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Published in:
Abstracts 25th European Workshop on Computational Geometry (EuroCG'09, Brussels, Belgium, March 16-18, 2009)

Published: 01/01/2009

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

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Link to publication

Citation for published version (APA):

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Visibility Maps of Realistic Terrains have Linear Smoothed Complexity

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Abstract

We study the complexity of the visibility map of so-called realistic terrains: terrains whose triangles are fat, not too steep and have roughly the same size. It is known that the complexity of the visibility map of such a terrain with n triangles is Θ(n²) in the worst case. We prove that if the elevations of the vertices of the terrain are subject to uniform noise which is proportional to the edge lengths, then the worst-case expected (smoothed) complexity is only Θ(n). This provides an explanation why visibility maps of superlinear complexity are unlikely to be encountered in practice.

1 Introduction

Motivation. A (triangulated) terrain is a polyhedral surface obtained by assigning elevations to the vertices of a planar triangulation. In geographic information science such terrain models are known as triangulated irregular networks, or TINs for short. Terrains can be used to model mountainous regions, as well as to approximate any scalar function defined over a planar region.

Often it is desirable to compute which parts of a terrain T are visible from a given viewing point pview. The projections of the visible triangle parts onto a viewing plane form the visibility map of T with respect to pview. Computing visibility maps is useful for visualization purposes (hiddensurface removal or shadow generation), for planning buildings under visibility constraints and for other tasks involving visibility analysis. There are several algorithms for computing visibility maps of terrains, the most efficient of which runs in time O((α(n) + k) log n) [6] where α(·) is the inverse Ackermann function. Here n is the number of triangles in T and k is the complexity of the visibility map, which can be defined1 as the number of vertices of the map. Each vertex of the map either corresponds to a triangle vertex, or to two edges whose projections onto the viewing plane intersect. In the worst case, Θ(n²) pairs of edges have visible intersecting projections. In most applications a quadratic complexity would make an explicit computation of the visibility map infeasible. Fortunately such high complexity is seldom encountered. In fact, in practice it seems that the complexity of visibility maps is closer to linear. Our goal is to understand why visibility maps of terrains in practice often have low complexity.

Realistic input models. One possible approach to explain the low complexity is using a so-called realistic input model [5]. Here one assumes that the input has certain properties that are hopefully satisfied by inputs encountered in practice, and that rule out contrived worst-case inputs. This approach works well for many problems, and Moet et al. [9] have applied it to visibility maps of terrains. Moet et al. make the following assumptions on the terrain: the triangles of the planar triangulation defining the terrain are fat (as defined below), the triangle edges have constant length ratio, and the domain of the triangulation is a rectangle of constant aspect ratio. Unfortunately, the assumptions do not explain why visibility maps of terrains would have near-linear complexity in practice: Moet et al. showed that the worst-case complexity of the visibility map of a terrain that

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Figure 1: Two views of the same terrain defined by a regular grid. The latter view has Θ(n√n) complexity.

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1Formally, the complexity would be defined as the total number of vertices, edges, and faces of the map. In our setting this is always linear in the number of vertices, so we restrict ourselves to this quantity.
satisfies their assumptions is $\Theta(n\sqrt{n})$. In fact, one can even assign elevations to the vertices of a triangulated grid in such a way that the triangles do not become steep while the visibility map has complexity $\Theta(n\sqrt{n})$ for certain viewing directions—see Fig. 1. To explain the linear behavior, an alternative approach is needed.

Smoothed analysis. For some problems, a small perturbation of a worst-case instance can reduce the complexity of the instance significantly. This may support the fact that such constructions are highly improbable to appear in practice. Smoothed analysis formalizes this idea.

Let $I(n)$ be the set of all possible input instances (in our case: terrains) of size $n$. For an input $I \in I(n)$, let $C(I)$ denote the quantity we want to analyze (the complexity of the visibility map). Furthermore, for any input $I \in I(n)$ we define a neighbourhood $\mathcal{N}(I) \subset I(n)$ of input instances, and we define a probability distribution over $\mathcal{N}(I)$ that indicates for every $I' \in \mathcal{N}(I)$ the probability that applying noise to the input $I$ will result in the input $I'$. The smoothed (that is worst-case expected) complexity is then defined as

$$C_{smooth}(n) = \sup_{I \in I(n)} E_{I' \in \mathcal{N}(I)}[C(I')]$$

where the expectation is according to the given probability distribution on $\mathcal{N}(I)$. Smoothed analysis was introduced by Spielman and Teng [10]. So far there have only been a few applications in computational geometry (see e.g. [2, 3, 4]), none of which deals with terrains. An up-to-date collection of the published works related to smoothed analysis is maintained online by Spielman [11].

Our result. We study the smoothed complexity of visibility maps of terrains under the following model:

- To each vertex’s elevation noise is added that follows a uniform distribution in an interval $[-c,c]$, where $c$ is a fixed constant fraction of the minimum edge length of the triangulation underlying the terrain.\(^2\)

Our noise model defines for each input terrain $T$ a neighbourhood $\mathcal{N}(T)$ consisting of those terrain instances that can be obtained by changing the elevation of each vertex by at most $c$, and a probability distribution on $\mathcal{N}(T)$. Yet, by applying a small perturbation one does not get rid of peaks that are unrealistically skinny and high, and so the smoothed visibility map complexity of arbitrary terrains is still quadratic.

Hence, we combine the power of smoothed analysis with the ideas of realistic input models. We define the following parameters of terrains:

- Fatness: the smallest angle of any triangle’s projection onto the horizontal plane;
- Steepness: the largest dihedral angle between any triangle and the horizontal plane;
- Scale factor: the length of the longest edge divided by the length of the shortest edge of the triangulation.

We assume that the fatness $\phi$, steepness $\theta$, and scale factor $\sigma$ of the unperturbed terrain are constants $\phi > 0$, $\theta < \pi/2$, $\sigma \geq 1$ that are independent of the number of triangles $n$. Some of these assumptions are also used in other papers on realistic terrains [1, 9]. In itself, our assumptions do not lead to the desired result: there are terrains satisfying these assumptions with quadratic-complexity visibility maps. For example, we can take the same construction as in Fig. 1, but with the aspect ratio of the domain being $\Theta(n)$. Our main result is that the smoothed complexity of any visibility map of a terrain satisfying the abovementioned assumptions is only $\Theta(n)$. This result holds for orthographic as well as perspective views.

2 Visibility maps resulting from perspective projection

Let $T$ be a terrain with $n$ triangles, and let $E$ be the set of edges of $T$. Let the coordinates of the vertices be specified by three coordinates $x$, $y$ and $z$, where the $z$-axis is the vertical axis on which the elevation is specified. Let $\mathbf{T}$ denote the triangulation in the $xy$-plane defining $T$. Without loss of generality we assume that the minimum edge length in $\mathbf{T}$ is 1. We study the smoothed complexity of the visibility map of $T$ for perspective views, that is, the map as it appears in the projection on a viewing plane $h_{view}$ as seen from a viewing point $p_{view}$. We assume that $p_{view}$ is located above the terrain. Our results can be shown to hold also for orthographic views.

We denote the projection of an object $o$ onto $h_{view}$ by $\text{pr}(o)$. For an edge $e \in E$ we use $h_{align}(e)$ to denote the plane containing $e$ and $p_{view}$. The
steepness $\theta(t)$ of a triangle $t$ is defined as the dihedral angle of the plane containing $t$ with the $xy$-plane, and the steepness $\theta(s)$ of a segment $s$ is defined as the angle of the line containing $s$ with the $xy$-plane. Observe that the steepness of a triangle is the maximum steepness of any segment contained in it. Recall that $\theta$ is the maximum steepness of any triangle in $\mathcal{T}$.

**Lemma 1** After perturbing the elevation of each terrain vertex independently by a distance of at most $c$, no terrain triangle is steeper than $\theta_{\text{max}} = \arctan(\tan(\theta) + \frac{2c}{\sin \phi})$.

As $\phi > 0$, and $\theta < \pi/2$, and $c$ are constants, $\theta_{\text{max}}$ is also a constant strictly smaller than $\pi/2$.

The **perceived steepness** $\theta_{\text{view}}(e)$ of an edge $e$ is the steepness of $pr(e)$ in the plane $h_{\text{view}}$. In other words, $\theta_{\text{view}}(e)$ is the smallest angle between the line containing $pr(e)$ and a horizontal line on $h_{\text{view}}$. Even though $\theta(e) \leq \theta_{\text{max}}$ by Lemma 1, an edge that is almost horizontal can appear vertical when projected onto $h_{\text{view}}$. We say that an edge $e$ appears steep when $\theta_{\text{view}}(e)$ is not a single point and $\theta_{\text{view}}(e) > \theta_{\text{max}}$, otherwise $e$ appears flat.

We say that an edge $e$ lies in front of an edge $e'$, if there is a ray from $p_{\text{view}}$ that, in the projection on the $xy$-plane, hits $e$ before hitting $e'$.

A **silhouette edge** is an edge $e$ such that the two triangles of $\mathcal{T}$ that share $e$ are on the same side of $h_{\text{align}}(e)$. Thus one can see past $e$ on the other side of $h_{\text{align}}(e)$. This means one of the incident triangles is front-facing while the other is back-facing. We say that two edges $e$ and $e'$ create a visible intersection if $pr(e) \cap pr(e')$ is a vertex of the visibility map. For this to be possible, the edge hit first by a ray from $p_{\text{view}}$—say this edge is $e$—must be a silhouette edge.

We observed above that even though the terrain edges are not steeper than $\theta_{\text{max}}$, they can still appear steep on $h_{\text{view}}$. Yet, we can show that this cannot happen for silhouette edges.

**Lemma 2** Let $\mathcal{T}$ be a terrain whose triangles have steepness at most $\theta_{\text{max}}$. Then the perceived steepness (on any viewing plane) of any silhouette edge of $\mathcal{T}$ is at most $\theta_{\text{max}}$.

**Counting intersections.** We will charge each visible intersection to the edge furthest from the viewer (of the two edges creating it). We denote with $K(e)$ the number of visible intersections an edge $e$ creates with edges in front of it.

Suppose that we have already perturbed the edges in front of $e$, and we wish to analyze the effect of perturbing $e$. Let $E_{\ell}(e)$ be the set of silhouette edges lying in front of $e$ in the projection onto the $xy$-plane, excluding the edges sharing a vertex with $e$; the latter cannot create a visible intersection with $e$. Then the visible intersections charged to $e$ are intersections of $pr(e)$ with the upper envelope of $\{pr(e') : e' \in E_{\ell}(e)\}$. Consider a fragment $f$ of an edge $e' \in E_{\ell}(e)$ that appears on this upper envelope. We wish to bound the probability that after perturbation of $e$, $pr(e)$ intersects $pr(f)$.

Define $\text{span}_c(o)$ to be the projection of an object $o$ onto a horizontal line in the vertical plane containing $e$, and define $\text{width}_c(e)$ to be the length of $\text{span}_c(e)$.

**Lemma 3** Consider a viewing plane $h_{\text{view}}$ that is vertical and parallel to $e$. Let $f$ be a fragment in front of $e$ such that $\text{span}_c(f) \subset \text{span}_c(e)$ and $\text{width}_c(f) \leq \text{width}_c(e)/3$. Suppose we independently perturb the elevations of the vertices of $e$, with perturbations chosen uniformly at random from $[-c,c]$. Then the probability that $e$ creates a visible intersection with $f$ is at most

$$\frac{3\text{width}_c(f) \cdot \tan \theta_{\text{max}}}{c}$$

**Proof.** Assume w.l.o.g that the projection of $e$ on the $xy$-plane is parallel to the $x$-axis. Let $v_1$ and $v_2$ be the vertices of $e$ such that $v_2$ is the vertex closest to $f$ in the projection onto the $x$-axis, with ties broken arbitrarily. Since $\text{width}_c(f) \leq \text{width}_c(e)/3$, the distance from $v_1$’s projection to $\text{span}_c(f)$ is at least $\text{width}_c(e)/3$.

Let $\ell$ be the vertical line on $h_{\text{view}}$ through $pr(v_2)$ and consider a fixed position of $v_1$. The set of all
possible positions of the perturbed $v_2$ that induce an intersection between $pr(f)$ and $pr(e)$ is a segment $qr$ on $\ell$; —see Fig. 2. The probability that $pr(e)$ intersects $pr(f)$ is at most $|qr|/(2c)$.

The triangle $\Delta(v_1qr)$ may not contain $f$ completely: there may be parts of $f$ where $e$ cannot create an intersection (for this position of $v_1$), because $v_2$ could not be perturbed that far. Let $f' = \frac{v_1q}{q\ell}$ be the part of $f$ inside $\Delta(v_1qr)$, with $v_3$ being the endpoint closest to $\ell$. Let $s$ be the point such that $\Delta(v_1v_3s)$ and $\Delta(v_1qr)$ are similar triangles. $\theta_{\text{view}}(T_{v_1}) \leq \theta_{\text{max}}$ since $e$ is projected onto a vertical viewing plane parallel to $e$; $f'$ has steepness at most $\theta_{\text{max}}$ by Lemma 2, so:

$$|v_3s| \leq 2 \text{width}_e(f') \cdot \tan \theta_{\text{max}}.$$ 

Moreover, we have $|v_1v_3| \geq |v_1q|/3$. Now

$$|qr| = |v_3s| \cdot \frac{|v_1q|}{|v_1v_3|} \leq 6 \text{width}_e(f) \cdot \tan \theta_{\text{max}}$$

Hence, the probability of intersection between $e$ and $f$ for any fixed position of $v_1$ is at most

$$\frac{|qr|}{2c} \leq \frac{3 \text{width}_e(f) \cdot \tan \theta_{\text{max}}}{c} \tag*{□}$$

Using lemma 3 it is easy to show that $K(e)$, the number of visible intersections of an edge $e$ of $T$ with edges in front of it, is expected to be small.

**Lemma 4**

$$E[K(e)] \leq \frac{3 \text{width}_e(e) \cdot \tan \theta_{\text{max}}}{c} + 2.$$ 

Using that $\text{width}_e(e) \leq \sigma$ and $\tan \theta_{\text{max}} \leq \tan(\theta) + \frac{2r}{\sin \phi}$ (by Lemma 1) we obtain our final result:

**Theorem 5** Let $T$ be a terrain of $n$ triangles with fatness $\phi$, steepness $\theta$, and scale factor $\sigma$. Then a visibility map of $T$ under perspective projection has smoothed complexity:

$$O \left( \left( \frac{\tan \theta}{c} + \frac{1}{\sin \phi} \right) \sigma n \right)$$

when adding noise to each vertex’s elevation that is uniformly distributed in an interval $[-c, c]$, with $c$ a fixed constant fraction of the minimum edge length of the underlying triangulation.

### 3 Concluding remarks

We proved that the smoothed complexity of the visibility map of not-too-steep terrains with fat triangles of similar size is $O(n)$. This is the first time that realistic input models are combined with smoothed analysis. Such an approach could also shed light on the complexity of other terrain structures. For example, the complexity of the river network on real-world terrains seems to be linear, while the worst-case complexity of the river network on a terrain with the above-mentioned properties is still $\Theta(n^2)$ [7]. Combining these properties with smoothed analysis may lead to better bounds.

### References


