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On accuracy of multivariate compound Poisson approximation

Abstract

We present multivariate generalisations of some classical results on accuracy of Poisson approximation for the distribution of a sum of 0–1 random variables.

1 Introduction

Let $X, X_1, X_2, \ldots$ be a stationary sequence of dependent random variables (r.v.s). The key object in Extreme Value Theory is the number of exceedances

$$N_n(u) = \sum_{i=1}^{n} \mathbb{I}\{X_i > u\}.$$ 

Investigation of $N_n(u)$ is motivated by applications in finance, insurance, network modelling, meteorology, etc. (cf. [11, 19]).

In the independent case, $N_n(u)$ has binomial $\text{B}(n, p)$ distribution, where $p = \mathbb{P}(X > u)$. If $p$ is "small" then $\mathcal{L}(N_n(u))$ may be approximated by the Poisson $\mathcal{P}(np)$ distribution. Accuracy of Poisson approximation for a binomial distribution has been investigated by famous authors (see, e.g., [17, 14, 10, 3] and references in [6]).

The case of a sum of dependent 0–1 random variables was the subject of [9, 2, 3] (see also references in [3]).

The natural measure of closeness of discrete distributions is the total variation distance (TVD). Recall the definition of the TVD between the distributions of random vectors $X$ and $Y$ taking values in $\mathbb{Z}^n_+$, where $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$:

$$d_{TV}(X; Y) \equiv d_{TV}(\mathcal{L}(X); \mathcal{L}(Y)) = \sup_{A \subseteq \mathbb{Z}} |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)|.$$ 

Let $\pi$ be a Poisson random variable with the parameter $np$. According to Barbour and Eagleson [2],

$$d_{TV}(N_n(u); \pi) \leq (1 - e^{-np}) p. \quad (1)$$ 

This is probably the best universal estimate of the TVD between binomial and Poisson distributions; it improves the results of Prokhorov [17] and LeCam [14]. Sharper bounds are available under extra restrictions (see [10, 20]).
Dependence can cause clustering of extremes, and the Poisson approximation may no longer be valid. It is known that under a mild mixing condition, the limiting distribution of $N_n(u)$ is compound Poisson.

Accuracy of compound Poisson approximation for $\mathcal{L}(N_n(u))$ has been evaluated in [1, 15, 18], among others. The feature of the estimate given in [15] is that it coincides with (1) in the particular case of independent r.v.s.

A natural problem is to investigate the distribution of the vector

$$N_n = (N_n(u_1), ..., N_n(u_m))$$

of the numbers of exceedances given a set of distinct levels $u_1, ..., u_m$. The problem has applications in insurance and finance. For instance, a stationary sequence $\{X_i\}$ of (dependent) random variables can represents claims to an insurance company. Let $N(u_i)$ denote the number of claims exceeding a level $u_i$. It can be of interest to approximate the probability that the number of claims exceeding $u_i$ equals $n_i$, $1 \leq i \leq m$. This question can be easily addressed if the distribution of the vector $N_n$ has been approximated.

We show that under natural conditions, the limiting distribution of $N_n$ is necessarily compound Poisson. We evaluate accuracy of multivariate compound Poisson approximation for the distribution of $N_n$. In particular, we improve the corresponding results of Barbour et al. [4] and Novak [15]. In the case of independent trials, our result yields an estimate of accuracy of multivariate Poisson approximation for a multinomial distribution.

### 2 Results

We may assume $u_1 > ... > u_m$. Let $\mathcal{F}_{a,b} \equiv \mathcal{F}_{a,b}(u_1, ..., u_m)$ be the $\sigma$-field generated by the events $\{X_i > u_j\}$, $a \leq i \leq b, 1 \leq j \leq m$. Denote

$$\alpha(l) \equiv \alpha(l, \{u_1, ..., u_m\}) = \sup | \text{IP}(AB) - \text{IP}(A)\text{IP}(B) |,$$

$$\beta(k) \equiv \beta(l, \{u_1, ..., u_m\}) = \sup \mathbb{E} \sup_B | \text{IP}(B|\mathcal{F}_{l,j}) - \text{IP}(B)|,$$

where the supremum is taken over all $A \in \mathcal{F}_{l,i}, B \in \mathcal{F}_{j+i+1,n}, j \geq 1$, such that $\text{IP}(A) > 0$.

**Condition** $\Delta_m \equiv \Delta_m(\{u_1, ..., u_m\})$ is said to hold if

$$\alpha_n \equiv \alpha(l_n, \{u_1, ..., u_m\}) \to 0$$

for some sequence $\{l_n\} \subset \mathbb{Z}_+$ such that $l_n/n \to 0$ as $n \to \infty$. A vector $Y$ has a multivariate compound Poisson distribution $\Pi(\lambda, \mathcal{L}(Z))$ if

$$Y = \sum_{i=1}^{\pi} Z_i,$$
where $Z, Z_1, \ldots$ are i.i.d. random vectors, $\pi$ is independent of $\{Z_i\}$ and has the Poisson distribution with parameter $\lambda$.

**Theorem 1** Assume condition $\Delta_m$, and suppose that $u_m \equiv u_m(n)$ obeys

$$\limsup nP(X > u_m) < \infty. \quad (2)$$

If $N_n$ converges weakly to a random vector $Y$ then $Y$ has a multivariate compound Poisson distribution.

Let $\zeta(n), \zeta_1(n), \zeta_2(n), \ldots$ be independent random vectors with the common distribution

$$\mathcal{L}(\zeta(n)) = \mathcal{L}(N_r | N_r(u_m) > 0), \quad (3)$$

where $r \in \{1, \ldots, n\}$. The proof of Theorem 1 shows that $Y \overset{d}{=} \Pi(\lambda, \mathcal{L}(Z))$, where $\lambda = -\lim_{n \to \infty} \ln P(N_n(u_m) = 0)$ and $\mathcal{L}(\zeta)$ is the weak limit of $\mathcal{L}(\zeta(n))$ for an appropriate sequence $r = r_n$.

Denote

$$p = P(X > u_m), \quad q = P(N_r(u_m) > 0), \quad k = \lfloor n/r \rfloor, \quad r' = n - rk,$$

and let $\pi$ be a Poisson random variable with parameter $kq$.

In Theorem 2 below we approximate the distribution of $N_n$ by the multivariate compound Poisson distribution $\mathcal{L}(N)$, where $N = \sum_{i=1}^r \zeta_i(n)$.

**Theorem 2** If $n > r > l \geq 0$ then

$$d_{\text{tv}}(N_n; N) \leq (1 - e^{-np})rp + (2nr^{-1}l + r')p + nr^{-1} \min\{\beta(l); \kappa(l)\}, \quad (4)$$

where $\kappa(l) = 2(1 + 2/m) \{2^{m-1}m^2\alpha^2(l)\}^{1/(2+m)}$ if $m2^{m\alpha(l)} \leq 1$, otherwise $\kappa(l) = 1$.

Barbour et al. [4] evaluated accuracy of compound Poisson approximation for general empirical point processes of exceedances in terms of a weaker Wasserstein-type distance $d_w$. Concerning the approximation $\mathcal{L}(N_n) \approx \mathcal{L}(N)$, Theorem 3.1 in [4] yields $d_w(N_n; N) \leq \left(1.65(1 - rp)^{-1/2} + e^{rp}\right)rp + 2(2rp + nr^{-1}l)p + nr^{-1}\beta(l)$. In the case $m = 1$ (the 1-dimensional situation), (4) improves a result from [15] (cf. also [1]). If $m = 1$ and the random variables $\{X_i\}$ are independent then (4) with $l = 0, r = 1$ yields (1).

As a consequence of Theorem 2, we derive an estimate of accuracy of multivariate Poisson approximation for a multinomial distribution.

Let $i = (i_1, \ldots, i_m)$, where $i_1 \leq \ldots \leq i_m$. Denote $i^* = (i_1, i_2 - i_1, \ldots, i_m - i_{m-1})$,

$$N^*_n = (N_n(u_1), N_n(u_1, u_2), \ldots, N_n(u_{m-1}, u_m)),$$
where \( N_n(u,v) = \sum_{i=1}^{n} \mathbb{I}\{u \geq X_i > v\} \) as \( u > v \). Evidently, the distribution of \( N_n \) determines that of \( N^*_n \) and vice versa.

The statement of Theorem 2 can be reformulated as follows: if \( n > r > l \geq 0 \) then

\[
d_{TV}(N^*_n; N^*_r) \leq (1 - e^{-np})rp + (2nr^{-1}l + r')p + nr^{-1} \min\{\beta(l); \kappa(l)\},
\]

where \( N^*_n = \sum_{i=1}^{n} \zeta^*_i(n) \), random vectors \( \zeta^*_i(n), \zeta^*_1(n), \ldots \) are independent and have the common distribution \( \mathbb{P}(\zeta^*_i(n) = i^*) = \mathbb{P}(\zeta_i(n) = i) \).

If the random variables \( \{X_i\} \) are independent and \( r = 1 \) then \( N^*_n \) has the multinomial distribution \( B(n, p_1, ..., p_m) \) with parameters \( p_1 = \mathbb{P}(X > u_1), p_2 = \mathbb{P}(u_1 \geq X > u_2), \ldots, p_m = \mathbb{P}(u_{m-1} \geq X > u_m) \):

\[
\mathbb{P}(N^*_n = (l_1, ..., l_m)) = \frac{n!}{l_1! ... l_m!(n - l)!} p_1^{l_1} ... p_m^{l_m} (1 - p)^{n-l},
\]

where \( l = l_1 + ... + l_m \leq n, p = p_1 + ... + p_m \). Theorem 2 yields an estimate of accuracy of multivariate Poisson approximation for the multinomial distribution \( B(n, p_1, ..., p_m) \).

**Corollary 3** Let \( \pi_1, ..., \pi_m \) be independent Poisson random variables with parameters \( np_1, ..., np_m \). Denote \( Y = (\pi_1, ..., \pi_m) \). If \( L(Y_n) = B(n, p_1, ..., p_m) \) then

\[
d_{TV}(Y_n; Y) \leq (1 - e^{-np})p.
\]

**3 Proofs**

**Proof** of Theorem 2 incorporates some ideas from [15] and results of Berbee [5] and Bradley [8].

Denote \( \mathbb{I}_i = (\mathbb{I}\{X > u_1\}, ..., \mathbb{I}\{X > u_m\}) \), and let

\[
N_{r,j} = \sum_{i=j+1}^{(j+1)r \wedge n} \mathbb{I}_i \quad (0 \leq j \leq k = \lfloor n/r \rfloor).
\]

Evidently, \( N_n = \sum_{j=0}^{k} N_{r,j} \). Notice that the last block \( N_{r,k} \) may be omitted:

\[
d_{TV}\left(N_n; \sum_{j=0}^{k-1} N_{r,j}\right) \leq \mathbb{P}(N_{r,k} \neq 0) \leq r'p.
\]

Following Bernstein's "blocks" approach, we subtract a subblock of length \( l \) from each block \( X_{jr+1}, ..., X_{(j+1)r} \) of length \( r \). Denote

\[
N^*_{r,j} = \sum_{i=j+1}^{(j+1)r-l} \mathbb{I}_i, \quad N^*_n = \sum_{j=0}^{k-1} N^*_{r,j} \quad (0 \leq j < k).
\]
Then \( \text{IP} \left( \sum_{j=0}^{k-1} N_{r,j} \neq \sum_{j=0}^{k-1} N_{r,j}^* \right) \leq k \text{IP} \left( N_{r,0} \neq N_{r,0}^* \right) \leq klp. \)

Let \( \{N_{r,j}^*\} \) be independent copies of \( N_{r,0}^* \). Denote

\[
S_i = \sum_{j=0}^{i-1} N_{r,j} + \sum_{j=i+1}^{k-1} \hat{N}_{r,j}^* \quad (0 < i < k).
\]

Notice that \( S_j + \hat{N}_{r,j}^* = S_{j-1} + N_{r,j-1}^* \). We apply Lindeberg's device (cf. [15]) in order to replace \( \{N_{r,j}^*\} \) by \( \{\hat{N}_{r,j}^*\} \):

\[
\text{IP} \left( \sum_{j=0}^{k-1} N_{r,j}^* \in A \right) - \text{IP} \left( \sum_{j=0}^{k-1} \hat{N}_{r,j}^* \in A \right) = \sum_{j=1}^{k-1} \{ \text{IP} (S_j + N_{r,j}^* \in A) - \text{IP} (S_j + \hat{N}_{r,j}^* \in A) \}.
\]

According to Berbee's lemma ([5], ch. 4), the random vectors \( \sum_{i=0}^{j-1} N_{r,i}^* \), \( N_{r,j}^* \) and \( \hat{N}_{r,j}^* \) can be defined on a common probability space so that \( \text{IP} (N_{r,j}^* \neq \hat{N}_{r,j}^*) \leq \beta(l) \). Therefore,

\[
\left| \text{IP} \left( \sum_{j=0}^{k-1} N_{r,j}^* \in A \right) - \text{IP} \left( \sum_{j=0}^{k-1} \hat{N}_{r,j}^* \in A \right) \right| \leq k \beta(l).
\]

The mixing coefficient \( \alpha \) is weaker than \( \beta \). Using Lemma 4 below, we evaluate

\[
\left| \text{IP} \left( \sum_{j=0}^{k-1} N_{r,j}^* \in A \right) - \text{IP} \left( \sum_{j=0}^{k-1} \hat{N}_{r,j}^* \in A \right) \right| \text{ in terms of } \alpha(l).\]

Note that \( \text{IE} |N_{r,0}^*| = rp \).

Inequality (10) with \( b = 1 \) and \( y = rp \) entails the random vectors \( \sum_{i=0}^{j-1} N_{r,i}^* \), \( N_{r,j}^* \) and \( \hat{N}_{r,j}^* \) can be defined on a common probability space so that \( \text{IP} (N_{r,j}^* \neq \hat{N}_{r,j}^*) = \text{IP} \left( |N_{r,j}^* - \hat{N}_{r,j}^*| \geq 1 \right) \leq \kappa(l) \) if \( m2^{(m-1)/2} \alpha(l) \leq 1 \). Hence

\[
\left| \text{IP} \left( \sum_{j=0}^{k-1} N_{r,j}^* \in A \right) - \text{IP} \left( \sum_{j=0}^{k-1} \hat{N}_{r,j}^* \in A \right) \right| \leq k \min\{\beta(l); \kappa(l)\}.
\]

Let \( \{\hat{N}_{r,j}\} \) be independent copies of \( N_{r,0} \), and set \( \hat{N}_n = \sum_{j=0}^{k-1} \hat{N}_{r,j} \). Evidently,

\[
\text{IP} \left( \sum_{j=0}^{k-1} \hat{N}_{r,j} \neq \sum_{j=0}^{k-1} \hat{N}_{r,j}^* \right) \leq klp. \]

Combining our estimates, we get

\[
d_{TV} \left( N_n; \hat{N}_n \right) \leq 2klp + r'p + k \min\{\beta(l); \kappa(l)\}.
\]

Denote \( \mu = \sum_{j=0}^{k-1} \mathbb{I}\{\hat{N}_{r,j} \neq 0\} \), and put

\[
Z_0 = 0, Z_j = \zeta_1(n) + \ldots + \zeta_j(n) \quad (j \geq 1).
\]

By Khintchin's formula (see [12], ch. 2), \( \hat{N}_n \overset{d}{=} Z_{\mu} \). According to (1), \( d_{TV} (\mu; \pi) \leq (1 - e^{-kq}) q \). Using this inequality and an idea from [15], we conclude that

\[
d_{TV} \left( Z_\mu, Z_\pi \right) = \frac{1}{2} \sum_i \left| \text{IP} \left( Z_\mu = \bar{i} \right) - \text{IP} \left( Z_\pi = \bar{i} \right) \right|.
\]
The result follows. □

The proof of Theorem 2 shows that the term \((1 - e^{-np})p\) in the right-hand side of (4) may be replaced by any other estimate of \(d_{TV}(\mu, \pi)\) (cf. [10, 20]).

**Proof** of Theorem 1. Let \(\{r = r_n\}\) be a sequence of natural numbers such that

\[
n \gg r_n \gg l_n + 1, \; nr_n^{-1}\alpha_n^{2/(2+m)} \to 0.
\]

Such a sequence exists: one can put \(r_n = \max\{[n\alpha_n^{1/(2+m)}]; [\sqrt{n(l_n + 1)}]\}\) (note that \(r \to 0\) because of (2)).

If \(N_n \Rightarrow \exists N\) then there exists the limit

\[
\lim \mathbb{P}(N_n(u_m) = 0) := e^{-t}.
\]

If \(t = 0\) then \(N_n(u_m) \to 0\), and the assertion of Theorem 1 trivially holds. Evidently, \(t < \infty\) (otherwise \(1 + o(1) = \mathbb{P}(N_n(u_m) \geq 1) \leq EN_n(u_m) = r\to 0\)). Thus, \(t \in (0; \infty)\).

It is known (cf. [13, 16]) that (8) with \(t \in (0; \infty)\) is equivalent to \(\mathbb{P}(N_r(u_m) > 0) \sim tr/n\). Therefore, if \(N_n \Rightarrow \exists N\) then Theorem 2 implies

\[
\mathbb{E}e^{iuN_n} = \exp \left( t \left( \varphi_{\zeta(n)}(v) - 1 \right) \right) + o(1) \to \mathbb{E}e^{ivN} \quad (\forall v \in \mathbb{R}^m)
\]

as \(n \to \infty\), where \(\varphi_{\zeta(n)}\) is the characteristic function of \(\zeta(n)\). Hence there exists the limit \(\lim_{n \to \infty} \varphi_{\zeta(n)}(v) := \varphi(v)\). As a limit of a sequence of characteristic functions, it is a characteristic function itself. Therefore,

\[
\mathbb{E}e^{ivN} = \exp \left( t(\varphi(v) - 1) \right).
\]

This is a characteristic function of a compound Poisson random vector with intensity \(t\) and multiplicity distribution \(\mathcal{L}(\zeta)\) such that \(\mathbb{E}e^{iv\zeta} = \varphi(v)\). □

**Proof** of Corollary 3. Let \(r = 1\) and \(l = 0\). Then \(\zeta^*(n)\) takes values \((1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\) with probabilities \(p_1/p, \ldots, p_m/p\) and \(\mathcal{L}(\pi) = \Pi(np)\). By Theorem 2,

\[
d_{TV} \left(Y_n; \sum_{j=1}^{\pi} \zeta_j^*(n)\right) \leq \left(1 - e^{-np}\right)p.
\]
It is easy to see that
\[ \mathbb{E} \exp \left( iv \sum_{j=1}^{\pi} \zeta_j^*(n) \right) = \exp \left( n \sum_{j=1}^{m} (e^{ivj} - 1) p_j \right) = \mathbb{E} e^{ivY} \]
for any \( v \in \mathbb{R}^m \). Hence \( \sum_{j=1}^{\pi} \zeta_j^*(n) \overset{d}{=} Y \). \( \square \)

For \( v \in \mathbb{R}^m \), we put \( |v| = \max_{i=1}^{m} |v_i| \). Let \((X, Y)\) be a random vector taking values in \( \mathbb{R}^l \times \mathbb{R}^m \), and let \( \alpha \) be the \( \alpha \)-mixing coefficient corresponding to the \( \sigma \)-fields \( \sigma(X) \) and \( \sigma(Y) \).

**Lemma 4** One can define random vectors \( X, Y \) and \( \hat{Y} \) on a common probability space in such a way that \( \hat{Y} \) is independent of \( X, \hat{Y} \overset{d}{=} Y \) and \((y > 0, K \in \mathbb{N})\)
\[ \mathbb{P} \left( |\hat{Y} - Y| > y \right) \leq 2^{(m+3)/2} K^{m/2} \alpha + 2 \mathbb{P}(|Y| > Ky) \tag{9} \]
In particular, if \( \nu = \mathbb{E}^{1/b} |Y|^b < \infty \) and \( \nu(b/\nu)^b \geq m^{(m-1)/2} \alpha \) then
\[ \mathbb{P} \left( |\hat{Y} - Y| > y \right) \leq 2 \left( 1 + 2b/m \right) \left[ \left( 2^{(m-1)/2} m / b \right)^{2b} (\nu/\nu)^{b} \right]^{1/(2b+m)} \tag{10} \]
If \( \nu_{\infty} \equiv \text{ess sup} |Y| < \infty \) then (10) yields
\[ \mathbb{P} \left( |\hat{Y} - Y| > y \right) \leq 2^{(m+3)/2} (\nu_{\infty}/\nu)^{m/2} \alpha \tag{11} \]
In the case \( m = 1 \), (10) improves the result of Theorem 3 in [8].

**Proof** of Lemma 4. Denote \( Y^\prec = Y \mathbb{1} \{|Y| \leq Ky\} \). Vector \( Y^\prec \) takes values in \([-Ky; Ky] \). Splitting \([-Ky; Ky]\) into \( 2K \) intervals of length \( y \) induces the partition of \([-Ky; Ky]\) into \( N = (2K)^m \) cubes \( H_1, \ldots, H_N \). According to Theorem 2 in [8], one can define \( X, Y^\prec \) and \( \hat{Y}^\prec \) on a common probability space so that \( \hat{Y}^\prec \) is independent of \( X, \hat{Y}^\prec \overset{d}{=} Y^\prec \) and
\[ \mathbb{P} \left( |\hat{Y}^\prec - Y^\prec| > y \right) = \mathbb{P}(A) \leq \sqrt{8N} \alpha \]
where \( A = \{ \hat{Y}^\prec \text{ and } \hat{Y}^\prec \text{ are not elements of the same } H_i \} \).

Now we construct a vector \( \hat{Y} \) on the base of \( \hat{Y}^\prec \) such that \( \hat{Y} \overset{d}{=} Y \). We put \( \hat{Y} = \hat{Y}^\prec + \mathbb{1}\{\hat{Y}^\prec = 0\} Y' \), where \( Y' \) is independent of all other random vectors, \( \mathcal{L}(Y') = \mathcal{L}(Y|B) \) and \( B = \{ Y^\prec = 0 \} = \{ Y = 0 \text{ or } |Y| > Ky \} \).

Evidently, \( \hat{Y} \overset{d}{=} Y \). Indeed, \( \mathbb{P}(\hat{Y} = 0) = \mathbb{P}(\hat{Y}^\prec = 0 = Y') = \mathbb{P}(B) \mathbb{P}(Y' = 0) = \mathbb{P}(Y = 0) \), and if \( z \neq 0 \) then
\[ \mathbb{P}(\hat{Y} \in dz) = \mathbb{P}(\hat{Y}^\prec \in dz) + \mathbb{P}(\hat{Y}^\prec = 0, Y' \in dz) \]
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\[
\mathbb{P}(B_c, Y \in dz) + \mathbb{P}(B)\mathbb{P}(Y \in dz | B) = \mathbb{P}(Y \in dz),
\]
where \(B_c = \{0 < |Y| < Ky\}\) is the complement to \(B\). It is easy to see that \(\mathbb{P}(\hat{Y} \neq \hat{Y}^\prec) = \mathbb{P}(\hat{Y}^\prec = 0 \neq Y') = \mathbb{P}(B)\mathbb{P}(Y \neq 0 | B) = \mathbb{P}(|Y| > Ky)\). Hence

\[
\mathbb{P}\left(|\hat{Y} - \hat{Y}^\prec| > y\right) \leq \sqrt{8N} + \mathbb{P}(\hat{Y} \neq \hat{Y}^\prec) \leq \sqrt{8N} + \mathbb{P}(|Y| > Ky).
\]

It remains to construct \((X, Y)\) on the base of \((X, Y^\prec)\). Let \(\{Y_x\}\) be independent random vectors with distributions \(\mathcal{L}(Y_x) = \mathcal{L}(Y|B, X = x)\). Denote \(Y^* = Y^\prec + \mathbb{I}\{Y^\prec = 0\}Y_x\). Then \((X, Y^*) \overset{d}{=} (X, Y)\). Indeed,

\[
\mathbb{P}(X \in dx, Y^* = 0) = \mathbb{P}(X \in dx, Y^\prec = 0, Y = 0) = \mathbb{P}(X \in dx, Y^\prec = 0)\mathbb{P}(Y_x = 0)
\]

\[
= \mathbb{P}(X \in dx, B, Y = 0) = \mathbb{P}(X \in dx, Y = 0).
\]

If \(z \neq 0\) then

\[
\mathbb{P}(X \in dx, Y^* = dz) = \mathbb{P}(X \in dx, Y^\prec = dz) + \mathbb{P}(X \in dx, Y^\prec = 0, Y_x \in dz)
\]

\[
= \mathbb{P}(X \in dx, B, Y \in dz) + \mathbb{P}(X \in dx, B)\mathbb{P}(Y_x \in dz) = \mathbb{P}(X \in dx, Y \in dz).
\]

Note that \(\mathbb{P}(Y^* \neq Y^\prec) = \mathbb{P}(Y^\prec = 0 \neq Y_x) = \mathbb{P}(|Y| > Ky)\). Therefore,

\[
\mathbb{P}\left(|\hat{Y} - Y^\prec| > y\right) \leq \mathbb{P}\left(|\hat{Y} - Y^\prec| > y\right) + \mathbb{P}(|Y| > Ky).
\]

Combining our estimates, we get (9).

Using Chebyshev's inequality, we deduce

\[
\mathbb{P}\left(|\hat{Y} - Y| > y\right) \leq cK^{m/2} + dK^{-b},
\]

where \(c = 2^{(m+3)/2}\alpha\) and \(d = 2(\nu/y)^b\). The function \(f(x) = cx^{m/2} + dx^{-b}\) takes its minimum in \(x \geq 1\) at \(x_\alpha = \max\{(2bd/cm)^{2/(m+2b)}; 1\}\). Since \(2bd/cm = \frac{b(\nu/y)^b}{2^{(m-1)/2m}}\), inequality (9) entails (10). The proof is complete.
References


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