A model for vortical plumes in rotating convection

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In turbulent rotating convection a typical flow structuring in columnar vortices is observed. In the internal structure of these vortices several symmetries are approximately satisfied. A model for these columnar vortices is derived by prescribing these symmetries. The symmetry constraints are applied to the Navier–Stokes equations with rotation in the Boussinesq approximation. It is found that the application of the symmetries results in a set of linearized equations. An investigation of the linearized equations leads to a model for the columnar vortices and a prediction for the heat flux (Nusselt number) that is very appropriate compared to the results from direct numerical simulations of the full governing equations. © 2008 American Institute of Physics. [DOI: 10.1063/1.2936313]

I. INTRODUCTION

Buoyant convection and the Coriolis force caused by the rotation of our Earth are important forces in the flows in the atmosphere and the oceans. A convenient model for such flows, although not fully compatible, is the rotating Rayleigh–Bénard setting: A horizontally infinite layer of fluid is vertically confined by solid walls rotating around a vertical axis, the bottom wall being at a higher temperature than the top wall. Although the lack of a top wall in the geophysical flows makes the model not directly applicable, the general behavior of the model flow shows considerable similarities to real flow in the atmosphere. Furthermore, in the atmosphere the tropopause can be regarded as a “top wall” to a certain extent.

Especially for the large-scale flows in the atmosphere, the effect of the rotation is dominant. The Rossby number, the ratio between inertial and Coriolis forces, is rather small (O(0.1)). A well-known theorem valid in rotation-dominated flows was formulated by Proudman and experimentally proven by Taylor; it is known as the Taylor–Proudman theorem, which states that for inviscid flow in the limit of small Rossby number (the geostrophic flow regime), the vertical component of the velocity gradient is zero, i.e., the flow is vertically uniform. If the influence of buoyancy is also incorporated, it is found that the situation is expanded to the so-called thermal-wind equilibrium, which, under the same conditions, essentially states that vertical gradients of the horizontal velocity components are only allowed when horizontal temperature gradients exist. Still, the condition of a uniform vertical velocity component remains. In view of these statements a columnar flow structuring is expected, with active (Ekman) boundary layers connecting the columns to the solid walls. The Ekman layers connect the horizontal motions in the bulk flow to the no-slip wall, thereby inducing vertical motion that is again independent of the vertical coordinate (as far as the bulk flow outside of the boundary layers is concerned).

Rotating Rayleigh–Bénard convection in the limit of small Rossby number has been the topic of many experimental, numerical, and theoretical studies. Chandrasekhar studied the onset of convective motion and the flow patterning at onset with linear perturbation theory. Asymptotic expansions in a small parameter are also used in, e.g., Refs. 5 and 6. Experiments with visualizations showed indeed a flow structuring consisting of many columnar vortices. In numerical studies these columnar vortices were also observed.

The prime inspiration for the current investigation stems from another numerical study, the results of which have been partly reported in Ref. 14, where the internal structure of the vortex columns was found to nearly obey certain symmetries. Vertical cross sections of such vortex columns showed that the vertical velocity is nearly symmetric in the midplane, while the vertical vorticity component is antisymmetric in the midplane. For a confined radial extent it seemed also reasonable to consider the vortex column to be independent of the azimuthal orientation. Thus the question arose whether demanding a compliance with these symmetries and conditions would perhaps give solutions to the governing equations, as these solutions would be very helpful for modeling the vortical-convection state and its heat transfer properties. The current work contains the results of this assignment.

Another application area is in deep convection, with related long-lasting vortical chimneys that have been observed in the Greenland Sea. Also, parts of this work might be of interest for problems in spherical-shell convection, relevant for celestial bodies.

In Sec. II the governing equations and the problem setting are introduced, along with the dimensionless numbers needed to specify the flow. Some results of a direct numerical simulation (DNS) of these equations are shown in Sec. III. The symmetry considerations and, subsequently, the so-
II. PROBLEM SETTING, GOVERNING EQUATIONS, AND DIMENSIONLESS PARAMETERS

Consider a horizontally infinite fluid layer, confined vertically by solid walls a distance \( H \) apart. The bottom wall is situated at \( z = -H/2 \) and is at a temperature \( T_0 + \Delta T \), while the top wall at \( z = H/2 \) is at a temperature \( T_0 \). The governing equations for the fluid motion are the Navier–Stokes and heat equations in a rotating reference frame with incompressibility and in the Boussinesq approximation,\(^4\) where the tildes indicate variables with dimension,

\[
\begin{align*}
\partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} + 2 \Omega \tilde{z} \times \tilde{u} &= -\nabla \tilde{p} + g \alpha \tilde{\theta} + \nu \nabla^2 \tilde{u}, \\
\partial_t \tilde{\theta} + \tilde{u} \cdot \nabla \tilde{\theta} - \frac{\tilde{w} \Delta T}{H} &= \kappa \nabla^2 \tilde{\theta}, \\
\nabla \cdot \tilde{u} &= 0,
\end{align*}
\]

with \( \mathbf{u} = (u, v, w) \) the velocity vector, \( \mathbf{z} \) the vertical unit vector (the rotation axis and the gravitational acceleration are also aligned vertically), \( \Omega \) the rotation rate, \( p \) the reduced pressure, \( g \) the gravitational acceleration, \( \theta \) the temperature deviation from the conductive profile \( T(z) = T_0 + \left( \frac{1}{2} - z/H \right) \Delta T \), and \( \alpha, v, \) and \( \kappa \) the thermal expansion coefficient, kinematic viscosity, and thermal diffusivity of the fluid, respectively. The equations can be made dimensionless with the length scale \( H \) (the separation of the plates), the temperature scale \( \Delta T \), and the time scale \( \tau = H/U = \sqrt{H/(g\alpha \Delta T)} \) based on the so-called free-fall velocity \( U = \sqrt{g\alpha \Delta T H} \). The resulting equations are

\[
\begin{align*}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\sigma \mathbf{Ta}}{\text{Ra}} \mathbf{z} \times \mathbf{u} &= -\nabla p + \theta \mathbf{z} + \sqrt{\frac{\sigma}{\text{Ra}}} \nabla^2 \mathbf{u}, \\
\partial_t \theta + \mathbf{u} \cdot \nabla \theta - \frac{w}{\sqrt{\sigma \text{Ra}}} &= \frac{1}{\sigma \text{Ra}} \nabla^2 \theta, \\
\nabla \cdot \mathbf{u} &= 0,
\end{align*}
\]

where all symbols now denote dimensionless variables. The following dimensionless parameters have been introduced: The Rayleigh number \( \text{Ra} = g\alpha \Delta TH^3/(\nu \kappa) \), the Taylor number \( \text{Ta} = (2\Omega H^2/\nu)^2 \), and the Prandtl number \( \sigma = \nu/\kappa \). Another important parameter in the following denotes the importance of the buoyancy force relative to the Coriolis force: The Rossby number \( \text{Ro} = U/(2\Omega H) = \sqrt{\text{Ra}/(\sigma \text{Ta})} \).

The boundary conditions on the plates are prescribed as follows:

\[
\mathbf{u} = 0, \quad \mathbf{z} = \pm \frac{1}{2}, \quad \theta = 0, \quad \mathbf{z} = \pm \frac{1}{2}.
\]

III. DNS OF THE EQUATIONS

The Equations (2a)–(2c) can be used for DNS. Such a study was undertaken before; the results have been partly reported in Ref. 14. The equations were solved on an \( L_x \times L_y \times L_z = 2 \times 2 \times 1 \) domain with a fourth-order accurate finite-difference scheme. Boundary conditions were as in Eq. (3). The horizontal directions were periodic to emulate a horizontally unbounded layer of fluid. For further details on the numerical procedure, we refer to Ref. 14.

Here we provide a typical image from a simulation using \( \text{Ra} = 2.5 \times 10^6 \), \( \sigma = 1 \), and \( \text{Ta} = 10^8 \). In this case \( \text{Ro} = 0.158 \), thus a strong rotational constraint is expected. An isosurface plot of the vertical velocity component in Fig. 1 shows the columnar flow structuring typical of rotation-dominated turbulent convection. The red (light gray) surfaces are for \( w = +0.07 \), so for columns with upward motion, while the blue (dark gray) surfaces at \( w = -0.07 \) are for columns with downward motion. The number of columns with upward and downward motions is roughly equal. Also the sizes are similar. This is expected given the inherent symmetries of the Boussinesq equations. The columnar structuring due to rotation was found in all simulations for which \( \text{Ro} \approx 0.5 \). This can be taken as the border of the range of validity for the current work.

From these simulations it was found that the vortex columns show approximate symmetry in the midplane. As a first-order approximation the vertical vorticity component \( \omega_z \) inside the column is antisymmetric in \( z = 0 \), while \( u_z \) is roughly symmetric in \( z = 0 \). The temperature, after subtraction of the conductive profile \( \theta = T - (1/2 - z) \), shows more deviations from symmetry, but we still assume symmetry in \( z = 0 \) as a first approximation. [The ensemble-averaged profiles of 40 columns with upward motion are shown later, in Fig. 6(a). These symmetries were found to be also valid for the “downward” columns.] The observation of the approximate symmetries was the principal motivation for the current work.

IV. A HEURISTIC MODEL

It is profitable to proceed with cylindrical coordinates \( (r, \phi, z) \) and the associated velocity vector \( (u_r, u_\phi, u_z) \). Our objective is thus to search for (i) axisymmetric, (ii) steady (viz., time independent), and (iii) vertically symmetric solutions according to
Under the assumption of axial symmetry all \( \phi \) derivatives are zero, and the steadiness of the solution implies vanishing of the time derivatives. In view of the prescribed symmetries, Eqs. (2a)–(2c) split up in even and odd parts that vanish separately. This results in

\[
\begin{align*}
  u_r(r,z) &= - u_r(r,-z), & (4a) \\
  u_\phi(r,z) &= - u_\phi(r,-z), & (4b) \\
  u_z(r,z) &= u_z(r,-z), & (4c) \end{align*}
\]

and the conditions that all nonlinear parts be zero. Thus the symmetries lead to the linearized versions of the governing equations. Note that a solution of Eqs. (5a)–(5e) probably does not satisfy the full equations including the nonlinear terms. Still, as a heuristic model we continue with the linearized equations. The linear set of equations is often used in studies concerning the onset of convection, e.g., Refs. 4, 19, and 20. In this case, however, an alternative solution procedure is applied in view of the axial symmetry, leading to a new solution.

Because of Eq. (5e) it is possible to introduce a streamfunction \( \psi(r,z) \) according to

\[
  u_r = - \frac{1}{r} \partial_z \psi, \quad u_z = - \frac{1}{r} \partial_r \psi. \tag{6}
\]

In view of Eqs. (4a) and (4c) \( \psi \) is even in \( z \). The boundary conditions for \( \psi \) are \( \partial_z \psi = \partial_z \psi = 0 \) at \( z = \pm \frac{1}{2} \).

We can now eliminate \( p \) by cross differentiation of Eqs. (5a) and (5c). Three equations remain,

Next, we try to separate variables and write the solution as the product of a function depending on \( z \) and a function depending on \( r \). For the latter, radial part we take either \( J_0(kr) \) (even symmetry) or \( J_1(kr) \) (odd symmetry). Here \( J_0 \) and \( J_1 \) are Bessel functions of orders of 0 and 1, respectively, and \( k \) is a constant to be determined later on. In particular, we use the following ansatz:

\[
\begin{align*}
  \psi &= k r J_1(kr) \Psi(z), \tag{8a} \\
  u_\phi &= J_1(kr) \Phi(z), \tag{8b} \\
  \theta &= J_0(kr) \Theta(z). \tag{8c}
\end{align*}
\]

The boundary conditions translate to

\[
\Phi = \Theta = \Psi = \Psi' = 0, \quad z = \pm \frac{1}{2}. \tag{9}
\]

We insert these trial functions into the set (7). By use of the well-known properties

\[
\partial_z \left[ x^m J_m(x) \right] = x^m J_{m-1}(x)
\]

and

\[
J_{-m}(x) = (-1)^m J_m(x),
\]

valid for integer values of \( m \), we find that the dependence on the radial coordinate \( r \) drops out. What remains are three linear ordinary differential equations in terms of the vertical coordinate \( z \),

\[
\begin{align*}
  \sqrt{\frac{\sigma \text{Ta}}{\text{Ra}}} \Phi' &= - k \Theta + k \sqrt{\frac{\sigma}{\text{Ra}}} (\partial_z - k^2)^2 \Psi, \tag{10a} \\
  - k \sqrt{\text{Ta}} \Psi' &= (\partial_z - k^2) \Phi, \tag{10b} \\
  - k^2 \frac{\sigma \text{Ta}}{\text{Ra}} \Psi &= (\partial_z - k^2) \Theta. \tag{10c}
\end{align*}
\]

By differentiating Eq. (10b) once, applying the operator \( (\partial_z - k^2) \) to Eq. (10a), and substituting, we arrive at a single sixth-order differential equation concerning just \( \Psi \),

\[
(\partial_z - k^2)^3 \Psi + \text{Ta} \Psi'' + k^2 \text{Ra} \Psi = 0. \tag{11}
\]

Equation (11) can be solved by substituting \( \Psi = \exp(kz) \) and finding the roots of the characteristic equation,
\[
\lambda^6 - 3k^2\lambda^4 + (3k^4 + Ta)\lambda^2 + k^2Ra - k^6 = 0.
\]
Eq. \(12\)
The roots of the cubic equation in \(x = \lambda^2\) are
\[
x_1 = k^2 - \frac{2i/3 Ta}{\zeta} + \frac{\zeta}{3(2^{1/3})},
\]
Eq. \(13a\)
\[
x_2 = k^2 + \frac{(1 + i \sqrt{3})Ta}{2^{2/3} \zeta} - \frac{(1 - i \sqrt{3})\zeta}{6(2^{1/3})},
\]
Eq. \(13b\)
\[
x_3 = \lambda^2^*.
\]
Eq. \(13c\)
where \(*\) denotes complex conjugation and the symbol \(\zeta\) is used to represent the real, positive constant,
\[
\zeta = [\sqrt{108}Ta^3 + 27k^2(Ra + Ta)^2 - 27k^2(Ra + Ta)]^{1/3}.
\]

Since only the real parts of these solutions are relevant for the current problem the roots are written in a slightly different way. We introduce the constants \(f\), \(g\), and \(h\) as
\[
f = \frac{1}{2} \sqrt{\sqrt{2}|x_2| + \Re(x_2)},
\]
\[
g = \frac{1}{2} \sqrt{\sqrt{2}|x_2| - \Re(x_2)},
\]
\[
h = \sqrt{-x_1}.
\]

The minus sign in the definition of \(h\) is put there since positive values of \(x_1\) give unphysical solutions, so that relevant values of \(h\) are now real. Concerning \(f\) and \(g\), it holds that
\[
x_2 = [\pm (f + ig)]^2,
\]
\[
x_3 = [\pm (f - ig)]^2.
\]
The general solution reads as
\[
\Psi = a_1 \cos(hz) + a_2 \sinh(fz) \sin(gz) + a_3 \cosh(fz) \cos(gz) + a_4 \sinh(fz) \cos(gz) + a_5 \cosh(fz) \sin(gz) + a_6 \sin(hz),
\]
Eq. \(14\)
where the coefficients \(a_{1,2,3}\) are for the even parts, while \(a_{4,5,6}\) are for the odd parts. The odd parts can be taken out since \(\Psi\) must be even, which implies that \(a_4 = a_5 = a_6 = 0\).

From Eq. \(10b\) a similar notation is now obtained for \(\Phi\), with the even parts already left out as \(\Phi\) is odd,
\[
\Phi = b_1 \sin(kz) + b_2 \sin(hz) + b_3 \sinh(fz) \cos(gz) + b_4 \cosh(fz) \sin(gz).
\]
Eq. \(15\)
Coefficients \(b_{2,3,4}\) are found in terms of \(a_{1,2,3}\) from Eq. \(10b\),
\[
b_2 = -a_1 \sqrt{\frac{hk}{h^2 + k^2}},
\]
\[
b_3 = k \sqrt{\frac{(a_2^2 - a_3^2)(f^2 + g^2)}{(f^2 - g^2 - k^2)^2 + 4f^2g^2}},
\]
\[
b_4 = -k \sqrt{\frac{(a_2^2 + a_3^2)(f^2 + g^2) - (a_2 f - a_3 g)k^2}{(f^2 - g^2 - k^2)^2 + 4f^2g^2}}.
\]
The coefficient \(b_1\) is not determined by \(a_{1,2,3}\) and needs to be solved along with these coefficients. Similarly, \(\Theta\) is written as
\[
\Theta = c_1 \cos(hz) + c_2 \sinh(fz) \sin(gz) + c_3 \cosh(fz) \cos(gz) + c_4 \cosh(kz).
\]
Eq. \(16\)
With Eqs. \(10a\) and \(10c\) \(c_1 - c_4\) can be fully expressed in terms of the previous unknowns,
\[
c_1 = k^2 \frac{\sqrt{\sigma Ra}}{h^2 + k^2} a_1,
\]
\[
c_2 = -k^2 \sqrt{\frac{2a_3 f g + a_2 (f^2 - g^2 - k^2)}{(f^2 - g^2 - k^2)^2 + 4f^2g^2}},
\]
\[
c_3 = -k^2 \sqrt{\frac{2a_3 f g - a_2 (f^2 - g^2 - k^2)}{(f^2 - g^2 - k^2)^2 + 4f^2g^2}},
\]
\[
c_4 = -k \sqrt{\frac{\sigma Ta}{Ra} b_1}.
\]

Now the boundary conditions are applied. Note that, because of symmetry, only the boundary conditions at \(z = \frac{1}{2}\) are evaluated; the conditions at \(z = -\frac{1}{2}\) are then automatically satisfied. We introduce a matrix \(M\) such that the boundary conditions are
\[
M \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ b_1 \end{pmatrix} = 0.
\]
Eq. \(17\)
\(M\) has the rather unwieldy shape
\[
M = \begin{pmatrix}
\cos(\frac{\zeta}{2}) & \sinh(\frac{\zeta}{2}) & \cosh(\frac{\zeta}{2}) & \cos(\frac{\zeta}{2}) & 0 \\
-k \sin(\frac{\zeta}{2}) & -\cos(\frac{\zeta}{2}) & -\sin(\frac{\zeta}{2}) & -\cosh(\frac{\zeta}{2}) & 0 \\
2\sqrt{\frac{\sigma H^4}{H^2 + k^2}} & 2\sqrt{\frac{\sigma H^4}{H^2 + k^2}} & 2\sqrt{\frac{\sigma H^4}{H^2 + k^2}} & 2\sqrt{\frac{\sigma H^4}{H^2 + k^2}} & 0 \\
\sqrt{H H^4} & \sqrt{H H^4} & \sqrt{H H^4} & \sqrt{H H^4} & -\sqrt{\frac{\sigma H^4}{H^2 + k^2}} \cos(\frac{\zeta}{2}) \\
\end{pmatrix}
\]
With given $Ra$, $Ta$, and $\sigma$, the determinant of $M$ vanishes for values of $k$ that form nontrivial solutions. These roots can be found with iterative numerical methods. Finally, to obtain a solution we must determine the constants $a_{1,2,3}$ and $b_{1}$. Three of these can be expressed in terms of $a_{1}$, where $M_{ij}$ denote the individual matrix elements of $M$,

\begin{align*}
  a_{2} &= \frac{M_{13}M_{21} - M_{11}M_{32}}{M_{12}M_{23} - M_{13}M_{22}} a_{1},
  a_{3} &= \frac{M_{12}M_{21} - M_{11}M_{22}}{M_{13}M_{23} - M_{12}M_{22}} a_{1},
  b_{1} &= \frac{a_{1}}{M_{12}M_{23}M_{34} - M_{13}M_{22}M_{34}} (M_{13}M_{22}M_{31} \\
  &- M_{12}M_{23}M_{31} - M_{13}M_{21}M_{32} + M_{11}M_{23}M_{32} \\
  &+ M_{12}M_{21}M_{33} - M_{13}M_{22}M_{33}).
\end{align*}

The last unknown constant $a_{1}$ is a free parameter. An upper bound for $a_{1}$ can be given on physical grounds by demanding that the temperature $\theta + \frac{1}{2} - z$ remains between 0 and 1 for all $z$. This will be used later on to give an upper bound for the heat flux transfer through the vortical plume.

**V. RESULTS OF THE MODEL**

In this section we first show the general shape of the model solution. A comparison to the DNS is carried out, with variation of the governing parameters. Finally, an upper bound for the heat flux through an array of vortices is derived.

The first case we consider is for fixed values $Ra=2.5 \times 10^{6}$, $\sigma=1$. This is equal to the parameters used in Ref. 14. Figure 2 shows the values of $k$ as a function of $Ta$. There are two limits to the parameters that are applied here. The first gives an upper limit for $Ta$ as a function of $Ra$; it is the Chandrasekhar stability criterion for onset of convection $Ra_{c} \approx 8.7Ta^{2.3}$. For given $Ta$, $Ra$ must be larger than $Ra_{c}$ for convection to set in. It is found that the highest allowable $Ta$ value that allows a solution in the model ($Ta=2.08 \times 10^{6}$ at $k=28.1$) matches rather well with the Chandrasekhar limit value $Ta_{c}=1.54 \times 10^{8}$. We choose a lower limit of $Ta$ by demanding that the flow is rotation dominated, i.e., $Ro \ll 1$ and thus $Ra \ll \sigma Ta$.

Another observation considers the growing number of possible solutions as $Ta$ decreases. Then, two branches are formed: One at $k \approx 39$, the “narrow vortex,” and another with a stronger dependence on $Ta$, the “wide vortex.” Of the two, the narrow-vortex solution is of interest here because it is most comparable to the DNS results. Indeed, for the wide vortex its radius $r_{0} \sim 1/k$ grows very large, even at moderate $Ta$. This is not what is captured in the DNS, where far narrower plumes are found at all relevant $Ta$ values. Furthermore, velocity values for the wide vortex are very small, while the motions in the DNS vortices are considerably more vigorous. Therefore, the focus will be on the narrow vortex, while the wide vortex is also a physically allowable solution. For $Ta \approx 1.5 \times 10^{7}$ another pair of $k$ values is found. These correspond to solutions that are less relevant in this context since the resulting structures are composed of more than a single cell in the vertical direction, i.e., the vertical velocity is not of one sign throughout the vertical extent. The growing number of allowed $k$ values is possibly an indication of the growing instability of the vortex solutions, as more and more three-dimensional structuring is allowed at lower dimensionless rotation rates $Ta$.

Variation of $Ra$ is covered in Fig. 3. For this plot fixed values $Ta=1 \times 10^{6}$ and $\sigma=1$ are chosen. Again the applicability of the current model is bounded by the Chandrasekhar limit $Ra_{c}$ and the condition of rotation-dominated flow $Ro \ll 1$, both indicated with dashed lines. A similar picture arises, with two branches of allowable $k$ values, deviating as $Ra$ increases. At higher $Ra \approx 8 \times 10^{6}$ additional solutions are found; these are again solutions with more complicated vertical structuring and thus are not relevant here.

The allowed $k$ values are independent of $\sigma$. The $\sigma$ dependence is present only in a single row of $M$, and hence can be factored out of $\text{det}(M)=0$.

As an example case for the spatial structure of the vortex solution we choose $Ta=10^{6}$, with $Ra=2.5 \times 10^{6}$ and $\sigma=1$ as before. The relevant $k$ value comes from the upper branch of solutions in Fig. 2; for the current parameters $k=36.8$. With the $k$ value the model solution is fully determined but for the
constant $a_1$. We wish to restrict the results to just the “warm” vortices that are responsible for upward transport of fluid and heat and are of higher-than-average temperature. The corresponding “cold” vortex solution follows straightforwardly from symmetry. For the warm vortex, by demanding that temperature $T$ remains below the boundary value $T = 1$ for all $z$, we can derive an upper bound for $a_1$. For the current parameter set we arrive at $a_1 = 1.28 \times 10^{-4}$.

To give a visual impression of the solution, contour plots of $u_r$, $u_\phi$, $u_z$, and $\theta$ are presented as a function of $r$ and $z$ in Fig. 4. Another important result for the vortex solution is the vertical component of vorticity $\omega_z$, easily calculated from $u_\phi$ as $\omega_z = (1/r)\partial_z(ru_\phi)$ given the symmetry. It is presented in Fig. 5. The solutions have vertically dominant sine-or cosine-like behavior in the bulk with boundary layers to connect to the walls. In the radial direction the Bessel function profiles are readily recognized.

We can compare the model solution at $Ra = 2.5 \times 10^5$, $Ta = 1 \times 10^5$, and $\sigma = 1$, as introduced in Figs. 4 and 5, to actual vortices found in the DNS. An ensemble average of 40 warm vortices, found in the DNS, has been carried out. Vertical profiles taken at the horizontal position of the center of the vortices of vertical velocity $u_z$, vertical vorticity $\omega_z$, and temperature $T$ are shown in Fig. 6(a). In Fig. 6(b) the corresponding profiles from the model are given. Similarly, in Fig. 7(a) a horizontal profile of the average DNS vortex at height $z = -0.25$ is presented, with the corresponding model profiles displayed in Fig. 7(b). The horizontal profiles also include azimuthal velocity $u_\phi$. Note that the model vortex is an “upper limit” in terms of the constant $a_1$, so that the DNS vortex will generally have lower temperature, velocity, and vorticity.

The qualitative comparison is quite favorable, but there are several differences to be found. The vertical symmetry is not exactly followed in DNS. However, as a first-order approximation it is definitely applicable. Also, the radius of the vortex in DNS is somewhat wider than in the model. The first zero crossings in the horizontal profiles of $u_z$ or $\omega_z$ are found at $r = 0.066$ in the model and at $r = 0.090$ in the DNS vortex. The vertical velocity is about a factor of 2 larger in the DNS vortex. Ver- the model than in DNS, while the maximal azimuthal veloc- ity is almost equal for the two cases. The vertical-vorticity profiles match well, apart from an asymmetric offset in the vertical profile in the DNS vortex. An oppositely signed shield is found for larger $r > 0.090$ in the DNS vortex. At the outer edge of this shield the velocities and vorticity tend to

![FIG. 4. Radial-vertical cross-sectional plots of the vortex solution at $Ra = 2.5 \times 10^5$, $Ta = 1 \times 10^5$, and $\sigma = 1$. The solid contours indicate positive values, while dashed contours are for negative values. The dotted lines are the zero contours.](image1)

![FIG. 5. Plot of vertical vorticity $\omega_z$ of the vortex solution. $Ra = 2.5 \times 10^5$, $Ta = 10^5$, and $\sigma = 1$. Lines as in Fig. 4, but now with contour increment of 0.4.](image2)
zero. The shield may be represented in the model by extending
the radial extent to \( r_0 = j_{0.2}/k \) (the notation \( j_{0.2} \) is used to
indicate the second zero of \( \int_0^{2\pi} J_0(r_0) dr_0 \approx 5.52 \), as is done in Fig.
7(b) with \( r_0 = 0.15 \). The sharp edge is unphysical, but again it
should be regarded a first-order approximation to the DNS vortex.

The heat flux of convective systems is usually expressed
by the Nusselt number \( Nu \), the total heat flux normalized by
the conductive heat flux that is present in a quiescent fluid. In
the current units \( Nu \) is defined as

\[
Nu = \left\langle \frac{\partial T}{\partial z} \right\rangle + \sqrt{\sigma Ra} \langle u_r T \rangle ,
\]

where the angular brackets indicate averaging over a hori-
izontal cross section of the fluid layer. The intersection is
taken at \( z = 0 \), the central plane. What remains of \( Nu \) is then

\[
Nu = 1 + \sqrt{\sigma Ra} \langle u_r \theta \rangle .
\]

To calculate \( Nu \) for the model, the following assumptions are
made.

1. As before, the value of \( a_1 \) is taken the maximal value that
still complies with the condition \( T \leq 1 \). This gives
an upper bound on \( a_1 \), and thus also on \( Nu \).

2. There are equally many warm as cold vortices. This has
been verified from the DNS.

3. Each vortex is radially terminated at \( r_0 \). This is done to
create a shield. We remark here that taking higher zeroes
of \( J_0 \) would lead to radial profiles incompatible with the
DNS results, as there is just one shield to be found.

4. The entire cross-sectional area is filled with a closest-
packed hexagonal grid of circular vortices. Again, this
situation provides an upper bound as real snapshots from
DNS show a sparser vortex distribution. By the model
symmetry, contributions from warm and cold vortices
are identical. Hence, this assumption allows the aver-
ing to be constrained to just a single model vortex, with
the area of consideration being one hexagon of inner
radius \( r_0 \) and area \( 2\sqrt{3}r_0^2 \). Another implication of this
assumption that will be discussed later is that the vortex
number density \( N \) scales as \( N = r_0^{-2} \).

We arrive at

\[
Nu = 1 + \frac{\sqrt{\sigma Ra}}{2\sqrt{3}r_0} \int_0^{r_0} \Theta(0)J_0(kr') \cdot \Psi(0) \frac{\partial J_1(kr')}{\partial r'} dr',
\]

which finally reduces to

\[
Nu = 1 + \frac{\pi \sqrt{\sigma Ra}}{2\sqrt{3}} \Theta(0) \Psi(0) k^2 J_1(j_{0.2})^2 .
\]

A calculation of the upper bound on \( Nu \) for various \( Ta \) at
\( Ra = 2.5 \times 10^6 \), \( \sigma = 1 \) has been carried out. The results are indi-
cated with the thick solid line in Fig. 8. Also included are

FIG. 6. (a) Vertical profiles of vertical velocity \( u_z \) (dashed line; multiplied by 5), vertical vorticity \( \omega_z \) (thick solid line; divided by 5), and temperature \( T \) (thin solid line), from the ensemble-
averaged DNS vortex column. (b) Corresponding profiles from the current vortex model.

FIG. 7. (a) Horizontal profiles of vertical velocity \( u_r \) (dashed line), temperature \( T \) (thin solid line; shifted downward by 0.5), vertical vorticity \( \omega_r \) (thick solid line; divided by 10), and azimuthal velocity \( u_\phi \) (dash-dotted line; it is the velocity component in the horizontal plane perpendicular to the cross-sectional line) taken at \( z = 0.25 \) through the ensemble-
averaged DNS vortex column. (b) Corresponding profiles from the current vortex model; the temperature curve (thin solid line) is shifted downward by 0.65 instead.
results from the DNS (Ref. 14) (crosses and circles), results from an experiment in water at $\sigma=7$ (Ref. 21) (thin solid line), and an upper bound at $\sigma\to\infty$ obtained with a variational method (Ref. 22) (dash-dotted line). It is found that the current model overestimates the heat flux by about a factor of 2. Given the unknown constant $a_1$ and the assumptions of the radial cutoff and tightly packed coverage of the fluid layer with vortices this is indeed a satisfying result. It is comparable to the result of Ref. 22 except for the small growth of $Nu$ at moderate $Ta$. At the larger $Ta$ values the steep decrease of $Nu$ is well represented.

We now compare the effects of variation of $Ta$ between the current theory and the DNS. As can be seen in Fig. 2, the relevant value of $k$ (the upper branch of solutions) remains almost constant when $Ta$ changes, so that vortices are expected to be of similar size for all $Ta$ considered. A comparison of DNS snapshots at $Ta=1\times10^7$ and $1\times10^8$ revealed that indeed in both cases on average roughly the same number of vortices can be identified (by visual inspection). However, at $Ta=1\times10^7$ the rotational constraint is less stringent and more vertical variation is found in the velocity, temperature, and vorticity profiles. Also, at this lower rotation rate the velocities inside the vortices are higher, as is the temperature contrast between the surroundings, such that the heat transfer ($Nu$) is considerably higher.

Variation of $Ra$ is also straightforward in the model, see Fig. 9. The solid line indicates the model result at $Ta=1\times10^8$, $\sigma=1$, while the circles are the corresponding values from DNS. The rapid drop in the model curve near $Ra_c$ is an indication of the model reaching the Rayleigh number where no vortex solution is allowed, see also Fig. 3 where for $Ra\leq1.5\times10^6$ no suitable $k$ value can be found. At higher $Ra$ it is found that the upper bound on $Nu$ obtained from the model has a somewhat larger separation from the DNS result, but is still very much applicable. Extension to even higher $Ra$ is unfeasible, as then the constraint $Ro=1$ is crossed.

An additional effect of the variation of $Ra$ can be appreciated from Fig. 3. With increasing $Ra$ the relevant $k$ value also increases. This means that the typical vortex size must decrease. In other words, the vortex density increases with increasing $Ra$. If we define a typical vortex density as $N\approx r_0^{-2}=(\bar{v}_0/k)^2$ and compare these for $Ra=2.5\times10^6$ and $2.5\times10^7$, then in the model the higher-$Ra$ case has 3.67 times as many vortices as the lower-$Ra$ case. Visual inspection of DNS snapshots indeed showed that the vortices are smaller at higher $Ra$, but the factor of 3.67 was not recovered. Instead, the higher-$Ra$ case had, on average slightly more than two times as many vortices as at the lower $Ra$. This may explain the larger model overprediction of $Nu$ at higher $Ra$ compared to the DNS result.

Another approach is to investigate variation of $Ra$ with an additional constraint that the ratio of buoyancy and rotation, i.e., the Rossby number $Ro$, remains the same. At a constant Prandtl number, $Ra$ and $Ta$ are varied such that $Ro\sim\sqrt{Ra/Ta}$ remains constant. This approach was followed before in literature, e.g., Refs. 12 and 23. In Fig. 10 the dependence of $Nu$ on $Ra$ (and thus, implicitly, $Ta$) is shown, while keeping $Ro$ constant. This is plotted for three separate values of $Ro$. Also included are our DNS results at $Ro=0.75$ (circles and crosses). The three curves at different $Ro$ are roughly parallel; a power-law fit of the slopes results in $Nu\sim Ra^{0.57}$ at $Ro=0.1$, $Nu\sim Ra^{0.48}$ at $Ro=0.2$, and $Nu\sim Ra^{0.47}$ at $Ro=0.75$. In the $Ro=0.1$ curve the sudden decrease to $Nu=1$ at $Ra\approx3.5\times10^6$ occurs because no solution of the model is found anymore, cf. Figs. 3 and 9. When comparing these slopes to results from the DNS of Ref. 12 or the experiments of Ref. 23 (in both studies $Nu\sim Ra^{0.27}$ is found) it is found that the model progressively overestimates the heat flux at larger $Ra$ for constants $Ro$ and $\sigma$; or the bound becomes less and less stringent.
VI. CONCLUDING REMARKS

Starting from the Navier–Stokes equations and some symmetry considerations, it was possible to derive a model for the vortical columnar plumes of rotation-dominated convection. The radial and vertical structures of the vortex as predicted by the model matched the features found in the DNS rather well. A calculation of the heat flux provided an upper bound that was found to be appropriate for the regime under study.

Still, the model has some shortcomings. There is an unknown constant in the model, for which only an upper bound is known. The termination of the radial extent of the vortex is unphysical. Also, the heat flux is dependent on the vertical coordinate and cannot match with the often-used definition of the averaged wall-normal temperature gradient (see, e.g., Ref. 14). This follows directly from the symmetry constraint on the temperature deviation from the conductive profile.

A comparison of the model results with DNS showed that the average vortex size was captured well. The dependence of the heat transfer on $\sigma$ was very satisfactorily represented in the model. When $Ra$ is increased the upper bound for heat transfer in the model has a larger margin over the DNS results, but for the parameter range considered, it still gives a reasonable result. The relation with $\sigma$ could not be checked directly to DNS results. A comparison to experiments and simulations in different geometries revealed that the Prandtl number dependence in the model is not well recovered. The range of validity of the model solution appears to be restricted to lower Prandtl numbers.

This study shows that the linearized Navier–Stokes equations not only provide information about the onset of instabilities but also possess interesting and relevant classes of solutions that fulfill certain symmetry requirements.

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