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The uniform measure on a Galton–Watson tree without the XlogX condition

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Summary. We consider a Galton–Watson tree with offspring distribution \( \nu \) of finite mean. The uniform measure on the boundary of the tree is obtained by putting mass 1 on each vertex of the \( n \)-th generation and taking the limit \( n \to \infty \). In the case \( E[\nu \ln(\nu)] < \infty \), this measure has been well studied, and it is known that the Hausdorff dimension of the measure is equal to \( \ln(m) \) ([2], [9]). When \( E[\nu \ln(\nu)] = \infty \), we show that the dimension drops to 0. This answers a question of Lyons, Pemantle and Peres [10].

Résumé. Nous considérons un arbre de Galton–Watson dont le nombre d’enfants \( \nu \) a une moyenne finie. La mesure uniforme sur la frontière de l’arbre s’obtient en chargeant chaque sommet de la \( n \)-ième génération avec une masse 1, puis en prenant la limite \( n \to \infty \). Dans le cas \( E[\nu \ln(\nu)] < \infty \), cette mesure a été très étudiée, et l’on sait que la dimension de Hausdorff de la mesure est égale à \( \ln(m) \) ([2], [9]). Lorsque \( E[\nu \ln(\nu)] = \infty \), nous montrons que la dimension est 0. Cela répond à une question posée par Lyons, Pemantle et Peres [10].

Keywords: Galton–Watson tree, Hausdorff dimension.

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1 Introduction

Let \( \mathcal{T} \) be a Galton–Watson tree of root \( e \), associated to the offspring distribution \( q := (q_k, k \geq 0) \). We denote by GW the distribution of \( \mathcal{T} \) on the space of rooted trees, and \( \nu \) a generic random variable on \( \mathbb{N} \) with distribution \( q \). We suppose that \( q_0 = 0 \) and \( m := \sum_{k \geq 0} kq_k \in (1, \infty) \):

\[ 1 \]
the tree has no leaf (hence survives forever) and is not degenerate. For any vertex $u$, we write $|u|$ for the height of vertex $u$ ($|e| = 0$), $\nu(u)$ for the number of children of $u$, and $Z_n$ is the population at height $n$. We define $S(\mathcal{T})$ as the set of all infinite self-avoiding paths of $\mathcal{T}$ starting from the root and we define a metric on $S(\mathcal{T})$ by $d(r, r') := e^{-|r \wedge r'|}$ where $r \wedge r'$ is the highest vertex belonging to $r$ and $r'$. The space $S(\mathcal{T})$ is called boundary of the tree, and elements of $S(\mathcal{T})$ are called rays.

When $E[\nu \ln(\nu)] < \infty$, it is well-known that the martingale $m^{-n}Z_n$ converges in $L^1$ and almost surely to a positive limit ([4]). Seneta [13] and Heyde [3] proved that in the general case (i.e allowing $E[\nu \ln(\nu)]$ to be infinite), there exist constants $(c_n)_{n \geq 0}$ such that

(a) $W_\infty := \lim_{n \to \infty} \frac{Z_n}{c_n}$ exists a.s.
(b) $W_\infty > 0$ a.s.
(c) $\lim_{n \to \infty} \frac{c_{n+1}}{c_n} = m$.

In particular, for each vertex $u \in \mathcal{T}$, if $Z_k(u)$ stands for the number of descendants $v$ of $u$ such that $|v| = |u| + k$, we can define

$$W_\infty(u) := \lim_{k \to \infty} \frac{Z_k(u)}{c_k}$$

and we notice that $m^{-n} \sum_{|u| = n} W_\infty(u) = W_\infty(e)$.

**Definition.** The uniform measure (also called branching measure) is the unique Borel measure on $S(\mathcal{T})$ such that

$$\text{UNIF}\{\{r \in S(\mathcal{T}), r_n = u\} := \frac{m^{-n}W_\infty(u)}{W_\infty(e)}$$

for any integer $n$ and any vertex $u$ of height $n$.

We observe that, for any vertex $u$ of height $n$,

$$\text{UNIF}\{\{r \in S(\mathcal{T}), r_n = u\} = \lim_{k \to \infty} \frac{Z_k(u)}{Z_{n+k}}.$$

Therefore the uniform measure can be seen informally as the probability distribution of a ray taken uniformly in the boundary. This paper is interested in the Hausdorff dimension of $\text{UNIF}$, defined by

$$\dim(\text{UNIF}) := \min\{\dim(E), \text{UNIF}(E) = 1\}.$$
where the minimum is taken over all subsets $E \subset S(T)$ and $\dim(E)$ is the Hausdorff dimension of set $E$. The case $E[\nu \ln(\nu)] < \infty$ has been well studied. In [2] and [9], it is shown that $\dim(UNIF) = \ln(m)$ almost surely. A description of the multifractal spectrum is available in [5],[9],[12],[14]. The case $E[\nu \ln(\nu)] = \infty$ presented as Question 3.1 in [10] was left open. This case is proved to display an extreme behaviour.

**Theorem 1.1.** If $E[\nu \ln(\nu)] = \infty$, then $\dim(UNIF) = 0$ for GW-a.e tree $T$.

The drop in the dimension comes from bursts of offspring at some places of the tree $T$. Namely, for UNIF-a.e. ray $r$, the number of children of $r_n$ will be greater than $(m - o(1))^n$ for infinitely many $n$. To prove it, we work with a particular measure $Q$, under which the distribution of the numbers of children of a uniformly chosen ray is more tractable. Section 2 contains the description of the new measure in terms of a spine decomposition. Then we prove Theorem 1.1 in Section 3.

## 2 A spine decomposition

For $k \geq 1$ and $s \in (0, 1)$, we call $\phi_k(s)$ the probability generating function of $Z_k$

$$\phi_k(s) := E[s^{Z_k}].$$

We denote by $\phi_k^{-1}(s)$ the inverse map on $(0, 1)$ and we let $s \in (0, 1)$. Then $M_n := \phi_n^{-1}(s)^{Z_n}$ defines a martingale and converges in $L^1$ to some $M_\infty > 0$ a.s ([3]). Therefore we can take in (a)

$$c_n := \frac{-1}{\ln(\phi_n^{-1}(s))}$$

which we will do from now on. Hence we can rewrite equivalently $M_n = e^{-Z_n/c_n}$ and $M_\infty = e^{-W_\infty(e)}$. For any vertex $u$ at generation $n$, we define similarly

$$M_\infty(u) := \frac{1}{\phi_n^{-1}(s)} e^{-m^{-n}W_\infty(u)} = e^{1/c_n} e^{-m^{-n}W_\infty(u)}$$

which is the limit of the martingale $M_k(u) := e^{1/c_n} e^{-Z_k(u)/c_n+k}$. In [6], Lynch introduces the so-called derivative martingale

$$\partial M_n := e^{1/c_n} \frac{Z_n}{\phi_n'(\phi_n^{-1}(s))} M_n$$
and shows that the derivative martingale also converges almost surely and in $L^1$ ($\partial M_n$ is in fact bounded). Moreover the limit $\partial M_\infty$ is positive almost surely. We deduce that the ratio $\phi'_n(\phi_n^{-1}(s))/c_n$ converges to some positive constant. In particular, it follows from (c) that

\begin{equation}
\lim_{n \to \infty} \frac{\phi'_{n+1}(\phi_{n+1}^{-1}(s))}{\phi'_n(\phi_n^{-1}(s))} = m.
\end{equation}

We are interested in the probability measure $Q$ on the space of rooted trees defined by

\[ \frac{dQ}{dGW} := \partial M_\infty. \]

Let us describe this change of measure. We call a marked tree a couple $(T, r)$ where $T$ is a rooted tree and $r$ a ray of the tree $T$. Let $(T, \xi)$ be a random variable in the space of all marked trees (equipped with some probability $P(\cdot)$), whose distribution is given by the following rules. Conditionally on the tree up to level $k$ and on the location of the ray at level $k$, (which we denote respectively by $T_k$ and $\xi_k$),

- the number of children of the vertices at generation $k$ are independent
- the vertex $\xi_k$ has a number $\nu(\xi_k)$ of children such that for any $\ell$

\begin{equation}
\begin{aligned}
P(\nu(\xi_k) = \ell) &= \tilde{q}_\ell := q_\ell \ell \exp \left( -\frac{\ell - 1}{c_{k+1}} \right) \frac{\phi'_k(\phi_k^{-1}(s))}{\phi'_{k+1}(\phi_{k+1}^{-1}(s))}.
\end{aligned}
\end{equation}

- the number of children of a vertex $u \neq \xi_k$ at generation $k$ verifies for any $\ell$

\begin{equation}
\begin{aligned}
P(\nu(u) = \ell) &= \tilde{q}_\ell := q_\ell e^{\ell/c_k} \exp \left( -\frac{\ell}{c_{k+1}} \right).
\end{aligned}
\end{equation}

- the vertex $\xi_{k+1}$ is chosen uniformly among the children of $\xi_k$

As often in the literature, we will call the ray $\xi$ the spine. We refer to [8], [7] for motivation on spine decompositions. In our case, we can see $T$ as a Galton-Watson tree in varying environment and with immigration. The fact that (2.2) and (2.3) define probabilities come from the equations (remember that by definition $e^{-1/c_k} = \phi_k^{-1}(s)$)

\begin{align*}
E \left[ (\phi_k^{-1}(s))^\nu \right] &= \phi_k^{-1}(s), \\
E \left[ \nu(\phi_k^{-1}(s))^{\nu-1} \right] &= \frac{\phi'_{k+1}(\phi_{k+1}^{-1}(s))}{\phi'_k(\phi_k^{-1}(s))}.
\end{align*}
We mention that in [8], a similar decomposition was presented using the martingale \( \frac{Z_n}{m} \). In this case, the offspring distribution of the spine is the size-biased distribution \( \left( \frac{m}{n} \right)_{\ell \geq 0} \) whereas the other particles generate offspring according to the original distribution \( q \). In particular, the offspring distributions do not depend on the generation. When \( E[\nu \ln(\nu)] < \infty \), the process, which is a Galton–Watson process with immigration, has a distribution equivalent to GW. It is no longer true when \( E[\nu \ln(\nu)] = \infty \), in which case the spine can give birth to a super-exponential number of children.

**Proposition 2.1.** Under \( Q \), the tree \( T \) has the distribution of \( T \). Besides, for \( P \)-almost every tree \( T \), the distribution of \( \xi \) conditionally on \( T \) is the uniform measure \( \text{UNIF} \).

**Proof.** For any tree \( T \), we define \( T_n \) the tree \( T \) obtained by keeping only the \( n \)-first generations. Let \( T \) be a tree. We will prove by induction that, for any integer \( n \) and any vertex \( u \) at generation \( n \),

\[
P(T_n = T_n, \xi_n = u) = \frac{\partial M_n}{Z_n} \text{GW}(T_n = T_n). \tag{2.4}
\]

For \( n = 0 \), it is straightforward since \( T_0 \) and \( T_0 \) are reduced to the root. We suppose that this is true for \( n-1 \), and we prove it for \( n \). Let \( \hat{u} \) denote the parent of \( u \), and, for any vertex \( v \) at height \( n-1 \), let \( k(v) \) denote the number of children of \( v \) in the tree \( T \). We have

\[
P(T_n = T_n, \xi_n = u \mid T_{n-1} = T_{n-1}, \xi_{n-1} = \hat{u}) = \frac{1}{k(\hat{u})} \prod_{|v|=n-1} q_k(v) \]

\[
= \frac{e^{1/c_n} \phi_{n-1}^{-1}(\phi_{n-1}^{-1}(s))}{e^{1/c_{n-1}} \phi_{n}^{-1}(s)} \frac{e^{Z_{n-1}/c_n}}{e^{Z_{n-1}/c_{n-1}}} \prod_{|v|=n-1} q_k(v) \]

\[
= \frac{e^{1/c_n} \phi_{n-1}^{-1}(\phi_{n-1}^{-1}(s))}{e^{1/c_{n-1}} \phi_{n}^{-1}(s)} \frac{e^{Z_{n-1}/c_n}}{e^{Z_{n-1}/c_{n-1}}} \text{GW}(T_n = T_n \mid T_{n-1} = T_{n-1}) \]

We use the induction assumption to get

\[
P(T_n = T_n, \xi_n = u) = e^{1/c_n} \frac{1}{\phi_{n}^{-1}(\phi_{n}^{-1}(s))} e^{-\frac{Z_n}{c_n}} \text{GW}(T_n = T_n)
\]

which proves (2.4). Summing over the \( n \)-th generation of \( T \) gives

\[
P(T_n = T_n) = \partial M_n \text{GW}(T_n = T_n) = Q(T = T_n).
\]
This computation also shows that \( P(\xi_n = u \mid T_n) = 1/Z_n \) which implies that \( \xi \) is uniformly distributed on the boundary \( S(T) \). □

**Remark A.** For \( u \) a vertex of \( T \) at generation \( n \), call \( T(u) \) the subtree rooted at \( u \). A similar computation shows that if \( u \notin \xi \), then the distribution \( P_u \) of \( T(u) \) (conditionally on \( T_n \) and on \( \xi_n \)) verifies

\[
\frac{dP_u}{dGW} = M_\infty(u).
\]

### 3 Proof of Theorem 1.1

The following proposition shows that in the tree \( T \), there exist infinitely many times when the ball \( \{ r \in S(T) : r_n = \xi_n \} \) has a 'big' weight.

**Proposition 3.1.** Suppose that \( E[\nu \ln(\nu)] = \infty \). Then we have \( P \)-a.s.

\[
\limsup_{n \to \infty} \frac{1}{n} \ln(W_\infty(\xi_n)) = \ln(m).
\]

**Proof.** Let \( 1 < a < b < m \) and \( n \geq 0 \). We get from (2.2)

\[
P(\nu(\xi_n) \in (a^n, b^n)) = \frac{\phi'_n(\phi^{-1}_n(s))}{\phi'_{n+1}(\phi^{-1}_{n+1}(s))} E[\nu e^{-(\nu-1)/c_{n+1}}, \nu \in (a^n, b^n)]
\]

\[
\geq \frac{\phi'_n(\phi^{-1}_n(s))}{\phi'_{n+1}(\phi^{-1}_{n+1}(s))} e^{-b^n/c_n} E[\nu, \nu \in (a^n, b^n)].
\]

From (c) and (2.1), we deduce that for \( n \) large enough, we have

\[
P(\nu(\xi_n) \in (a^n, b^n)) \geq \frac{1}{2m} E[\nu, \nu \in (a^n, b^n)].
\]

Therefore, under the condition \( E[\nu \ln(\nu)] = \infty \), we have

\[
\sum_{n \geq 0} P(\nu(\xi_n) \in (a^n, b^n)) = \infty.
\]

(3.1)

Let \( H(\xi_n) := \{ u \in T : u \text{ child of } \xi_n, u \neq \xi_{n+1} \} \). By Remark A, we have

\[
P\left( \sum_{u \in H(\xi_n)} W_\infty(u) \leq a^n \mid T_{n+1}, \xi_{n+1} \right) = E_{GW} \left[ \prod_{u \in H} M_\infty(u), \sum_{u \in H} W_\infty(u) \leq a^n \right].
\]
Since $M_{\infty}(u) \leq e^{1/c_n}$ for any $|u| = n$, we get
\[
P \left( \sum_{u \in H(\xi_n)} W_{\infty}(u) \leq a^n \mid T_{n+1}, \xi_{n+1} \right) \leq e^{(\nu(\xi_n)-1)/c_n} GW \left( \sum_{u \in H} W_{\infty}(u) \leq a^n \right)_{H=H(\xi_n)}.
\]

Let $(W_{\infty}^i, i \geq 1)$ be independent random variables distributed as $W_{\infty}(e)$ under GW. It follows that on the event $\{\nu(\xi_n) \in (a^n, b^n)\}$, we have
\[
P \left( \sum_{u \in H(\xi_n)} W_{\infty}(u) \leq a^n \right) \leq e^{(b^n-1)/c_n} GW \left( \sum_{i=1}^{a^n} W_{\infty}^i \leq a^n \right) =: d_n.
\]
We obtain that
\[
P \left( \sum_{u \in H(\xi_n)} W_{\infty}(u) > a^n \right) \geq P \left( \sum_{u \in H(\xi_n)} W_{\infty}(u) > a^n, \nu(\xi_n) \in (a^n, b^n) \right)
\geq P (\nu(\xi_n) \in (a^n, b^n)) (1 - d_n).
\]
By (c), $e^{(b^n-1)/c_n}$ goes to 1. Furthermore, we know from [13] that $E_{GW}[W_{\infty}(e)] = \infty$, which ensures by the law of large numbers that $d_n$ goes to 0. By equation (3.1), we deduce that
\[
\sum_{n \geq 0} P \left( \sum_{u \in H(\xi_n)} W_{\infty}(u) > a^n \right) = \infty.
\]
We use the Borel-Cantelli lemma to see that $\sum_{u \in H(\xi_n)} W_{\infty}(u) > a^n$ infinitely often. Since $W_{\infty}(\xi_n) \geq \frac{1}{m} \sum_{u \in H(\xi_n)} W_{\infty}(u)$, we get that $W_{\infty}(\xi_n) \geq a^n/m$ for infinitely many $n$, P-a.s. Hence
\[
\limsup_{n \to \infty} \frac{1}{n} \ln(W_{\infty}(\xi_n)) \geq \ln(a).
\]
Let $a$ go to $m$ to have the lower bound. Since $W_{\infty}(\xi_n) \leq \sum_{|u|=n} W_{\infty}(u) = m^n W_{\infty}(e)$, we have $\limsup_{n \to \infty} \frac{1}{n} \ln(W_{\infty}(\xi_n)) \leq \ln(m)$ hence Proposition 3.1. □

We turn to the proof of the theorem.

**Proof of Theorem 1.1.** By Proposition 3.1, we have
\[
P \left( \limsup_{n \to \infty} \frac{1}{n} \ln(W_{\infty}(\xi_n)) = \ln(m) \right) = 1.
\]
In particular, for P-a.e. $T$, 

$$P \left( \limsup_{n \to \infty} \frac{1}{n} \ln(W_\infty(\xi_n)) = \ln(m) \bigg| T \right) = 1.$$ 

By Proposition 2.1, the distribution of $\xi$ given $T$ is UNIF. Therefore, for P-a.e. $T$, 

$$\text{UNIF} \left( r \in S(T) : \limsup_{n \to \infty} \frac{1}{n} \ln(W_\infty(r_n)) = \ln(m) \right) = 1. \tag{3.2}$$

Again by Proposition 2.1, the distribution of $T$ is the one of $\mathcal{T}$ under $Q$. We deduce that (3.2) holds for $Q$-a.e. tree $T$. Since $Q$ and GW are equivalent, equation (3.2) holds for GW-a.e. tree $T$. We call the Hölder exponent of $\text{UNIF}$ at ray $r^*$ the quantity 

$$\text{Hö}(\text{UNIF})(r^*) := \liminf_{n \to \infty} -\frac{1}{n} \ln \left( \text{UNIF}(\{r \in S(\mathcal{T}) : r_n = r^*_n\}) \right).$$

By definition of $\text{UNIF}$, we can rewrite it 

$$\text{Hö}(\text{UNIF})(r^*) = \liminf_{n \to \infty} -\frac{1}{n} \ln \left( m^{-n} W_\infty(r^*_n)/W_\infty(e) \right).$$

Therefore, for $\text{UNIF}$-a.e. ray $r$, $\text{Hö}(\text{UNIF})(r) = 0$. By Theorem 14.15 of [11] (or § 14 of [1]), it implies that dim($\text{UNIF}$) = 0 GW-almost surely. □

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