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Summary. We consider a Galton–Watson tree with offspring distribution $\nu$ of finite mean. The uniform measure on the boundary of the tree is obtained by putting mass 1 on each vertex of the $n$-th generation and taking the limit $n \to \infty$. In the case $E[\nu \ln(\nu)] < \infty$, this measure has been well studied, and it is known that the Hausdorff dimension of the measure is equal to $\ln(m)$ ([2], [9]). When $E[\nu \ln(\nu)] = \infty$, we show that the dimension drops to 0. This answers a question of Lyons, Pemantle and Peres [10].

Résumé. Nous considérons un arbre de Galton–Watson dont le nombre d’enfants $\nu$ a une moyenne finie. La mesure uniforme sur la frontière de l’arbre s’obtient en chargeant chaque sommet de la $n$-ième génération avec une masse 1, puis en prenant la limite $n \to \infty$. Dans le cas $E[\nu \ln(\nu)] < \infty$, cette mesure a été très étudiée, et l’on sait que la dimension de Hausdorff de la mesure est égale à $\ln(m)$ ([2], [9]). Lorsque $E[\nu \ln(\nu)] = \infty$, nous montrons que la dimension est 0. Cela répond à une question posée par Lyons, Pemantle et Peres [10].

Keywords: Galton–Watson tree, Hausdorff dimension.

AMS subject classifications: 60J80, 28A78.

1 Introduction

Let $T$ be a Galton–Watson tree of root $e$, associated to the offspring distribution $q := (q_k, k \geq 0)$. We denote by $GW$ the distribution of $T$ on the space of rooted trees, and $\nu$ a generic random variable on $\mathbb{N}$ with distribution $q$. We suppose that $q_0 = 0$ and $m := \sum_{k \geq 0} kq_k \in (1, \infty)$:

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the tree has no leaf (hence survives forever) and is not degenerate. For any vertex \( u \), we write \(|u|\) for the height of vertex \( u \) (\(|e| = 0\)), \( \nu(u) \) for the number of children of \( u \), and \( Z_n \) is the population at height \( n \). We define \( S(\mathcal{T}) \) as the set of all infinite self-avoiding paths of \( \mathcal{T} \) starting from the root and we define a metric on \( S(\mathcal{T}) \) by \( d(r, r') := e^{-|r \land r'|} \) where \( r \land r' \) is the highest vertex belonging to \( r \) and \( r' \). The space \( S(\mathcal{T}) \) is called boundary of the tree, and elements of \( S(\mathcal{T}) \) are called rays.

When \( E[\nu \ln(\nu)] < \infty \), it is well-known that the martingale \( m^{-n} Z_n \) converges in \( L^1 \) and almost surely to a positive limit ([4]). Seneta [13] and Heyde [3] proved that in the general case (i.e allowing \( E[\nu \ln(\nu)] \) to be infinite), there exist constants \((c_n)_{n \geq 0}\) such that

(a) \( W_\infty := \lim_{n \to \infty} \frac{Z_n}{c_n} \) exists a.s.

(b) \( W_\infty > 0 \) a.s.

(c) \( \lim_{n \to \infty} \frac{c_{n+1}}{c_n} = m \).

In particular, for each vertex \( u \in \mathcal{T} \), if \( Z_k(u) \) stands for the number of descendants \( v \) of \( u \) such that \(|v| = |u| + k\), we can define

\[
W_\infty(u) := \lim_{k \to \infty} \frac{Z_k(u)}{c_k}
\]

and we notice that \( m^{-n} \sum_{|u| = n} W_\infty(u) = W_\infty(e) \).

**Definition.** The uniform measure (also called branching measure) is the unique Borel measure on \( S(\mathcal{T}) \) such that

\[
\text{UNIF}\{(r \in S(\mathcal{T}), r_n = u)\} := \frac{m^{-n}W_\infty(u)}{W_\infty(e)}
\]

for any integer \( n \) and any vertex \( u \) of height \( n \).

We observe that, for any vertex \( u \) of height \( n \),

\[
\text{UNIF}\{(r \in S(\mathcal{T}), r_n = u)\} = \lim_{k \to \infty} \frac{Z_k(u)}{Z_{n+k}}.
\]

Therefore the uniform measure can be seen informally as the probability distribution of a ray taken uniformly in the boundary. This paper is interested in the Hausdorff dimension of \( \text{UNIF} \), defined by

\[
\dim(\text{UNIF}) := \min\{\dim(E), \text{UNIF}(E) = 1\}
\]
where the minimum is taken over all subsets $E \subset S(\mathcal{T})$ and $\dim(E)$ is the Hausdorff dimension of set $E$. The case $E[\nu \ln(\nu)] < \infty$ has been well studied. In [2] and [9], it is shown that $\dim(\text{UNIF}) = \ln(m)$ almost surely. A description of the multifractal spectrum is available in [5],[9],[12],[14]. The case $E[\nu \ln(\nu)] = \infty$ presented as Question 3.1 in [10] was left open. This case is proved to display an extreme behaviour.

**Theorem 1.1.** If $E[\nu \ln(\nu)] = \infty$, then $\dim(\text{UNIF}) = 0$ for GW-a.e tree $\mathcal{T}$.

The drop in the dimension comes from bursts of offspring at some places of the tree $\mathcal{T}$. Namely, for UNIF-a.e. ray $r$, the number of children of $r_n$ will be greater than $(m - o(1))^n$ for infinitely many $n$. To prove it, we work with a particular measure $Q$, under which the distribution of the numbers of children of a uniformly chosen ray is more tractable. Section 2 contains the description of the new measure in terms of a spine decomposition. Then we prove Theorem 1.1 in Section 3.

## 2 A spine decomposition

For $k \geq 1$ and $s \in (0,1)$, we call $\phi_k(s)$ the probability generating function of $Z_k$

$$\phi_k(s) := E[s^{Z_k}].$$

We denote by $\phi_k^{-1}(s)$ the inverse map on $(0,1)$ and we let $s \in (0,1)$. Then $M_n := \phi_n^{-1}(s)^{Z_n}$ defines a martingale and converges in $L^1$ to some $M_\infty > 0$ a.s ([3]). Therefore we can take in (a)

$$c_n := \frac{-1}{\ln(\phi_n^{-1}(s))}$$

which we will do from now on. Hence we can rewrite equivalently $M_n = e^{-Z_n/c_n}$ and $M_\infty = e^{-W_\infty(e)}$. For any vertex $u$ at generation $n$, we define similarly

$$M_\infty(u) := \frac{1}{\phi_n^{-1}(s)} e^{-m^{-n}W_\infty(u)} = e^{1/c_n} e^{-m^{-n}W_\infty(u)}$$

which is the limit of the martingale $M_k(u) := e^{1/c_n} e^{-Z_k(u)/c_n+k}$. In [6], Lynch introduces the so-called derivative martingale

$$\partial M_n := e^{1/c_n} \frac{Z_n}{\phi_n'(\phi_n^{-1}(s))} M_n.$$
and shows that the derivative martingale also converges almost surely and in $L^1$ ($\partial M_n$ is in fact bounded). Moreover the limit $\partial M_\infty$ is positive almost surely. We deduce that the ratio $\phi'_n(\phi_n^{-1}(s))/c_n$ converges to some positive constant. In particular, it follows from (c) that

$$\lim_{n \to \infty} \frac{\phi'_{n+1}(\phi_n^{-1}(s))}{\phi'_n(\phi_n^{-1}(s))} = m. \tag{2.1}$$

We are interested in the probability measure $Q$ on the space of rooted trees defined by

$$\frac{dQ}{dGW} := \partial M_\infty.$$

Let us describe this change of measure. We call a marked tree a couple $(T, r)$ where $T$ is a rooted tree and $r$ a ray of the tree $T$. Let $(T, \xi)$ be a random variable in the space of all marked trees (equipped with some probability $P(\cdot)$), whose distribution is given by the following rules. Conditionally on the tree up to level $k$ and on the location of the ray at level $k$, (which we denote respectively by $T_k$ and $\xi_k$),

- the number of children of the vertices at generation $k$ are independent
- the vertex $\xi_k$ has a number $\nu(\xi_k)$ of children such that for any $\ell$

$$P(\nu(\xi_k) = \ell) = \tilde{q}_\ell := q_\ell \ell \exp \left( -\frac{\ell}{c_{k+1}} \right) \frac{\phi'_{k}(\phi_k^{-1}(s))}{\phi'_{k+1}(\phi_{k+1}^{-1}(s))} \tag{2.2}$$

- the number of children of a vertex $u \neq \xi_k$ at generation $k$ verifies for any $\ell$

$$P(\nu(u) = \ell) = \tilde{q}_\ell := q_\ell e^{1/c_k} \exp \left( -\frac{\ell}{c_{k+1}} \right) \tag{2.3}$$

- the vertex $\xi_{k+1}$ is chosen uniformly among the children of $\xi_k$

As often in the literature, we will call the ray $\xi$ the spine. We refer to [8], [7] for motivation on spine decompositions. In our case, we can see $T$ as a Galton-Watson tree in varying environment and with immigration. The fact that (2.2) and (2.3) define probabilities come from the equations (remember that by definition $e^{-1/c_k} = \phi_k^{-1}(s)$)

$$E \left[ (\phi_{k+1}^{-1}(s))^\nu \right] = \phi_{k+1}^{-1}(s),$$
$$E \left[ \nu(\phi_{k+1}^{-1}(s))^{\nu-1} \right] = \frac{\phi'_{k+1}(\phi_{k+1}^{-1}(s))}{\phi'_k(\phi_k^{-1}(s))}.$$
We mention that in [8], a similar decomposition was presented using the martingale $\frac{Z_n}{m}$. In this case, the offspring distribution of the spine is the size-biased distribution $(\frac{\ell q}{m})_{\ell \geq 0}$ whereas the other particles generate offspring according to the original distribution $q$. In particular, the offspring distributions do not depend on the generation. When $E[\nu \ln(\nu)] < \infty$, the process, which is a Galton–Watson process with immigration, has a distribution equivalent to GW. It is no longer true when $E[\nu \ln(\nu)] = \infty$, in which case the spine can give birth to a super-exponential number of children.

**Proposition 2.1.** Under $Q$, the tree $T$ has the distribution of $T$. Besides, for $P$-almost every tree $T$, the distribution of $\xi$ conditionally on $T$ is the uniform measure UNIF.

**Proof.** For any tree $T$, we define $T_n$ the tree $T$ obtained by keeping only the $n$-first generations. Let $T$ be a tree. We will prove by induction that, for any integer $n$ and any vertex $u$ at generation $n$,

$$P(T_n = T_n, \xi_n = u) = \frac{\partial M_n}{Z_n} GW(T_n = T_n). \tag{2.4}$$

For $n = 0$, it is straightforward since $T_0$ and $T_0$ are reduced to the root. We suppose that this is true for $n - 1$, and we prove it for $n$. Let $\tilde{u}$ denote the parent of $u$, and, for any vertex $v$ at height $n - 1$, let $k(v)$ denote the number of children of $v$ in the tree $T$. We have

$$P(T_n = T_n, \xi_n = u | T_{n-1} = T_{n-1}, \xi_{n-1} = \tilde{u})$$

$$= \frac{1}{k(\tilde{u})} \prod_{|v|=n-1} q_k(v)$$

$$= \frac{1}{e^{1/c_n}} \frac{\phi_n'(s)}{\phi_n(s)} e^{Z_n/c_n} \prod_{|v|=n-1} q_k(v)$$

$$= \frac{1}{e^{1/c_n}} \frac{\phi_n'(s)}{\phi_n(s)} e^{Z_n/c_n} GW(T_n = T_n | T_{n-1} = T_{n-1}).$$

We use the induction assumption to get

$$P(T_n = T_n, \xi_n = u) = e^{1/c_n} \frac{1}{\phi_n'(s)} e^{-Z_n/c_n} GW(T_n = T_n)$$

which proves (2.4). Summing over the $n$-th generation of $T$ gives

$$P(T_n = T_n) = \partial M_n GW(T_n = T_n) = Q(T = T_n).$$
This computation also shows that \( P(\xi_n = u \mid \mathbb{T}_n) = 1/Z_n \) which implies that \( \xi \) is uniformly distributed on the boundary \( S(\mathbb{T}) \). □

**Remark A.** For \( u \) a vertex of \( \mathbb{T} \) at generation \( n \), call \( \mathbb{T}(u) \) the subtree rooted at \( u \). A similar computation shows that if \( u \not\in \xi \), then the distribution \( P_u \) of \( \mathbb{T}(u) \) (conditionally on \( \mathbb{T}_n \) and on \( \xi_n \)) verifies

\[
\frac{dP_u}{dGW} = M_\infty(u) .
\]

### 3 Proof of Theorem 1.1

The following proposition shows that in the tree \( \mathbb{T} \), there exist infinitely many times when the ball \( \{ r \in S(\mathbb{T}) : r_n = \xi_n \} \) has a ‘big’ weight.

**Proposition 3.1.** Suppose that \( E[\nu \ln(\nu)] = \infty \). Then we have \( P \)-a.s.

\[
\limsup_{n \to \infty} \frac{1}{n} \ln(W_\infty(\xi_n)) = \ln(m) .
\]

**Proof.** Let \( 1 < a < b < m \) and \( n \geq 0 \). We get from (2.2)

\[
P(\nu(\xi_n) \in (a^n, b^n)) = \frac{\phi'_n(\phi^{-1}_n(s))}{\phi'_{n+1}(\phi^{-1}_{n+1}(s))} E[\nu e^{-\nu/c_n+1}, \nu \in (a^n, b^n)]
\geq \frac{\phi'_n(\phi^{-1}_n(s))}{\phi'_{n+1}(\phi^{-1}_{n+1}(s))} e^{-b^n/c_n} E[\nu, \nu \in (a^n, b^n)] .
\]

From (c) and (2.1), we deduce that for \( n \) large enough, we have

\[
P(\nu(\xi_n) \in (a^n, b^n)) \geq \frac{1}{2m} E[\nu, \nu \in (a^n, b^n)] .
\]

Therefore, under the condition \( E[\nu \ln(\nu)] = \infty \), we have

\[
\sum_{n \geq 0} P(\nu(\xi_n) \in (a^n, b^n)) = \infty .
\]

Let \( H(\xi_n) := \{ u \in \mathbb{T} : u \text{ child of } \xi_n, u \not= \xi_{n+1} \} \). By Remark A, we have

\[
P \left( \sum_{u \in H(\xi_n)} W_\infty(u) \leq a^n \bigg| \mathbb{T}_{n+1}, \xi_{n+1} \right) = E_{GW} \left[ \prod_{u \in H} M_\infty(u), \sum_{u \in H} W_\infty(u) \leq a^n \right]_{\mathcal{H}=H(\xi_n)} .
\]
Since $M_\infty(u) \leq e^{1/cn}$ for any $|u| = n$, we get

$$P \left( \sum_{u \in H(\xi_n)} W_\infty(u) \leq a^n \bigg| \mathcal{T}_{n+1}, \xi_{n+1} \right) \leq e^{(\nu(\xi_n) - 1)/cn} \text{GW} \left( \sum_{u \in H} W_\infty(u) \leq a^n \right)_{H = H(\xi_n)}.$$ 

Let $(W_{\infty,i}, i \geq 1)$ be independent random variables distributed as $W_\infty(e)$ under GW. It follows that on the event $\{\nu(\xi_n) \in (a^n, b^n)\}$, we have

$$P \left( \sum_{u \in H(\xi_n)} W_\infty(u) \leq a^n \bigg| \mathcal{T}_{n+1}, \xi_{n+1} \right) \leq e^{(b^n - 1)/cn} \text{GW} \left( \sum_{i=1}^{a^n} W_{\infty,i} \leq a^n \right) =: d_n.$$ 

We obtain that

$$P \left( \sum_{u \in H(\xi_n)} W_\infty(u) > a^n \right) \geq P \left( \sum_{u \in H(\xi_n)} W_\infty(u) > a^n, \nu(\xi_n) \in (a^n, b^n) \right) \geq P (\nu(\xi_n) \in (a^n, b^n))(1 - d_n).$$

By (c), $e^{(b^n - 1)/cn}$ goes to 1. Furthermore, we know from [13] that $E_{GW}[W_\infty(e)] = \infty$, which ensures by the law of large numbers that $d_n$ goes to 0. By equation (3.1), we deduce that

$$\sum_{n \geq 0} P \left( \sum_{u \in H(\xi_n)} W_\infty(u) > a^n \right) = \infty.$$ 

We use the Borel-Cantelli lemma to see that $\sum_{u \in H(\xi_n)} W_\infty(u) > a^n$ infinitely often. Since $W_\infty(\xi_n) \geq \frac{1}{m} \sum_{u \in H(\xi_n)} W_\infty(u)$, we get that $W_\infty(\xi_n) \geq a^n/m$ for infinitely many $n$, P-a.s. Hence

$$\limsup_{n \to \infty} \frac{1}{n} \ln(W_\infty(\xi_n)) \geq \ln(a).$$

Let $a$ go to $m$ to have the lower bound. Since $W_\infty(\xi_n) \leq \sum_{|u| = n} W_\infty(u) = m^n W_\infty(e)$, we have $\limsup_{n \to \infty} \frac{1}{n} \ln(W_\infty(\xi_n)) \leq \ln(m)$ hence Proposition 3.1. $\square$

We turn to the proof of the theorem.

**Proof of Theorem 1.1.** By Proposition 3.1, we have

$$P \left( \limsup_{n \to \infty} \frac{1}{n} \ln(W_\infty(\xi_n)) = \ln(m) \right) = 1.$$
In particular, for P-a.e. T,

$$P \left( \limsup_{n \to \infty} \frac{1}{n} \ln(W_\infty(\xi_n)) = \ln(m) \mid T \right) = 1.$$ 

By Proposition 2.1, the distribution of \( \xi \) given T is UNIF. Therefore, for P-a.e. T,

$$(3.2) \quad \text{UNIF} \left( r \in S(T) : \limsup_{n \to \infty} \frac{1}{n} \ln(W_\infty(r_n)) = \ln(m) \right) = 1.$$ 

Again by Proposition 2.1, the distribution of T is the one of \( T \) under Q. We deduce that (3.2) holds for Q-a.e. tree T. Since Q and GW are equivalent, equation (3.2) holds for GW-a.e. tree T. We call the H"older exponent of UNIF at ray \( r^* \) the quantity

$$H_\circ(\text{UNIF})(r^*) := \liminf_{n \to \infty} \frac{-1}{n} \ln \left( \text{UNIF}( \{ r \in S(T) : r_n = r_n^* \} ) \right).$$ 

By definition of UNIF, we can rewrite it

$$H_\circ(\text{UNIF})(r^*) = \liminf_{n \to \infty} \frac{-1}{n} \ln \left( m^{-n} W_\infty(r_n^*) / W_\infty(e) \right).$$ 

Therefore, for UNIF-a.e. ray r, \( H_\circ(\text{UNIF})(r) = 0 \). By Theorem 14.15 of [11] (or § 14 of [1]), it implies that \( \dim(\text{UNIF}) = 0 \) GW-almost surely. □

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References


