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On accuracy of multivariate compound Poisson approximation

Abstract

We present multivariate generalisations of some classical results on accuracy of Poisson approximation for the distribution of a sum of 0–1 random variables.

1 Introduction

Let $X, X_1, X_2, \ldots$ be a stationary sequence of dependent random variables (r.v.s). The key object in Extreme Value Theory is the number of exceedances

$$N_n(u) = \sum_{i=1}^{n} \mathbb{I}\{X_i > u\}.$$ 

Investigation of $N_n(u)$ is motivated by applications in finance, insurance, network modelling, meteorology, etc. (cf. [11, 19]).

In the independent case, $N_n(u)$ has binomial $B(n, p)$ distribution, where $p = \mathbb{P}(X > u)$. If $p$ is "small" then $\mathcal{L}(N_n(u))$ may be approximated by the Poisson $\Pi(np)$ distribution. Accuracy of Poisson approximation for a binomial distribution has been investigated by famous authors (see, e.g., [17, 14, 10, 3] and references in [6]). The case of a sum of dependent 0–1 random variables was the subject of [9, 2, 3] (see also references in [3]).

The natural measure of closeness of discrete distributions is the total variation distance (TVD). Recall the definition of the TVD between the distributions of random vectors $X$ and $Y$ taking values in $\mathbb{Z}^m_+$, where $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$:

$$d_{\text{TVD}}(X; Y) = d_{\text{TVD}}(\mathcal{L}(X); \mathcal{L}(Y)) = \sup_{A \in \mathbb{Z}} |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)|.$$

Let $\pi$ be a Poisson random variable with the parameter $np$. According to Barbour and Eagleson [2],

$$d_{\text{TVD}}(N_n(u); \pi) \leq \left(1 - e^{-np}\right) p. \quad (1)$$

This is probably the best universal estimate of the TVD between binomial and Poisson distributions; it improves the results of Prokhorov [17] and LeCam [14]. Sharper bounds are available under extra restrictions (see [10, 20]).

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Dependence can cause clustering of extremes, and the Poisson approximation may no longer be valid. It is known that under a mild mixing condition, the limiting distribution of $N_n(u)$ is compound Poisson.

Accuracy of compound Poisson approximation for $L(N_n(u))$ has been evaluated in [1, 15, 18], among others. The feature of the estimate given in [15] is that it coincides with (1) in the particular case of independent r.v.s.

A natural problem is to investigate the distribution of the vector

$$N_n = (N_n(u_1), \ldots, N_n(u_m))$$

of the numbers of exceedances given a set of distinct levels $u_1, \ldots, u_m$. The problem has applications in insurance and finance. For instance, a stationary sequence $\{X_i\}$ of (dependent) random variables can represent claims to an insurance company. Let $N(u_i)$ denote the number of claims exceeding a level $u_i$. It can be of interest to approximate the probability that the number of claims exceeding $u_i$ equals $n_i$, $1 \leq i \leq m$. This question can be easily addressed if the distribution of the vector $N_n$ has been approximated.

We show that under natural conditions, the limiting distribution of $N_n$ is necessarily compound Poisson. We evaluate accuracy of multivariate compound Poisson approximation for the distribution of $N_n$. In particular, we improve the corresponding results of Barbour et al. [4] and Novak [15]. In the case of independent trials, our result yields an estimate of accuracy of multivariate Poisson approximation for a multinomial distribution.

2 Results

We may assume $u_1 > \ldots > u_m$. Let $F_{a,b} \equiv F_{a,b}(u_1, \ldots, u_m)$ be the $\sigma$-field generated by the events $\{X_i > u_j\}$, $a \leq i \leq b$, $1 \leq j \leq m$. Denote

$$\alpha(l) \equiv \alpha(l, \{u_1, \ldots, u_m\}) = \sup |P(AB) - P(A)P(B)|,$$

$$\beta(k) \equiv \beta(l, \{u_1, \ldots, u_m\}) = \sup \mathbb{E} \sup_B \left| P(B|F_{l,j}) - P(B) \right|,$$

where the supremum is taken over all $A \in F_{i,j}$, $B \in F_{j+1}, j \geq 1$, such that $P(A) > 0$.

Condition $\Delta_m \equiv \Delta_m\{u_1, \ldots, u_m\}$ is said to hold if

$$\alpha_n \equiv \alpha(l_n, \{u_1, \ldots, u_m\}) \to 0$$

for some sequence $\{l_n\} \subset \mathbb{Z}_+$ such that $l_n/n \to 0$ as $n \to \infty$. A vector $Y$ has a multivariate compound Poisson distribution $\Pi(\lambda, \mathcal{L}(Z))$ if

$$Y = \sum_{i=1}^{\pi} Z_i,$$
where \( Z, Z_1, \ldots \) are i.i.d. random vectors, \( \pi \) is independent of \( \{Z_i\} \) and has the Poisson distribution with parameter \( \lambda \).

**Theorem 1** Assume condition \( \Delta_m \), and suppose that \( u_m \equiv u_m(n) \) obeys

\[
\limsup n \mathbb{P}(X > u_m) < \infty. \tag{2}
\]

If \( N_n \) converges weakly to a random vector \( Y \) then \( Y \) has a multivariate compound Poisson distribution.

Let \( \zeta(n), \zeta_1(n), \zeta_2(n), \ldots \) be independent random vectors with the common distribution

\[
\mathcal{L}(\zeta(n)) = \mathcal{L}(N_r|N_r(u_m) > 0), \tag{3}
\]

where \( r \in \{1, \ldots, n\} \). The proof of Theorem 1 shows that \( Y \stackrel{d}{=} \Pi(\lambda, \mathcal{L}(Z)) \), where \( \lambda = -\lim_{n \to \infty} \ln \mathbb{P}(N_n(u_m) = 0) \) and \( \mathcal{L}(\zeta) \) is the weak limit of \( \mathcal{L}(\zeta(n)) \) for an appropriate sequence \( r = r_n \).

Denote

\[
p = \mathbb{P}(X > u_m), \quad q = \mathbb{P}(N_r(u_m) > 0), \quad k = \lfloor n/r \rfloor, \quad r' = n - rk,
\]

and let \( \pi \) be a Poisson random variable with parameter \( kq \).

In Theorem 2 below we approximate the distribution of \( N_n \) by the multivariate compound Poisson distribution \( \mathcal{L}(N) \), where \( N = \sum_{i=1}^r \zeta_i(n) \).

**Theorem 2** If \( n > r > l \geq 0 \) then

\[
d_{tv}(N_n; N) \leq (1 - e^{-np})rp + (2nr^{-1}l + r')p + nr^{-1} \min\{\beta(l); \kappa(l)\}, \tag{4}
\]

where \( \kappa(l) = 2(1 + 2/m) \left\{ 2^{m-1}m^2\alpha^2(l) \right\}^{1/(2+m)} \) if \( m2^{(m-1)/2}\alpha(l) \leq 1 \), otherwise \( \kappa(l) = 1 \).

Barbour et al. [4] evaluated accuracy of compound Poisson approximation for general empirical point processes of exceedances in terms of a weaker Wasserstein-type distance \( d_w \). Concerning the approximation \( \mathcal{L}(N_n) \approx \mathcal{L}(N) \), Theorem 3.1 in [4] yields \( d_w(N_n; N) \leq \left( 1.65(1 - rp)^{-1/2} + e^{rp} \right) rp + 2(2rp + nr^{-1}l)p + nr^{-1}\beta(l) \). In the case \( m = 1 \) (the 1-dimensional situation), (4) improves a result from [15] (cf. also [1]). If \( m = 1 \) and the random variables \( \{X_i\} \) are independent then (4) with \( l = 0, r = 1 \) yields (1).

As a consequence of Theorem 2, we derive an estimate of accuracy of multivariate Poisson approximation for a multinomial distribution.

Let \( i = (i_1, \ldots, i_m) \), where \( i_1 \leq \ldots \leq i_m \). Denote \( i^* = (i_1, i_2 - i_1, \ldots, i_m - i_{m-1}) \),

\[
N_n^* = (N_n(u_1), N_n(u_1, u_2), \ldots, N_n(u_{m-1}, u_m)),
\]
where $N_n(u,v) = \sum_{i=1}^{n} \mathbb{I}\{u \geq X_i > v\}$ as $u > v$. Evidently, the distribution of $N_n$ determines that of $N_n^*$ and vice versa.

The statement of Theorem 2 can be reformulated as follows: if $n > r > l \geq 0$ then

$$d_{TV}(N_n^*; N^*) \leq (1 - e^{-np})r p + (2nr^{-1}l + r')p + nr^{-1}\min\{\beta(l); \kappa(l)\},$$

where $N^* = \sum_{i=1}^{n} \zeta_i^*(n)$, random vectors $\zeta_i^*(n)$, $\zeta_k^*(n)$, ... are independent and have the

common distribution $\mathbb{P}(\zeta^*(n) = i^*) = \mathbb{P}(\zeta(n) = i)$.

If the random variables $\{X_i\}$ are independent and $r = 1$ then $N_n^*$ has the multinomial distribution $B(n, p_1, ..., p_m)$ with parameters $p_1 = \mathbb{P}(X > u_1)$, $p_2 = \mathbb{P}(u_1 \geq X > u_2)$, ..., $p_m = \mathbb{P}(u_{m-1} \geq X > u_m)$:

$$\mathbb{P}(N^*_n = (l_1, ..., l_m)) = \frac{n!}{l_1!...l_m!(n-l)!} p_{l_1}^{l_1}...p_{l_m}^{l_m}(1-p)^{n-l},$$

where $l = l_1 + ... + l_m \leq n$, $p = p_1 + ... + p_m$. Theorem 2 yields an estimate of accuracy of multivariate Poisson approximation for the multinomial distribution $B(n, p_1, ..., p_m)$.

Corollary 3 Let $\pi_1, ..., \pi_m$ be independent Poisson random variables with parameters $np_1, ..., np_m$. Denote $Y = (\pi_1, ..., \pi_m)$. If $\mathcal{L}(Y_n) = B(n, p_1, ..., p_m)$ then

$$d_{TV}(Y_n; Y) \leq (1 - e^{-np})p.$$  

3 Proofs

Proof of Theorem 2 incorporates some ideas from [15] and results of Berbee [5] and Bradley [8].

Denote $\Pi_i = (\mathbb{I}\{X > u_1\}, ..., \mathbb{I}\{X > u_m\})$, and let

$$N_{r,j} = \sum_{i=j+1}^{(j+1)r\wedge n} \Pi_i \quad (0 \leq j \leq k = \lfloor n/r \rfloor).$$

Evidently, $N_n = \sum_{j=0}^{k} N_{r,j}$. Notice that the last block $N_{r,k}$ may be omitted:

$$d_{TV}(N_n; \sum_{j=0}^{k-1} N_{r,j}) \leq \mathbb{P}(N_{r,k} \neq \emptyset) \leq r'p.$$

Following Bernstein's "blocks" approach, we subtract a subblock of length $l$ from each block $X_{jr+1}, ..., X_{(j+1)r}$ of length $r$. Denote

$$N^*_{r,j} = \sum_{i=j+1}^{(j+1)r-l} \Pi_i, \quad N^*_n = \sum_{j=0}^{k-1} N^*_{r,j} \quad (0 \leq j < k).$$
Then $\mathbb{P} \left( \sum_{j=0}^{k-1} R_{r,j} \neq \sum_{j=0}^{k-1} R_{*r,j} \right) \leq k \mathbb{P} \left( R_{r,0} \neq R_{*r,0} \right) \leq k p$.

Let $\{ R_{*r,j} \}$ be independent copies of $R_{r,0}$. Denote

$$S_i = \sum_{j=0}^{i-1} R_{r,j} + \sum_{j=i+1}^{k-1} \tilde{R}_{r,j} \quad (0 < i < k).$$

Notice that $S_j + \tilde{R}_{r,j} = S_{j-1} + R_{r,j-1}$. We apply Lindeberg’s device (cf. [15]) in order to replace $\{ R_{r,j} \}$ by $\{ \tilde{R}_{r,j} \}$:

$$\mathbb{P} \left( \sum_{j=0}^{k-1} R_{r,j} \in A \right) - \mathbb{P} \left( \sum_{j=0}^{k-1} \tilde{R}_{r,j} \in A \right) = \sum_{i=1}^{k-1} \left\{ \mathbb{P} \left( S_j + R_{r,j} \in A \right) - \mathbb{P} \left( S_j + \tilde{R}_{r,j} \in A \right) \right\}.$$ 

According to Berbee’s lemma ([5], ch. 4), the random vectors $\sum_{j=0}^{i-1} R_{r,j}, R_{r,j}$ and $\tilde{R}_{r,j}$ can be defined on a common probability space so that $\mathbb{P} \left( R_{r,j} \neq \tilde{R}_{r,j} \right) \leq \beta(l)$. Therefore,

$$\left| \mathbb{P} \left( \sum_{j=0}^{k-1} R_{r,j} \in A \right) - \mathbb{P} \left( \sum_{j=0}^{k-1} \tilde{R}_{r,j} \in A \right) \right| \leq k \beta(l).$$

The mixing coefficient $\alpha$ is weaker than $\beta$. Using Lemma 4 below, we evaluate $\mathbb{P} \left( \sum_{j=0}^{k-1} R_{r,j} \in A \right) - \mathbb{P} \left( \sum_{j=0}^{k-1} \tilde{R}_{r,j} \in A \right)$ in terms of $\alpha(l)$. Note that $\mathbb{E} |R_{r,0}| = rp$.

Inequality (10) with $b = 1$ and $y = rp$ entails the random vectors $\sum_{j=0}^{i-1} R_{*r,j}, R_{r,j}$ and $\tilde{R}_{r,j}$ can be defined on a common probability space so that $\mathbb{P} \left( R_{r,j} \neq \tilde{R}_{r,j} \right) = \mathbb{P} \left( |R_{r,j} - \tilde{R}_{r,j}| \geq 1 \right) \leq \kappa(l)$ if $m^{2(m-1)/2} \alpha(l) \leq 1$. Hence

$$\left| \mathbb{P} \left( \sum_{j=0}^{k-1} R_{*r,j} \in A \right) - \mathbb{P} \left( \sum_{j=0}^{k-1} \tilde{R}_{r,j} \in A \right) \right| \leq k \min\{ \beta(l); \kappa(l) \}.$$ 

Let $\{ \tilde{R}_{r,j} \}$ be independent copies of $R_{r,0}$, and set $\tilde{N}_n = \sum_{j=0}^{k-1} \tilde{R}_{r,j}$. Evidently, $\mathbb{P} \left( \sum_{j=0}^{k-1} \tilde{R}_{r,j} \neq \sum_{j=0}^{k-1} \tilde{R}_{r,j} \right) \leq kl p$. Combining our estimates, we get

$$d_{TV} \left( N_n; \tilde{N}_n \right) \leq 2k l p + r' p + k \min\{ \beta(l); \kappa(l) \}.$$ 

Denote $\mu = \sum_{j=0}^{k-1} \mathbb{I} \{ \tilde{R}_{r,j} \neq 0 \}$, and put

$$Z_0 = 0, \ Z_j = \zeta_1(n) + \ldots + \zeta_j(n) \quad (j \geq 1).$$

By Khintchin’s formula (see [12], ch. 2), $\tilde{N}_n \overset{d}{=} Z_\mu$. According to (1), $d_{TV} (\mu; \pi) \leq (1 - e^{-kq}) q$. Using this inequality and an idea from [15], we conclude that

$$d_{TV} (Z_\mu, Z_\pi) = \frac{1}{2} \sum \left| \mathbb{P} (Z_\mu = \bar{i}) - \mathbb{P} (Z_\pi = \bar{i}) \right|.$$
\[ \leq \frac{1}{2} \sum_{i} \sum_{m=0}^{\infty} \mathbb{P}(Z_m = i) \left| \mathbb{P}(\mu = m) - \mathbb{P}(\pi = m) \right| \]
\[ = d_{TV}(\mu, \pi) \leq (1 - e^{-kp})q \leq (1 - e^{-np})r_p. \]

The result follows. \( \square \)

The proof of Theorem 2 shows that the term \((1 - e^{-np})r_p\) in the right-hand side of (4) may be replaced by any other estimate of \(d_{TV}(\mu, \pi)\) (cf. \([10, 20]\)).

**Proof** of Theorem 1. Let \(\{ r = r_n \} \) be a sequence of natural numbers such that
\[ n \gg r_n \gg l_n + 1, \quad nr_n^{-1} \alpha_n^{2/(2+m)} \to 0. \] (7)

Such a sequence exists: one can put \(r_n = \max \left\{ \left[ n \alpha_n^{1/(2+m)} \right], \left\lfloor \sqrt{n(l_n + 1)} \right\rfloor \right\} \) (note that \(r_p \to 0\) because of (2)).

If \(N_n \Rightarrow \exists N\) then there exists the limit
\[ \lim \mathbb{P}(N_n(u_m) = 0) := e^{-t}. \] (8)

If \(t = 0\) then \(N_n(u_m) \to 0\), and the assertion of Theorem 1 trivially holds. Evidently, \(t < \infty\) (otherwise \(1 + o(1) = \mathbb{P}(N_n(u_m) \geq 1) \leq \mathbb{E}N_n(u_m) = r_p \to 0\)). Thus, \(t \in (0; \infty)\).

It is known (cf. \([13, 16]\)) that (8) with \(t \in (0; \infty)\) is equivalent to \(\mathbb{P}(N_r(u_m) > 0) \sim tr/n\). Therefore, if \(N_n \Rightarrow \exists N\) then Theorem 2 implies
\[ \mathbb{E}e^{iuN_n} = \exp \left( t \left( \varphi_{\zeta(n)}(v) - 1 \right) \right) + o(1) \to \mathbb{E}e^{iuN} \quad (\forall v \in \mathbb{R}^m) \]

as \(n \to \infty\), where \(\varphi_{\zeta(n)}\) is the characteristic function of \(\zeta(n)\). Hence there exists the limit \(\lim_{n \to \infty} \varphi_{\zeta(n)}(v) := \varphi(v)\). As a limit of a sequence of characteristic functions, it is a characteristic function itself. Therefore,
\[ \mathbb{E}e^{iuN} = \exp \left( t(\varphi(v) - 1) \right). \]

This is a characteristic function of a compound Poisson random vector with intensity \(t\) and multiplicity distribution \(\mathcal{L}(\zeta)\) such that \(\mathbb{E}e^{iu\zeta} = \varphi(v)\). \( \square \)

**Proof** of Corollary 3. Let \(r = 1\) and \(l = 0\). Then \(\zeta^*(n)\) takes values \((1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\) with probabilities \(p_1/p, \ldots, p_m/p\) and \(\mathcal{L}(\pi) = \Pi(np)\). By Theorem 2,
\[ d_{TV}(Y_n; \sum_{j=1}^{\pi} \zeta_j^*(n)) \leq (1 - e^{-np})p. \]
It is easy to see that
\[ \mathbb{E} \exp \left( iv \sum_{j=1}^{\pi} \zeta_j^*(n) \right) = \exp \left( n \sum_{j=1}^{m} \left( e^{ivj} - 1 \right) p_j \right) = \mathbb{E} e^{ivY} \]
for any \( v \in \mathbb{R}^m \). Hence \( \sum_{j=1}^{\pi} \zeta_j^*(n) \leq Y \). \( \square \)

For \( v \in \mathbb{R}^m \), we put \( |v| = \max_{i \leq m} |v_i| \). Let \((X,Y)\) be a random vector taking values in \( \mathbb{R}^l \times \mathbb{R}^m \), and let \( \alpha \) be the \( \alpha \)-mixing coefficient corresponding to the \( \sigma \)-fields \( \sigma(X) \) and \( \sigma(Y) \).

**Lemma 4** One can define random vectors \( X, Y \) and \( \hat{Y} \) on a common probability space in such a way that \( \hat{Y} \) is independent of \( X \), \( Y \) and \( (y > 0, K \in \mathbb{N}) \)

\[ \mathbb{P} \left( |\hat{Y} - Y| > y \right) \leq 2^{(m+3)/2} K^{m/2} + 2 \mathbb{P}(|Y| > Ky) . \]  \tag{9}

In particular, if \( \nu = \mathbb{E}^{1/b}|Y|^b < \infty \) and \( b(\nu/y)^b \geq m^{2(m-1)/2} \alpha \) then

\[ \mathbb{P} \left( |\hat{Y} - Y| > y \right) \leq 2(1 + 2b/m) \left[ (2^{(m-1)/2} m/b^{b(\nu/y)^b})^{m/2m} \right]^{1/(2b+m)} . \]  \tag{10}

If \( \nu_\infty \equiv \text{ess sup} |Y| < \infty \) then (10) yields

\[ \mathbb{P} \left( |\hat{Y} - Y| > y \right) \leq 2^{(m+3)/2}(\nu_\infty/y)^{m/2} \alpha . \]  \tag{11}

In the case \( m = 1 \), (10) improves the result of Theorem 3 in [8].

**Proof** of Lemma 4. Denote \( Y < = Y 1_{\{|Y| \leq Ky\}} \). Vector \( Y < \) takes values in \([-Ky, Ky]^m \). Splitting \([-Ky, Ky] \) into 2K intervals of length \( y \) induces the partition of \([-Ky, Ky]^m \) into \( N = (2K)^m \) cubes \( H_1, ..., H_N \). According to Theorem 2 in [8], one can define \( X, Y < \) and \( \hat{Y} < \) on a common probability space so that \( \hat{Y} < \) is independent of \( X \), \( \hat{Y} \lessdot Y < \) and

\[ \mathbb{P} \left( |\hat{Y} < - Y < | > y \right) = \mathbb{P}(A) \leq \sqrt{8N} \alpha , \]

where \( A = \{ \hat{Y} < \text{ and } \hat{Y} < \text{ are not elements of the same } H_i \} \).

Now we construct a vector \( \hat{Y} \) on the base of \( \hat{Y} < \) such that \( \hat{Y} \lessdot Y \). We put \( \hat{Y} = \hat{Y} < + 1\{\hat{Y} < = 0\} Y' \), where \( Y' \) is independent of all other random vectors, \( \mathcal{L}(Y') = \mathcal{L}(Y|B) \) and \( B = \{Y < = 0\} = \{Y = 0 \text{ or } |Y| > Ky\} \).

Evidently, \( \hat{Y} \lessdot Y \). Indeed, \( \mathbb{P}(\hat{Y} = 0) = \mathbb{P}(\hat{Y} < = 0 = Y') = \mathbb{P}(B) \mathbb{P}(Y' = 0) = \mathbb{P}(Y = 0) \), and if \( z \neq 0 \) then

\[ \mathbb{P}(\hat{Y} \in dz) = \mathbb{P}(\hat{Y} < \in dz) + \mathbb{P}(\hat{Y} < = 0, Y' \in dz) \]
\[ = \mathbb{P}(B_c, Y \in dz) + \mathbb{P}(B)\mathbb{P}(Y \in dz | B) = \mathbb{P}(Y \in dz), \]

where \( B_c = \{0 < |Y| < Ky\} \) is the complement to \( B \). It is easy to see that \( \mathbb{P}(\hat{Y} \neq \hat{Y}^c) = \mathbb{P}(\hat{Y}^c = 0 \neq Y') = \mathbb{P}(B)\mathbb{P}(Y \neq 0 | B) = \mathbb{P}(|Y| > Ky) \). Hence

\[
\mathbb{P}
\left|
\begin{array}{l}
|\hat{Y} - Y^c| > y \\
\end{array}
\right|
\leq \sqrt{8N} \alpha + \mathbb{P}(\hat{Y} \neq \hat{Y}^c) \leq \sqrt{8N} \alpha + \mathbb{P}(|Y| > Ky).
\]

It remains to construct \((X, Y)\) on the base of \((X, Y^c)\). Let \( \{Y_x\} \) be independent random vectors with distributions \( \mathcal{L}(Y_x) = \mathcal{L}(Y | B, X = x) \). Denote \( Y^* = Y^c + \mathbb{I}\{Y^c = 0\} Y_x \). Then \((X, Y^*) \equiv (X, Y)\). Indeed,

\[
\mathbb{P}(X \in dx, Y^* = 0) = \mathbb{P}(X \in dx, Y^c = 0 = Y_x) = \mathbb{P}(X \in dx, Y^c = 0)\mathbb{P}(Y_x = 0)
= \mathbb{P}(X \in dx, B, Y = 0) = \mathbb{P}(X \in dx, Y = 0).
\]

If \( z \neq 0 \) then

\[
\mathbb{P}(X \in dx, Y^* \in dz) = \mathbb{P}(X \in dx, Y^c \in dz) + \mathbb{P}(X \in dx, Y^c = 0, Y_x \in dz)
= \mathbb{P}(X \in dx, B_c, Y \in dz) + \mathbb{P}(X \in dx, B)\mathbb{P}(Y_x \in dz) = \mathbb{P}(X \in dx, Y \in dz).
\]

Note that \( \mathbb{P}(Y^* \neq Y^c) = \mathbb{P}(Y^c = 0 \neq Y_x) = \mathbb{P}(|Y| > Ky) \). Therefore,

\[
\mathbb{P}
\left|
\begin{array}{l}
|\hat{Y} - Y| > y \\
\end{array}
\right|
\leq \mathbb{P}
\left|
\begin{array}{l}
|\hat{Y} - Y^c| > y \\
|Y| > Ky
\end{array}
\right|.
\]

Combining our estimates, we get (9).

Using Chebyshev’s inequality, we deduce

\[
\mathbb{P}
\left|
\begin{array}{l}
|\hat{Y} - Y| > y \\
\end{array}
\right| \leq cK^{m/2} + dK^{-b},
\]

where \( c = 2^{(m+3)/2} \alpha \) and \( d = 2(\nu/y)^b \). The function \( f(x) = cx^{m/2} + dx^{-b} \) takes its minimum in \( x \geq 1 \) at \( x_0 = \max\{(2bd/cm)^{2/(m+2b)}; 1\} \). Since \( \frac{2bd}{cm} = \frac{b(\nu/y)^b}{2(m-1)/m} \), inequality (9) entails (10). The proof is complete. \( \Box \)
References