$ let $\lambda_{\min(\max)}(Z)$ denote its smallest (largest) eigenvalue. Moreover, $\star$ is used to denote the symmetric part of a matrix, i.e., $\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} a & \star \\ \star & c \end{bmatrix}$. A function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to class $\mathcal{K}_\infty$ if $\varphi \in \mathcal{K}$ and $\lim_{s \rightarrow \infty} \varphi(s) = \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is decreasing and $\lim_{k \rightarrow \infty} \beta(s, k) = 0$.

B. Stability notions

Consider the discrete-time nonlinear system

$$x(k+1) = f(k, x(k)), \quad k \in \mathbb{Z}_+, \quad (1)$$

where $f : \mathbb{Z}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an arbitrary nonlinear map with $f(k, 0) = 0$ for all $k \in \mathbb{Z}_+$ and $x(k) \in \mathbb{X} \subseteq \mathbb{R}^n$ is the system state at time $k \in \mathbb{Z}_+$. For brevity we employ the following standing assumption.

Assumption II.1 The set $\mathbb{X} \subseteq \mathbb{R}^n$ with $0 \in \text{int}(\mathbb{X})$ is positively invariant for system (1), i.e., for all $k \in \mathbb{Z}_+$ and all $x \in \mathbb{X}$ it holds that $f(k, x) \in \mathbb{X}$.

In what follows we will focus on a subclass of time-varying systems, namely periodically time-varying systems.

Definition II.2 System (1) is called *periodic* if there exists an $N \in \mathbb{Z}_{\geq 1}$ such that $f(k, x) = f(k+N, x)$ for all $(k, x) \in \mathbb{Z}_+ \times \mathbb{X}$. Furthermore, the smallest value of $N \in \mathbb{Z}_{\geq 1}$ for which $f(k, x) = f(k+N, x)$ is called the *period* of system (1).

Definition II.3 System (1) is called asymptotically stable in $\mathbb{X}$, or shortly, $\text{AS}(\mathbb{X})$, if there exists a $\mathcal{KL}$ -function $\beta(\cdot, \cdot)$ such that, for each $x(0) \in \mathbb{X}$ the corresponding state trajectory of (1) satisfies $\|x(k)\| \leq \beta(\|x(0)\|, k)$, $\forall k \in \mathbb{Z}_+$. System (1) is called exponentially stable in $\mathbb{X}$, or shortly, $\text{ES}(\mathbb{X})$, if $\beta(s, k) := \theta \mu^k s$ for some $\theta \in \mathbb{R}_{\geq 1}$, $\mu \in \mathbb{R}_{[0,1)}$.

In the case of linear periodic systems, i.e.,

$$x(k+1) = A(k)x(k), \quad k \in \mathbb{Z}_+, \quad (2)$$

where $A(k) = A(k+N) \in \mathbb{R}^{n \times n}$, $\forall k \in \mathbb{Z}_+$, the periodic Lyapunov lemma (PLL) [5] states the following.

Lemma II.4 (Periodic Lyapunov Lemma [5]) *The linear, periodically time-varying system (2) is exponentially stable in $\mathbb{R}^n$ if and only if for a given N -periodic scalar $\rho(k) = \rho(k+N) \in \mathbb{R}_{[0,1)}$, $\forall k \in \mathbb{Z}_+$, there exists an N -periodic matrix $P(k) = P(k+N) \in \mathbb{R}^{n \times n}$, with $P(k) \succ 0$, $\forall k \in \mathbb{Z}_+$, such that the following linear matrix inequality holds:*

$$A(k)^\top P(k+1)A(k) - \rho(k)P(k) \preceq 0, \quad \forall k \in \mathbb{Z}_{[0,N-1]}. \quad (3)$$

In [5] an alternative condition to (3) was used, i.e.,

$$A(k)^\top P(k+1)A(k) - P(k) = -Q(k), \quad \forall k \in \mathbb{Z}_{[0,N-1]},$$

which is called the periodic Lyapunov equation, and where $Q(k) = Q(k+N) \in \mathbb{R}^{n \times n}$, with $Q(k) \succ 0$, $\forall k \in \mathbb{Z}_+$. This condition is equivalent to (3) when “=” is replaced by “ $\preceq$ ” and $\rho(k) := \frac{\lambda_{\min}(Q(k))}{\lambda_{\max}(P(k))}$, $k \in \mathbb{Z}_{[0,N-1]}$.

To the authors’ best knowledge, the only results available for stability analysis of periodic *nonlinear* systems make use of standard time-varying Lyapunov functions [6]–[8], which do not exploit the periodicity of the system dynamics. Motivated by this, in the following section we will introduce the concept of a *periodic Lyapunov function*.

III. PERIODIC LYAPUNOV FUNCTIONS

The results presented in this section rely on the following boundedness assumption.

Assumption III.1 $\exists \bar{\alpha}_j \in \mathcal{K}_\infty$, $j \in \mathbb{Z}_{[1,N-1]}$, such that

$$\|x(j)\| \leq \bar{\alpha}_j(\|x(j-1)\|), \quad \forall j \in \mathbb{Z}_{[1,N-1]}. \quad (4)$$

For example, Assumption III.1 is satisfied for systems where $f(k, x)$ is continuous in x for each fixed k .

Theorem III.2 Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and suppose that Assumption III.1 holds. Furthermore, let $\eta \in \mathbb{R}_{[0,1)}$ and let $V : \mathbb{Z}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a function such that

$$\alpha_1(\|x\|) \leq V(k, x) \leq \alpha_2(\|x\|), \quad (5a)$$

$$V(k+N, x(k+N)) \leq \eta V(k, x(k)), \quad (5b)$$

for all $x \in \mathbb{X}$ and all $k \in \mathbb{Z}_+$. Then, system (1) is $\text{AS}(\mathbb{X})$.

For brevity, the interested reader is referred to [14] for the proof of Theorem III.2. A function $V : \mathbb{Z}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ that satisfies the hypothesis of Theorem III.2 is called a time-varying *periodic Lyapunov function* (PLF) for system (1). It is important to point out the difference with respect to a *periodically time-varying Lyapunov function*, which is used in [6]–[8] or in the PLL [5]. A periodically time-varying Lyapunov function is simply a particular type of a standard time-varying Lyapunov function, which is still required to decrease at each time instant. In contrast, a time-varying PLF as proposed in this paper is only required to decrease at every period. Notice that Theorem III.2 allows for a periodically time-varying PLF, i.e., $V(k, x) = V(k+N, x)$ for all $k \in \mathbb{Z}_+$ and all $x \in \mathbb{X}$, as a particular type of a time-varying PLF.

By requiring $V(\cdot, \cdot)$ to decrease at *each* time instant for all $k \in \mathbb{Z}_{[0,N-1]}$, Assumption III.1 can be omitted, as shown in [14]. Also, therein it is shown that for linear periodic systems with polytopic constraints, the novel concept of PLFs can be exploited to calculate a larger region of attraction compared to the one obtained using the PLL. The interested reader is referred to [14] for further details.

IV. SYNTHESIS OF PERIODIC CLFS

In what follows we will make use of the results presented in Theorem III.2 to synthesize a stabilizing controller. As such, consider constrained periodic nonlinear systems of the form

$$x(k+1) = f(k, x(k), u(k)), \quad k \in \mathbb{Z}_+, \quad (6)$$

where $f : \mathbb{Z}_+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is an arbitrary nonlinear map with $f(k, x, u) = f(k+N, x, u)$ for all $(k, x, u) \in \mathbb{Z}_+ \times \mathbb{R}^n \times \mathbb{R}^m$ and $f(k, 0, 0) = 0$ for all $k \in \mathbb{Z}_+$. As usual, $x(k) \in \mathbb{X} \subseteq \mathbb{R}^n$ and $u(k) \in \mathbb{U} \subseteq \mathbb{R}^m$ are the state and the input of the system, respectively, at time $k \in \mathbb{Z}_+$. The sets $\mathbb{X}$ and $\mathbb{U}$ contain the origin in their interior.

Assumption IV.1 The set $\mathbb{X}$ is constrained control invariant for system (6), i.e., for all $k \in \mathbb{Z}_+$ and all $x \in \mathbb{X}$ there exists a $u \in \mathbb{U}$ such that $f(k, x, u) \in \mathbb{X}$.

Assumption IV.2 There exist functions $\bar{\alpha}_j \in \mathcal{K}_\infty$, $j \in \mathbb{Z}_{[1, N-1]}$, such that for each $x(0) \in \mathbb{X}$ there is a sequence $\{u(j)\}_{j \in \mathbb{Z}_{[0, N-2]}} \in \mathbb{U}^{N-1}$ such that

$$\|x(j)\| \leq \bar{\alpha}_j(\|x(j-1)\|), \quad j \in \mathbb{Z}_{[1, N-1]}, \quad (7)$$

where $x(j)$ satisfies (6).

For example, Assumption IV.2 is satisfied for systems with $f(k, x, u)$ a continuous function of x for each fixed k and u .

Definition IV.3 A function $V : \mathbb{Z}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ that satisfies (5a) for all $x \in \mathbb{X}$ and all $k \in \mathbb{Z}_+$ is called a *periodic control Lyapunov function (PCLF)* in $\mathbb{X}$ for system (6) if for all $x(0) \in \mathbb{X}$ there exists a sequence $\{u(k)\}_{k \in \mathbb{Z}_+}$ with $u(k) \in \mathbb{U}$ for all $k \in \mathbb{Z}_+$ such that

$$V(k+N, x(k+N)) - \eta V(k, x(k)) \leq 0, \quad \forall k \in \mathbb{Z}_+. \quad (8)$$

Next, we will present two RHC problem set-ups. For any $k \in \mathbb{Z}_+$ let $x(k|k) := x(k)$ and $x(k+i+1|k) := f(k+i, x(k+i|k), u(k+i|k))$ for all $i \in \mathbb{Z}_+$, where $u(k+i|k) \in \mathbb{U}$ for all $i \in \mathbb{Z}_+$. Similarly, for any $k \in \mathbb{Z}_+$ and $x \in \mathbb{X}$ let $V(k|k, x) := V(k, x)$ and, for any $i \in \mathbb{Z}_{\geq 1}$ let $V(k+i|k, x)$ denote the prediction of the actual function $V(k+i, x)$ that is calculated at time $k \in \mathbb{Z}_+$.

Problem IV.4 Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\eta \in \mathbb{R}_{[0,1]}$ be given. At time $k \in \mathbb{Z}_+$ measure the state $x(k)$ and calculate $V(k, x(k))$, $V(k+N|k, x(k+N|k))$ and a sequence $\{u(k+i|k)\}_{i \in \mathbb{Z}_{[0, N-1]}} \in \mathbb{U}^N$ such that for all $i \in \mathbb{Z}_{[0, N-1]}$

$$V(k+N|k, x(k+N|k)) - \eta V(k, x(k)) \leq 0, \quad (9a)$$

$$\alpha_1(\|x(k+N|k)\|) - V(k+N|k, x(k+N|k)) \leq 0, \quad (9b)$$

$$V(k+N|k, x(k+N|k)) - \alpha_2(\|x(k+N|k)\|) \leq 0, \quad (9c)$$

$$\alpha_1(\|x(k)\|) \leq V(k, x(k)) \leq \alpha_2(\|x(k)\|), \quad (9d)$$

$$u(k+i|k) \in \mathbb{U}, \quad x(k+i+1|k) \in \mathbb{X}, \quad (9e)$$

holds, and such that, if $k \in \mathbb{Z}_{\geq N}$,

$$V(k, x(k)) - V(k|k-N, x(k|k-N)) \leq 0 \quad (9f)$$

holds.

Let $\sigma_1(x(k)) := \{u(k+i|k)\}_{i \in \mathbb{Z}_{[0, N-1]}} | \exists V(k, x(k)), V(k+N|k, x(k+N|k))$ s.t. (9a)-(9e) for $k \in \mathbb{Z}_+$ and (9f) for $k \in \mathbb{Z}_{\geq N}$ hold}. Further, let

$$\pi_1(x(k)) := \{u(k|k) | \{u(k+i|k)\}_{i \in \mathbb{Z}_{[0, N-1]}} \in \sigma_1(x(k))\}$$

and $\Phi_1(k, x, \pi_1(x)) := \{f(k, x, u) | u \in \pi_1(x)\}$. Notice that for a given $x(0) \in \mathbb{X}$ the inequalities (9) generate, besides a sequence of sets of feasible control actions $\{\pi_1(x(k))\}_{k \in \mathbb{Z}_+}$ and sequences $\{\sigma_1(x(k))\}_{k \in \mathbb{Z}_+}$, also a sequence of sets of feasible realizations of a PCLF, i.e.,

$$\mathcal{V}_1(x(k)) := \{V(k, x(k)), V(k+N|k, x(k+N|k)) | \sigma_1(x(k)) \neq \emptyset\}.$$

Due to the dependency of $\mathcal{V}_1(x(k))$ on $x(k)$, Problem IV.4 yields a *trajectory-dependent* [15] PCLF (tdPCLF). Implicitly, $\sigma_1(x(k))$, $\pi_1(x(k))$ and $\mathcal{V}_1(x(k))$ also depend on $V(k|k-N, x(k|k-N))$, but we omitted this dependency for brevity of notation. For $k \in \mathbb{Z}_{[0, N-1]}$, $\sigma_1(x(k))$, $\pi_1(x(k))$ and $\mathcal{V}_1(x(k))$ depend on $x(k)$ only and their definition is recovered by removing (9f) for $k \in \mathbb{Z}_{[0, N-1]}$.

The next result shows how the sequence of sets of feasible realizations of a tdPCLF, $\{\mathcal{V}_1(x(k))\}_{k \in \mathbb{Z}_+}$, can be used to establish asymptotic stability of system (6).

Theorem IV.5 Let $\eta \in \mathbb{R}_{[0,1]}$ and $\alpha_1, \alpha_2, \bar{\alpha}_j \in \mathcal{K}_\infty$, $j \in \mathbb{Z}_{[1, N-1]}$, be given. Suppose Assumption IV.2 holds and Assumption III.1 holds for all $x(j) \in \Phi_1(j-1, x(j-1), \pi_1(x(j-1)))$ and for all $j \in \mathbb{Z}_{[1, N-1]}$. Further, suppose that Problem IV.4 is feasible for all states in $\mathbb{X}$. Then, the difference inclusion

$$x(k+1) \in \Phi_1(k, x(k), \pi_1(x(k))), \quad k \in \mathbb{Z}_+, \quad (10)$$

is AS($\mathbb{X}$).

Proof: Let $x(k) \in \mathbb{X}$ for some $k \in \mathbb{Z}_+$. Then, feasibility of Problem IV.4 ensures that $x(k+1) \in \Phi_1(k, x(k), \pi_1(x(k))) \subseteq \mathbb{X}$ due to constraint (9e). Thus, $x(k) \in \mathbb{X}$ for all $k \in \mathbb{Z}_+$ and Problem IV.4 is feasible for all $k \in \mathbb{Z}_+$ for any $x(0) \in \mathbb{X}$. This implies that $\mathbb{X}$ is a positively invariant set for (10). Then, any $\{u(k+i|k)\}_{i \in \mathbb{Z}_{[0, N-1]}} \in \sigma_1(x(k))$ yields

$$V(k+N|k, x(k+N|k)) - \eta V(k, x(k)) \leq 0, \quad (11)$$

for all $k \in \mathbb{Z}_+$ due to satisfaction of (9a) at time k . Furthermore, at time $k+N$ it follows from (9f) that any $\{u(k+N+i|k+N)\}_{i \in \mathbb{Z}_{[0, N-1]}} \in \sigma_1(x(k+N))$ is such that

$$V(k+N, x(k+N)) - V(k+N|k, x(k+N|k)) \leq 0 \quad (12)$$

for all $k \in \mathbb{Z}_+$. Combining (11) and (12) yields

$$V(k+N, x(k+N)) - \eta V(k, x(k)) \leq 0, \quad \forall k \in \mathbb{Z}_+, \quad (13)$$

which is equivalent to (5b). Furthermore, any $V(k, x(k)) \in \mathcal{V}_1(x(k))$ satisfies (9d) and thus (5a) for all $k \in \mathbb{Z}_+$. The result then follows from Theorem III.2. ■

In Problem IV.4, at time $k \in \mathbb{Z}_{\geq N}$ condition (9f) is required to link the actual tdPCLF $V(k, x(k))$ with the predicted tdPCLF $V(k|k-N, x(k|k-N))$ calculated at time $k-N \in \mathbb{Z}_+$. The following problem set-up presents an alternative to Problem IV.4 that enables the removal of condition (9f). The idea is to determine the actual tdPCLF for time $k+N \in \mathbb{Z}_+$ already at time $k \in \mathbb{Z}_+$ based on the predicted state $x(k+N|k)$. Then, at time $k+N-1 \in \mathbb{Z}_+$, the actual state $x(k+N)$ is linked with the predicted state $x(k+N|k)$ such that the required condition on the tdPCLF at time $k+N \in \mathbb{Z}_+$ is satisfied.

In what follows, for brevity of exposition we assume that $V(k, \cdot)$ is known for all $k \in \mathbb{Z}_{[0, N-1]}$. However, the next problem can easily be modified to include calculation of $V(k, x(k))$ for the first N time instants.

Problem IV.6 Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\eta \in \mathbb{R}_{[0,1]}$ be given. At time $k \in \mathbb{Z}_+$ measure the state $x(k)$ and calculate $V(k+N, x(k+N|k))$ and a sequence $\{u(k+i|k)\}_{i \in \mathbb{Z}_{[0, N-1]}} \in \mathbb{U}^N$ such that for all $i \in \mathbb{Z}_{[0, N-1]}$

$$V(k+N, x(k+N|k)) - \eta V(k, x(k)) \leq 0, \quad (14a)$$

$$\alpha_1(\|x(k+N|k)\|) - V(k+N, x(k+N|k)) \leq 0, \quad (14b)$$

$$V(k+N, x(k+N|k)) - \alpha_2(\|x(k+N|k)\|) \leq 0, \quad (14c)$$

$$u(k+i|k) \in \mathbb{U}, \quad x(k+i+1|k) \in \mathbb{X}, \quad (14d)$$

holds, and such that, if $k \in \mathbb{Z}_{\geq N-1}$,

$$\begin{aligned} V(k+1, x(k+1|k)) - \\ \eta V(k+1-N, x(k+1-N)) \leq 0, \end{aligned} \quad (14e)$$

$$\alpha_1(\|x(k+1|k)\|) - V(k+1, x(k+1|k)) \leq 0, \quad (14f)$$

$$V(k+1, x(k+1|k)) - \alpha_2(\|x(k+1|k)\|) \leq 0, \quad (14g)$$

holds.

Notice that in the above problem we made the assignment

$$V(k+N, x) := V(k+N|k, x), \quad \forall x \in \mathbb{X}, \quad \forall k \in \mathbb{Z}_+$$

implicit. As such, at time $k \in \mathbb{Z}_+$ the “future” tdPCLF $V(k+N, x(k+N|k))$ is already calculated. As a result, in contrast to Problem IV.4, where condition (9f) links the actual and the predicted realization of a tdPCLF at a specific time instant for a fixed state, in Problem IV.6 the actual and the predicted state must be linked for a fixed realization of a tdPCLF at a particular time instant.

Let $\sigma_2(x(k)) := \{\{u(k+i|k)\}_{i \in \mathbb{Z}_{[0, N-1]}} | \exists V(k+N, x(k+N|k)) \text{ s.t. (14a)-(14d) for } k \in \mathbb{Z}_+ \text{ and (14e)-(14g) for } k \in \mathbb{Z}_{\geq N-1} \text{ hold}\}$. Further, let

$$\pi_2(x(k)) := \{u(k|k) | \{u(k+i|k)\}_{i \in \mathbb{Z}_{[0, N-1]}} \in \sigma_2(x(k))\}$$

and let $\Phi_2(k, x, \pi_2(x)) := \{f(k, x, u) | u \in \pi_2(x)\}$. Let

$$\mathcal{V}_2(x(k)) :=$$

$$\{V(k+1, x(k+1)), V(k+N, x(k+N|k)) | \sigma_2(x(k)) \neq \emptyset\}.$$

Theorem IV.7 Let $\eta \in \mathbb{R}_{[0,1]}$ and $\alpha_1, \alpha_2, \bar{\alpha}_j \in \mathcal{K}_\infty$, $j \in \mathbb{Z}_{[1, N-1]}$, be given. Suppose Assumption IV.2 holds and Assumption III.1 holds for all $x(j) \in \Phi_2(j-1, x(j-1), \pi_2(x(j-1)))$ and for all $j \in \mathbb{Z}_{[1, N-1]}$. Further, suppose that Problem IV.6 is feasible for all states in $\mathbb{X}$. Then, the difference inclusion

$$x(k+1) \in \Phi_2(k, x(k), \pi_2(x(k))), \quad k \in \mathbb{Z}_+, \quad (15)$$

is AS($\mathbb{X}$).

The proof is omitted as it parallels *mutatis mutandis* the proof of Theorem IV.5. In the next section we will present a method to formulate each of the developed problem set-ups as a low-complexity SDP for linear periodic systems.

V. IMPLEMENTATION VIA SDP

In general, the conditions stated in Problem IV.4 and Problem IV.6 are non-convex. However, in the following we show that these conditions can be formulated as a computationally attractive SDP in the case of linear periodic systems, i.e.,

$$x(k+1) = A(k)x(k) + B(k)u(k), \quad (16)$$

where $A(k) = A(k+N) \in \mathbb{R}^{n \times n}$ and $B(k) = B(k+N) \in \mathbb{R}^{n \times m}$, $\forall k \in \mathbb{Z}_+$. For any $k \in \mathbb{Z}_+$ let $x(k|k) := x(k)$ and $x(k+i+1|k) := A(k+i)x(k+i|k) + B(k+i)u(k+i|k)$ for all $i \in \mathbb{Z}_+$, where $u(k+i|k) \in \mathbb{U}$ for all $i \in \mathbb{Z}_+$.

The following lemma formulates the conditions of Problem IV.4 as an SDP.

Lemma V.1 Let $\eta \in \mathbb{R}_{[0,1]}$, $\epsilon \in \mathbb{R}_+$, $\delta \in \mathbb{R}_{>\epsilon}$ and $x(k)$, $\forall k \in \mathbb{Z}_+$, be known. Consider the inequalities:

$$\begin{bmatrix} \eta x(k)^\top P(k) x(k) & \star \\ x(k+N|k) & X(k+N|k) \end{bmatrix} \succeq 0, \quad (17a)$$

$$P(k) - \epsilon I_n \succeq 0, \quad (17b)$$

$$\delta I_n - P(k) \succeq 0, \quad (17c)$$

$$\epsilon^{-1} I_n - X(k+N|k) \succeq 0, \quad (17d)$$

$$X(k+N|k) - \delta^{-1} I_n \succeq 0, \quad (17e)$$

$$u(k+i|k) \in \mathbb{U}, \quad x(k+i+1|k) \in \mathbb{X}, \quad (17f)$$

for all $i \in \mathbb{Z}_{[0, N-1]}$ and

$$x(k)^\top P(k) x(k) \leq x(k|k-N)^\top P(k|k-N) x(k|k-N). \quad (17g)$$

Let $P(k) \in \mathbb{R}^{n \times n}$, $X(k+N|k) \in \mathbb{R}^{n \times n}$ and $\{u(k+i|k)\}_{i \in \mathbb{Z}_{[0, N-1]}} \in \mathbb{U}^N$ denote a feasible solution to (17) for all $k \in \mathbb{Z}_+$, with (17g) omitted for all $k \in \mathbb{Z}_{[0, N-1]}$, and set $P(k+N|k) := X^{-1}(k+N|k)$. Then,

$$V(k, x(k)) := x(k)^\top P(k) x(k),$$

$$V(k+N|k, x(k+N|k)) :=$$

$$x(k+N|k)^\top P(k+N|k) x(k+N|k)$$

and $\{u(k+i|k)\}_{i \in \mathbb{Z}_{[0, N-1]}}$ are a feasible solution to Problem IV.4 for state $x(k)$, for all $k \in \mathbb{Z}_+$.

Proof: Applying the Schur complement to (17a) yields

$$\begin{aligned} x(k+N|k)^\top P(k+N|k)x(k+N|k) & - \\ \eta x(k)^\top P(k)x(k) & \leq 0, \end{aligned} \quad (18)$$

which is equivalent to (9a) in Problem IV.4. Combining (17b) and (17c), and pre-and post-multiplying with $x \in \mathbb{X}$ and its transposed yields

$$\epsilon \|x\|^2 \leq x^\top P(k)x \leq \delta \|x\|^2, \quad \forall x \in \mathbb{X}. \quad (19)$$

Thus, with $\alpha_1(\|x\|) := \epsilon \|x\|^2$ and $\alpha_2(\|x\|) := \delta \|x\|^2$ condition (9d) holds. Multiplying (17d) with $P(k+N|k)$ and ϵ yields $P(k+N|k) - \epsilon I \succeq 0$. Similarly, from (17e) we obtain $\delta I - P(k+N|k) \succeq 0$. With the functions $\alpha_1(\|x\|)$ and $\alpha_2(\|x\|)$ as defined above this yields

$$\alpha_1(\|x\|) \leq x^\top P(k+N|k)x \leq \alpha_2(\|x\|), \quad \forall x \in \mathbb{X}, \quad (20)$$

and thus satisfaction of (9b) and (9c). Condition (9e) directly follows from (17f). Finally, if $k \in \mathbb{Z}_{\geq N}$, (17g) is equivalent to (9f), which concludes the proof. ■

The conditions (17) are linear in the unknowns $P(k)$, $X(k+N|k)$, and $\{u(k+i|k)\}_{i \in \mathbb{Z}_{[0, N-1]}}$, and can be solved via semi-definite programming, provided that the sets $\mathbb{X}$ and $\mathbb{U}$ are polytopes, i.e., closed and bounded polyhedra. Similarly, the following lemma formulates the conditions of Problem IV.6 as an SDP.

Lemma V.2 Let $\eta \in \mathbb{R}_{(0,1)}$, $\epsilon \in \mathbb{R}_+$, $\delta \in \mathbb{R}_{>\epsilon}$ and $x(k)$, $\forall k \in \mathbb{Z}_+$, be known. Consider the inequalities:

$$\begin{bmatrix} \eta x(k)^\top P(k)x(k) & \star \\ x(k+N|k) & X(k+N) \end{bmatrix} \succeq 0, \quad (21a)$$

$$\epsilon^{-1} I_n - X(k+N) \succeq 0, \quad (21b)$$

$$X(k+N) - \delta^{-1} I_n \succeq 0, \quad (21c)$$

$$u(k+i|k) \in \mathbb{U}, \quad x(k+i+1|k) \in \mathbb{X}, \quad (21d)$$

for all $i \in \mathbb{Z}_{[0, N-1]}$ and

$$\begin{aligned} x(k+1|k)^\top P(k+1)x(k+1|k) & - \\ \eta x(k+1-N)^\top P(k+1-N)x(k+1-N) & \leq 0. \end{aligned} \quad (21e)$$

Let $X(k+N) \in \mathbb{R}^{n \times n}$ and $\{u(k+i|k)\}_{i \in \mathbb{Z}_{[0, N-1]}} \in \mathbb{U}^N$ be a feasible solution to (21) for all $k \in \mathbb{Z}_+$, with (21e) omitted for $k \in \mathbb{Z}_{[0, N-2]}$, and set $P(k+N) := X^{-1}(k+N)$. Then,

$$V(k+N, x(k+N|k)) := x(k+N|k)^\top P(k+N)x(k+N|k)$$

and $\{u(k+i|k)\}_{i \in \mathbb{Z}_{[0, N-1]}}$ are a feasible solution to Problem IV.6 for state $x(k)$, for all $k \in \mathbb{Z}_+$.

The proof is omitted as it parallels *mutatis mutandis* the proof of Lemma V.2. Both Lemma V.1 and Lemma V.2 can be employed to calculate a tdPCLF along with a corresponding input sequence by solving a single SDP online. Compared to the RHC scheme [10], in both approaches the SDP is significantly less complex, besides being less restrictive due to the relaxed stability conditions. Furthermore, [10] requires the calculation of a global CLF at each time instant, whereas in Lemma V.1 and Lemma V.2 the tdPCLF depends on

the actual trajectory. This further reduces conservativeness. Concerning constraint handling, [10] requires the calculation of invariant ellipsoids lying in the interior of *polytopic* constraint sets. Here, we consider more general constraint formulations, see (17f) and (21d), which leads to a less conservative set-up.

Remark V.3 Problem IV.4 and Problem IV.6 can be modified such that a control input and an associated tdPCLF can be calculated also in the case of *nonlinear* periodic systems by solving an SDP on-line. However, this approach only allows for a prediction for one time instant into the future, which might reduce performance. □

VI. SIMULATION RESULTS

To illustrate the effectiveness of the results presented in the previous section, we consider the double integrator

$$\dot{y}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_E y(t) + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_F v(t) \quad (22)$$

subject to the constraints $y(t) \in \mathbb{X} = \{x \in \mathbb{R}^2 \mid -0.5 \leq [x]_1 \leq 1.5, -0.3 \leq [x]_2 \leq 2\}$ and $v(t) \in \mathbb{U} = \{u \in \mathbb{R} \mid -1 \leq u \leq 1\}$ for all $t \in \mathbb{R}_+$. The input is updated at every sampling instant t_k and kept constant in between, i.e., $v(t) = v(t_k)$, $\forall t \in \mathbb{R}_{[t_k, t_{k+1})}$. We assume that the sampling rate $\delta(k) = t_{k+1} - t_k$ is periodic with period $N = 2$, i.e., $\delta(k+2) = \delta(k)$, $\forall k \in \mathbb{Z}_+$. We chose $\delta(0) = \delta_0 = 0.19$ and $\delta(1) = \delta_1 = 0.12$. Discretizing system (22) via exact integration yields a discrete-time periodic system as in (16) with

$$A(i) := e^{E\delta_i}, \quad B(i) := \int_0^{\delta_i} e^{E(\delta_i-\tau)} F d\tau, \quad i \in \{0, 1\}, \quad (23)$$

$$x(k) := y(t_k) \text{ and } u(k) := v(t_k).$$

Figure 1 presents simulation results obtained for the RHC scheme that corresponds to Lemma V.2 and initial condition $x(0) = [-0.5 \ 2]^\top$. A feasible solution to the LMIs in Lemma V.2 was selected at each sampling instant by minimizing the quadratic cost function

$$\begin{aligned} J(x(k), \{u(k+i|k)\}_{i \in \{0,1\}}) := & \\ & \sum_{i=1}^2 x^\top(k+i|k) Q x(k+i|k) \\ & + \sum_{i=0}^1 u^\top(k+i|k) R u(k+i|k), \end{aligned} \quad (24)$$

where $Q = \text{diag}(10, 2)$ and $R = 0.01$. Notice that minimization of this cost subject to the LMIs in Lemma V.2 can be cast as an SDP problem. In the simulation we chose $\eta = 0.98$, $\epsilon = 0.1$ and $\delta = 0.5$. The proposed control scheme drives the system state to the origin while satisfying state and input constraints. The resulting tdPCLF is increasing at some time instants, which validates the proposed relaxation. At several time instants both the input and the state constraints are active, which demonstrates the non-trivial nature of

Fig. 1. Four small plots on the left: Time history of the states $[x]_1$ and $[x]_2$, the control input u and the tdPCLF V . Large plot on the right: State trajectory in the $[x]_1$ - $[x]_2$ -plane. In all plots dashed lines represent state or input constraints.

the considered simulation scenario. The ellipsoids in the large plot of Figure 1 represent periodically invariant sets calculated as in [13]. These sets indicate the region of validity of a state-feedback control law calculated using the PLL. Notice that this control law was not used in the design of the proposed receding horizon controller.

VII. CONCLUSIONS

This paper considered stabilization of discrete-time periodic systems. Firstly, we defined the concept of a periodic Lyapunov function for a periodically time-varying nonlinear system, which is a relaxation of a standard Lyapunov function. Secondly, by means of a periodic control Lyapunov function we constructed two stabilizing RHC schemes for periodic systems. In the case of linear periodic systems, we have shown that these algorithms can be implemented as a single semi-definite program. An example illustrated the effectiveness of the developed methodology.

VIII. ACKNOWLEDGEMENTS

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