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HOW TO EXTEND ROBERTS' LAW FOR ECCENTRICALLY
DRIVEN, INVERTED SLIDER-CRANKS

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Abstract. The extension of Roberts' Law concerns inverted slider-cranks with an
eccentricity and a general tracing-point attached to the moving plane of the slider. As
both curve-cognates degenerate in this case, the infinite turning-joint has first been
replaced by a finite far-away joint, but such that the coupler-plane also meets two
accuracy-positions of the erstwhile slider.
One of the curve-cognates may then be adapted or modified until up to six accuracy­
positions are met. They seem to be sufficient to attain an extraordinary good approximati­
on of the entire branch of the original curve. Besides, even the transmission-angle attains
a value about twice as large as the one obtained with the auxiliary four-bar representing
the first approximation.
Cases, where the inverted slider-crank turns into a crank-and-rocker, into a double crank
or even into a stretchable four-bar, are shown. The latter corresponds with one of
Grashof's border cases for the inverted slider-crank. Roberts' Law then ensures that each
inverted slider-crank, having a fully revolving crank, possesses three four-bar branch­
cognates this way.

1. Introduction

Though Roberts' Law (ref. [1]) describes the general existence of three four-bar curve
 cognates, all able to produce the same four-bar coupler curve, its application seems to
collapse when applied on a degenerated four-bar such as the inverted slider-crank.
However, a previous paper [4] showed a practical circumvention in the special case
where the inverted slider-cranks produced symmetrical curves. Then, a very good
approximation was obtained based on stretch-rotation and symmetrization. However, for
the more general type of the (eccentric) inverted slider-crank, symmetrization isn't
applicable and has to be replaced by another procedure, apparently leading to a similar
concurrence between the curves as in the symmetrical case.
In the eccentric case, we are going to start from in this paper, also non-Grashof types
occur. Then, there is no crank making a complete revolution, whereas the curve produced
will be singular branched as then only one branch occurs as the gradual merging result of
two different branches originally produced by Grashof types.
In the symmetrical case the replacement four-bar reproduced only one branch generated
by the then centrically driven inverted slider-crank. Naturally, something alike is to be
expected for the eccentric case. Thus, when an eccentrically driven inverted slider-crank
is of the Grasfof-type, two branches appear, each possibly replaceable by one coupler­
branch of a Grashof four-bar. For the Non-Grashof type only part of the curve may be
reproduced. This part then corresponds with either a forward - or, otherwise, a backward
stroke of the "input-rocker" between its extreme positions (Fig. 9).
Fig. 1: Initial coupler branch produced by point $K$ attached to the slider of an inverted slider-crank (4 accuracy positions chosen to design a 4-bar branch-cognate with)

Fig. 2: Design of 4-bar branch cognate (nearly) producing the same coupler branch as the one produced by the eccentric inverted slider-crank (4 position synthesis with position reduction)
2. Investigated Types

It is possible to distinguish between four types of eccentric inverted slider cranks. Two of them are of the Grashof type, whereas two others are Non-Grashof. For the Grashof-type either \( a \cdot e_i < d \) (Fig. 1), or \( d \cdot e_i < a \) (Figs. 3, 10).

For the Non-Grashof type, the crank-circle about \( A_0 \) has to intersect the eccentricity-circle (about \( B_0 \)) at real points \( R \) and \( \bar{R} \), leading to the very existence of \( \Delta A_0 RB_0 \) meeting the three Non-Grashof conditions:

\[
\begin{align*}
a < d \cdot e_i \\
d < e_i \cdot a \\
e_i < a \cdot d
\end{align*}
\]

Of these, the last one has to be met anyway, otherwise no real mechanism is to be drawn with real tangents to the eccentricity-circle. Thus, as \( e_i < a \cdot d \) represents an almost trivial condition, true also for Grashof linkages, only two cases are left for the Non-Grashof type. They are:

- case I: \( d - e_i < a < d' \) (Fig. 9)
- case II: \( d < a < d \cdot e_i \)

Note that for the centric case, for which \( e_1 = 0 \), the Non-Grashof types disappear. Further, border-cases appear when either \( e_i < a \cdot d \) (Fig. 6) or \( d \cdot e_i < a \).

Of course, in order to be complete, all occurring cases are of interest when looking for the possible existence of four-bar branch cognates when Grashof linkages are at hand, or "half curve cognates" when Non-Grashof inverted slider cranks are presented (Fig. 9).

3. Design of the 4-bar branch-cognate

The actual design of the four-bar cognate replacing the inverted slider-crank occurs in two stages: (Figs. 2, 4, 7, (9) and 10)

In the 1\(^{st}\) stage a first approximation is adopted leading to an auxiliary four-bar, based on two specifically chosen accuracy positions of the coupler-point and on a simultaneous replacement of the slider.

In the 2\(^{nd}\) stage, part of Roberts' Configuration is used to find another crank-circle, simultaneously giving the designer the opportunity to improve his first approximation of the branch through two other pairs of specifically chosen accuracy positions.

As a result, only one branch of the 4-bar coupler curve approximates the chosen branch, initially produced by the inverted slider-crank. For the other branch one finds another branch-cognate somewhat different from the first.

Accuracy positions, as will be proved, occur in pairs. That is to say, for each accuracy-position another exists leading to the same result. So, basically, the design uses only three accuracy positions, in reality being six.

The first pair is to be allocated for the auxiliary four-bar, whereas the remaining pairs are to be used for the final mechanism found in the 2\(^{nd}\) stage.
Fig. 3: Original coupler-branch produced by a coupler point $K$ of an inverted slider-crank with eccentricity $e_1$ (4 accuracy positions are used to obtain the 4-bar branch cognate)

Fig. 4: The ecc. inverted slider-crank as well as the double crank (being the branch cognate) producing the same coupler branch
3.1 THE AUXILIARY FOUR-BAR

In order to simplify matters and to attain a unified method for all cases, the first pair of accuracy-positions are taken to correspond with the perpendicular positions of the input-crank with respect to the frame. (Figs. 1, 3, 6, 10). Then, \( P_{12} \), the virtual rotation center of the positions, \( A_1K_1 \) and \( A_2K_2 \), coincides at the center \( B_0 \) of the eccentricity-circle. (The mid-normal of \( B_1B_2 \) always joins \( B_0 \)).

One further replaces the slider \( A_1R_1 \), touching the eccentricity-circle at \( R_1 \), by a basically required large "circle" with center \( B_1 \). Then, the last point, becoming a finite turning-joint, has to comply with the first two accuracy-positions, whence \( B_1 \) joins the normal of \( A_1K_1 \), simultaneously meeting point \( P_{12} = B_0 \).

In order to have a large length for \( c - BP_0 - h \cdot e \), (1)

one chooses the angle \( \alpha = \angle B_1A_1R_1 = 80^\circ \). (In case \( \alpha = 90^\circ \), one reattains the inverted slider-crank.)

The dimensions of the auxiliary mechanism \( (A_0-A_1K_1B_1-B_0) \) having the same crank, the same coupler-point, and also the same center - \( B_0 \) - as the inverted slider-crank we started from, is then to be established from the initial dimensions: \( AIB_o - d \); \( AIA_o - a \); \( B_2R_1 - s_1 \); \( AIB_1 - p \) and \( s_1 \), the latter being the shortest distance from \( K_1 \) to \( A_1R_1 \).

We so find that:

\[
\begin{align*}
A_1R_1 &= \sqrt{d^2 + a^2 - s_1^2} \quad (2) \\
B_1R_1 &= h \cdot A_1R_1 \tan \alpha \quad (3) \\
A_1B_1 &= b \cdot A_1R_1 \cos^{-1} \alpha \quad (4) \\
B_1K_1 &= q \cdot \sqrt{(h \pm s_1)^2 - (A_1B_1)^2} \quad (5) \\
\beta &= Arccos(L) \quad (6)
\end{align*}
\]

Clearly, the design of the auxiliary mechanism is the same for all Grashof-types having a fully revolving crank. Thus,

for \( a \leq d - e \), with \( d > e \), (Figs. 1, 6) \nfor \( a \geq d \), with \( d > e \), (Fig. 3) \nand for \( a \geq d \), with \( d < e \), (Fig. 10)

Of course, as \( \alpha < 90^\circ \), the curve produced by the auxiliary mechanism will deviate a bit from the initial curve as produced by the eccentric inverted slider-crank. However, the deviation is to be corrected by application of the 2nd stage leading to the final replacement mechanism.
Fig. 5: The double crank as a four-bar branch cognate of an eccentrically driven inverted slider-crank (the 2 mechanisms produce about the same branch with nearly the same input velocity)

\[ \frac{a\, (d' + b' - a'' - c'')}{a\, (a'' - c'')} = 0.81 \]

\[ a' = 37.927 \text{ m} \]
\[ a'' = 66.372 \text{ m} \]
\[ b' = 215.904 \text{ m} \]
\[ c' = 181.205 \text{ m} \]

Fig. 6: Eccentrically driven, inverted slider crank (4 accuracy positions to be used for the design of a 4-bar branch cognate)
3.2 FINAL FOUR-BAR BRANCH COGNATE

Naturally, further enlargement of the angle $\alpha$ leads to lower values for the transmission angle $\mu_1 = \alpha B_0 B_1 A_1$. (For instance, if $\alpha = 90^0$, then $\mu_1 = 0^0$)

As this is undesired with regard to force-transmission, other ways have to be found. Whence, we abide by $\alpha = 80^0$, but look at Roberts' Law and ditto Configuration instead. Though Roberts' Law may then be applied caused by the now finite location of point $B_1$, the transformation does not change the curve itself.

However, if we consider a particular curve-cognate of Roberts', namely the one having a crank rotating with the same speed as the angular velocity of the initial crank, it becomes feasible to change three of its cognate-dimensions, in order to meet three different accuracy positions. To carry this out in detail, the following procedure is adopted:

1. First, change the auxiliary four-bar into its Roberts'curve-cognate having a crank $A_0'A'$ rotating with the same speed as the initial crank $A_0A$,

2. Secondly, replace turning-joint $B_1'$ or $B_2'$ until $B''$ meets the three (or six) accuracy-positions being chosen.

In more detail, this would lead to the stretch-rotation of $\Delta A_1A_0B_0$ about $B_0$ with the complex multiplication-factor $\frac{2}{b}e^{i\theta}$, yielding

$$\Delta A_1''A_0''B_0 - \frac{2}{b}e^{i\theta}(\Delta A_1A_0B_0) \quad (7)$$

For the 2nd accuracy-position, a similar equation holds:

$$\Delta A_2''A_0''B_0 - \frac{2}{b}e^{i\theta}(\Delta A_2A_0B_0) \quad (8)$$

Hence, we so obtain the simple formulas:

$$A_0''B_0 = d'' - \frac{q}{b}d \quad (9)$$

$$A_0''A_0 = a'' - \frac{q}{b}a \quad (10)$$

Further, $\angle B_1A_0''A_1'' = 90^0 - \angle B_1A_0''A_2'' \quad (11)$

Whereas, similarly according to Roberts' Configuration, also

$$A_1K_2 = \Delta A_1A_2''K_2 = p'' - \frac{c}{b}p \quad (12)$$

Naturally, the perpendicular bisectors of $\overline{A_1A_2''}$ and of $\overline{K_1K_2}$ intersect at $B_0$. Thus,

$$\Delta A_2''B_0K_2 = \Delta A_1''B_0K_1 \quad (13)$$

yielding $B_{01} = B_0 - B_{02} \quad (14)$

Then, with two other chosen accuracy positions from the initially given mechanism, say $A_3K_3$ and $A_4K_4$, it becomes quite possible to determine accurate corresponding positions,
Fig. 7: Design of four-bar Branch-Cognate of inverted slider-crank
(Grashof's Border case $a + e_i = d$ leading to $a' + b' > d' + c'$)

Fig. 8: Four-bar Branch Cognate of inverted slider-crank
$A_3''$ and $A_4''$, at the cognated crank-circle about $A_0''$. This may be realized by intersection of circles having radius $p'$ respectively about $K_3$ and $K_4$, with the cognated crank-circle about $A_0''$.

When, for instance, $A_3$ as well as $A_4$ join the initial fixed link $A_0B_0$, we find that both, $A_3''$ as well as $A_4''$, slightly deviates from the line containing the cognated fixed link, that is to say from $A_0''B_0$. Anyway, having determined their exact locations at the cognated crank-circle, we may establish the locations of the points, $B_{03}$ and $B_{04}$. The way to find these points, will be based on the so-called attachment-method, hence, on the equations:

$$
\Delta A_3''B_0K_3 = \Delta A_1''B_{03}K_1 \tag{15}
$$

$$
\Delta A_4''B_0K_4 = \Delta A_1''B_{04}K_1 \tag{16}
$$

The circle joining the three points $B_{01} = B_{02}$, $B_{03}$ and $B_{04}$ then has the required turning-joint $B_1''$ for its center.

As a result, one obtains the new dimensions:

$$
A_i''B_i'' - b' \quad B_i''B_i'' - c' \quad K_i''B_i'' - q' \tag{17}
$$

In the particular case for which the initial branch is symmetrical, which occurs when the two eccentricities are zero, $B_1''$ coincides at the center of a circle joining $K_1$, $B_0$ and $A_i''$. In that case $b' - c' - q'$, meaning that the four-bar is of the $\lambda$-type (ref.[4]).

The accuracy-positions chosen, corresponded with the initial crank-positions dividing $360^\circ$ into equal parts of $90^\circ$.

A measure for the accuracy of the cognate curve may be found by comparing the Grashof-distances of the two mechanisms. Then, the best approximation will be the one having its Grashof-distance-ratio much nearer the value 1 than with other approximations. For the mechanisms of Fig. 2, for instance, the auxiliary four-bar yields the Grashof-distance-ratio $\frac{a - c - d - b}{a - d \cdot e_i} = 1.57$, whereas the final branch-cognate results into the ratio $\frac{d(a' - b' - c' - d')}{d'(a - d \cdot e_i)} = 0.924$. Thus, the latter represents the better mechanism. (Note that the Grashof-distance-ratio equals 1 when two Roberts' 4-bar curve cognates are compared.)

For the auxiliary mechanism of Fig. 4, we obtain the Grashof-distance-ratio
Fig. 9: Non-Grashof Curve-Cognate

Fig. 10: Inverted slider crank and its 4-bar branch-cognate producing the same branch (4 position-synthesis using a point position reduction method: $B_{01} = B_{02}$)
\[ \frac{d \cdot c - a \cdot b}{a - d \cdot e_t} = 2.34, \] whereas the branch-cognate gives rise to the ratio

\[ \frac{d(d' \cdot b' - a' - c')}{d'(a - d \cdot e_t)} = 0.81. \]

Even for Non-Grashof linkages, namely for those without revolving bars, the method remains applicable. Fig. 9, for instance, yields for the auxiliary mechanism the Grashof-distance-ratio

\[ \frac{a \cdot b - c - d}{a - d \cdot e_t} = 1.65, \] whereas the final curve-cognate gives the ratio

\[ \frac{d(a' \cdot c' - d' - b')}{d'(a - d \cdot e_t)} = 1.35. \]

It is quite possible that an other distribution of our accuracy-positions, gives a better Grashof-ratio. Remarkable better results though, are not to be expected. For instance, if we choose point A\(_4\) of Fig. 9 at the other intersection of the crank - and the eccentricity-circle, we obtain the ratio

\[ \frac{d(a' \cdot c' - d' - b')}{d'(a - d \cdot e_t)} = 0.73. \] Now, its distance to 1 is only slightly less than with the 1.35-value belonging to the mechanism demonstrated in Fig. 9.

In Grashof's border case, such as the particular one demonstrated in the Figs. 6 and 7, we rather observe the difference between these Grashof-distances.

Then, for the auxiliary mechanism we precisely observe the value

\[ (a \cdot c - b \cdot d) - (a \cdot e_t - d) = \sqrt{d^2 - a^2 - e_t^2} \cdot (\sin \alpha - 1) \cos^2 \alpha = -4 \sqrt{3} \tan \frac{\pi}{2} \left[(\pi/2) - \alpha\right] = -4 \sqrt{3} \tan 5^\circ = -6.0614. \]

As in this case each Grashof-distance should tend to zero, a considerable improvement is obtained with the final branch-cognate as demonstrated in Fig. 8. Then, namely

\[ \frac{d}{d'} \cdot (a' \cdot b' - c' - d') - (a \cdot e_t - d) = 0.037. \]

A random circle about B\(_0\) intersecting the cognated crank-circle at possible accuracy-positions \(A''_m\) and \(A''_n\), yields a virtual rotation-centre \(P_{mn} = B_0\), giving \(B_{0m} = B_{0n}\). Indeed, accuracy-positions only occur in pairs. Thus, each accuracy-position has to be counted twice, unless of course the cognated crank-positions already are at a circle about \(B_0\), as will be the case for the accuracy-positions, \(A''_m\) and \(A''_n\).

Generally, the transmission angle \(\mu_1 = \angle A''_mB''_1B_0\) appears to be about twice as large as the one obtained with the auxiliary four-bar or with its Roberts' curve-cognate. Complete application of Roberts' Law would have lead to a point \((B''_1)_{\text{cognate}}\) as the fourth vertex of a linkage parallelogram \(BB_0K_0(B''_1)_{\text{cognate}}\). However, such a point \((B''_1)_{\text{cognate}}\) would have been much farther away, giving about half the transmission-angle. All examples demonstrated show this same phenomenon. We conclude, that the 2nd stage leads to a better approximation of the generated branch, simultaneously giving about twice the transmission angle in comparision to the one obtained in the 1st stage.
Theoretically, the *circular* curve described by point \( B_0 \) of the Stephenson-2 six-bar 
\[
(A_0 - A - B\prime - B_0 - A_5\prime - A_1'' - K_1)
\]
with respect to the link \( A_1''K_1 \) will be a Stephenson-2 six-bar curve of order 16, (ref.[5]). Clearly, this curve approximates a circle very neatly, although in reality not more than 16 intersections exist. (We count 2 times 16, minus the (8 times 2) intersections of the circle at the two circular or isotropic points of the curve.)

4. Conclusions

Branches of curves produced by inverted slider-cranks having an eccentricity and/or an eccentrically located coupler point, are to be reproduced by even *three* different four-bar branch-cognates. In the two cases the curves are singular branched, the three four-bar curve-cognates reproduce only part of the curve, namely the part corresponding to either a forward - or, otherwise a backward stroke of the input-rocker. Better transmission angles may be attained at the cost of the accuracy of the reproduction. In most cases though a high accuracy is obtained with acceptable transmission angles. (The accuracy of the replacement-method has been measured with a newly introduced *Grashof-distance-ratio*.)

References