Turbulent parametric surface waves

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We study disordered capillary waves on the surface of a vertically oscillated fluid layer and try to establish their relation with weak wave turbulence. We measure the surface gradient in space and time and argue that gradient spectra are better suited for comparison to the predictions of weak wave turbulence theory than spectra of the surface elevation. Because the gradient is a vector quantity, we must distinguish longitudinal and transverse spectra. However, they prove to be related trivially through isotropy. From the measured wavenumber-frequency spectrum it appears that the dispersion relation is only satisfied approximately. In the wavenumber direction the spectral features are strongly broadened due to spatial disorder. This disagrees with weak wave turbulence theory where exact satisfaction of the dispersion relation is pivotal. We find approximate algebraic frequency and wavenumber spectra but with exponents that are different from those predicted by weak wave turbulence theory. However, other findings, such as the broadening of harmonics proportional to their frequency and the emergence of Gaussian statistics at selected wavenumber bands, point to weak wave turbulence. © 2009 American Institute of Physics. [DOI: 10.1063/1.3075951]

I. INTRODUCTION

Weak wave turbulence is the random motion of waves caused by their weakly nonlinear interaction. Weak wave turbulence is characterized by a cascade of energy from large to small scales, a phenomenon it shares with fully developed three-dimensional hydrodynamic turbulence. For example, two interacting capillary waves with wavenumbers \( k_1 \) and \( k_2 \) nonlinearly spawn waves with wavenumber \( k = k_1 + k_2 \), and so on, until the wavelength becomes so small that the energy transported downscale is dissipated by viscosity. Weak wave turbulence is characterized by algebraic Kolmogorov spectra, \( E(k) = C_k k^{-\gamma} \), with an exponent \( \gamma \) that depends on the nature of the interaction. A beautiful theory of weak wave turbulence exists\(^1\) which predicts the spectrum, including the value of the Kolmogorov constant \( C_k \). As this theory offers a remarkable and universal prediction of random wave fields, it is worthwhile to test it in a laboratory experiment.

One of the systems which have received considerable attention is the weak turbulence formed by a system of interacting gravity waves, as they are encountered at sea. In a recent numerical study of the two-dimensional Euler equations the exponent of the wavenumber spectrum agreed with weak wave turbulence, but only if the inertial range is large enough.\(^2\) A numerical study of the associated kinetic equation was discussed in Ref. 3. Several authors\(^4-6\) compared measurements of the wave height to the theoretical predictions, but the comparison has not been very conclusive. One of the problems is that the main source of energy at sea is the wind, which causes the injection of energy to be fundamentally anisotropic. Finally, weak wave turbulence has been applied to electromagnetic waves in plasmas, acoustic waves, and optical turbulence in lasers (for references, see Ref. 1).

Since the first observation of Faraday, it is well known that parametrically driven waves on the surface of a fluid can become disordered at a critical driving strength.\(^7\) The properties of this transition have been studied through the behavior of correlation functions.\(^3-10\) For large enough driving frequencies, waves are capillary. Such disordered Faraday crisps have been interpreted as a manifestation of weak wave turbulence; this has inspired the current work.

For capillary waves, weak wave turbulence theory predicts a power spectrum of the surface elevation \( E(k) \sim k^{-15/4} \). The excitation of these waves in the case of Faraday turbulence is in a narrow band of frequencies around the subharmonic response. Both this narrow band excitation and the effect of dissipation have been considered in Ref. 11. A key element of weak wave turbulence theory is the dispersion relation \( k_d(\omega) \): At a given frequency there is a single wavenumber. Therefore, the spectrum is an infinitely sharp ridge in \( (k, \omega) \) space, which broadens with increasing nonlinearity. The dispersion relation of capillary surface waves is \( k_d(\omega) = (p \omega^2 / \sigma)^{1/3} \), with \( p \) the mass density and \( \sigma \) the surface tension, and the frequency spectrum \( E(\omega) \) of the surface elevation measured in a point follows from its wavenumber companion \( E(k) \) as \( E(\omega) = E[k_d(\omega)](dk_d/d\omega) \sim \omega^{-17/6} \). Within the framework of weak wave turbulence theory, therefore, it suffices to measure the time dependence of the surface in a single point; through the dispersion relation it also tells the statistics of its spatial dependence. The instrumentation of a point measurement is relatively easy and many frequency spectra of point measurements have been published.\(^12-16\) However, since the spectral energy drops very steeply with increasing frequency, \( [E(\omega) \sim \omega^{-17/6}] \), the height detector should be linear and should have a large dynamical range.

Experiments on parametrically excited surface waves have been reported by Wright et al.\(^17,18\) who measured the surface height of an opaque fluid from the extinction of light passed through it. Both space \( E(k) \) and time \( E(\omega) \) spectra...
were measured and were found to agree with the prediction of weak wave turbulence. However, the dispersion relation, which requires a joint space-time measurement of \( E(k, \omega) \), was not tested.

Other experiments on parametrically driven waves were done by Lommer and Levinsen who measured the fluctuating height of a fluorescing laser spot on the surface. Approximate agreement was found with weak wave turbulence theory, but their wave detection method critically depends on spatial averaging. A similar experimental technique was used on directly driven water waves, but the agreement with weak wave turbulence theory was only reached after averaging over many sharp resonances in the registered frequency spectra.

Weak wave turbulence theory pertains to a system without boundaries. A laboratory experiment is necessarily confined, and wavenumbers are subject to a quantization condition. Consequently, also the wave frequencies are quantized. This is, in particular, important for gravity waves, which have little damping. Wave quantization leads to a special constraint in weak wave turbulence as at the same time energy, \( \omega = \omega_1 + \omega_2 + \omega_3 \), and momentum, \( k = k_1 + k_2 + k_3 \), must be conserved in the four-wave interactions relevant for interacting gravity waves. Only when the nonlinear resonance broadening outgrows the spacing of the discrete wavenumber grid can the system be considered unbounded.

Experiments on gravity waves in a very large basin were done by Dennisenko et al., who found that scaling exponents of the frequency spectrum of wave heights depend on the energy input into the wave system, which they ascribed to finite size effects. We will argue that in small-scale experiments on capillary-gravity waves, such as described in this paper, quantization is not an issue because the resonance tongues near wave onset are broadened by viscous damping.

Quite recent experiments on directly driven gravity-capillary waves on the surface of mercury were reported by Falcon et al. Unlike all previous experiments in this regime of wavelengths, those experiments were noise driven. The surface elevation was measured using a capacitive wire sensor piercing the surface. However, the dynamic response of such a sensor is influenced by the fluid meniscus around it whose extent is comparable to the capillary length (=1.7 mm in their experiments) which limits the measurable response to frequencies \( \leq 190 \) Hz. Their excitation had a frequency content which ranged from 0 to 6 Hz, and a log-log plot of the spectrum showed algebraic behavior with two exponents, one for the gravity regime and one for the capillary regime, the latter one agreeing with the prediction of weak wave turbulence theory. In the gravity wave regime the exponent depends on the driving strength, a behavior also found in Ref. 15 but in a much larger system. Also the energy input \( \epsilon \) was measured in Ref. 16 but, in contrast to weak wave turbulence, which predicts the power spectral density to be proportional to \( \epsilon^{1/2} \) for capillary waves, they found \( E(\omega) \sim \epsilon \). The point signal of surface gradients was found to be strongly non-Gaussian, and anomalous scaling exponents were determined. A detailed comparison with these findings will be discussed in Sec. VI of this paper.

This paper is organized as follows. In Sec. II we will discuss the experimental setup and the technique used to measure the surface gradient field. The surface gradient field is two dimensional, which on an isotropic surface leads to two distinct spectra. Their relation is discussed in Sec. III. Measurements of the time-dependent surface gradient in a point are described in Sec. IV, while space-time measurements and the dispersion relation are discussed in Sec. V. In Sec. VI we argue that time signals are intermittent which will be quantified using structure functions.

**II. EXPERIMENTAL SETUP**

A schematic view of the experiment is given in Fig. 1. We use a circular container with a diameter of 0.44 m and a vertical wall boundary. The bottom of the container is a 15 mm thick glass plate and the wall consists of a Perspex ring of 25 mm height. For experiments at high frequencies, where the mass of the large container becomes prohibitive for high excitation levels, we have used a circular container of 0.14 m diameter, with a 5 mm thick glass bottom. The fluid we use is a low-viscosity, low surface tension silicon oil. The brand...
name of the oil is Tegiloxan 3 and is produced by Goldschmidt AG (Essen, Germany). At 21 °C its specified
surface tension is $\sigma = 18.3 \times 10^{-3}$ J m$^{-2}$, which is within 0.5% of the measured value. The measured viscosity is $\nu = (3.63 \pm 0.03) \times 10^{-6}$ ms$^{-2}$ and specified density is $\rho = 892.4$ kg m$^{-3}$.

We introduce a nondimensional driving strength $\epsilon$ in terms of the driving amplitude $A$ and the amplitude $A_0$ where waves appear for the first time when $A$ is increased monotonically from 0, $\epsilon = (A/A_0)/A_0$. Therefore, the onset of waves is at $\epsilon = 0$.

The measurement of the surface gradient is based on the refraction of a weakly focused laser beam by the curved fluid surface. The refracted laser beam hits a position sensing device (PSD) which provides the $x$ and $y$ coordinates of the center of gravity of the laser spot. Therefore, the laser beam does not need to be focused on the detector for a precise measurement of its position. In fact, it will be defocused by the short surface waves. Using straightforward geometry, the surface gradient $\nabla h = (h_x, h_y)$ in the $x$ and $y$ directions follows from the measured coordinates on the PSD,

$$E_h(k) = \frac{dk}{d\omega} \sim \omega^{-3/2}.$$  \hspace{1cm} (2)

Compared to the very steep spectra of Eq. (1), whose measurement is an experimental challenge, the exponents of the gradient spectra, Eq. (2), are much larger and the experimental requirements of linearity and spectral resolution are much less severe. Therefore, a test of weak wave turbulence should preferably be done through the gradient field.

For a point measurement on an isotropic surface, also the spectra should be isotropic, that is, $E_h(\omega)$ and $E_h(k)$ should be the same. However, for spatial spectra the direction of $k$ matters. In the isotropic case, there are two distinct spectra, the longitudinal spectrum, $E^L(k)$, in which the $k$ vector points in the same direction as the component of the gradient $[E_{h_x}(k_x)$ and $E_{h_y}(k_y)]$ and the transverse one, $E^T(k)$, where they are perpendicular $[E_{h_x}(k_x)$ and $E_{h_y}(k_y)]$. The surface gradient field is potential, $\nabla \times \nabla h = 0$, and it can be shown that in the isotropic case the longitudinal and transverse spectra are related as

$$E^T(k,\omega) = -k \frac{d}{dk} E^L(k,\omega).$$  \hspace{1cm} (3)

When the surface is driven parametrically at frequency $\omega_0$, the dominant response is at the subharmonic frequency $\omega_0/2$ and wavenumber $k_0 = k_0(\omega_0/2)$. Since the waves are traveling in a direction perpendicular to their crests, the longitudinal spectrum $E^L(k)$ will show a sharp peak at $k = k_0$ but not the transverse one. If the waves satisfy the dispersion relation exactly and the longitudinal spectrum is a delta function, $E^L(k) = \delta(k - k_0)$ (it is actually a square-root singularity for the isotropic transverse spectrum), the transverse spectrum is a step function, $E^T(k) = 1 - H[k - k_0]$, with $H$ the Heaviside function. Accordingly, in wavenumber-frequency space, the longitudinal spectrum is a delta ridge, $E^L(k,\omega) = \delta(k - k_0)$, while the transverse spectrum is a step, $E^T(k,\omega) = 1 - H[k - k_0]$. An experimental test of the dispersion relation should therefore be done using the longitudinal spectrum.

**IV. MEASUREMENTS IN A POINT**

At low driving amplitudes the surface is perfectly time periodic and the standing wave patterns selected can be explained completely by weak-nonlinear theory. Its predictions agree very well with experiments. The surface becomes first disordered at $\epsilon = 0.5$. The onset pattern breaks up and the surface becomes slowly time dependent when it is observed stroboscopically at $f_0/2$.

Because the theory for pattern formation at onset does not involve lateral boundaries, we have verified previously that our system is effectively infinite already at small values of $\epsilon$. Considerations of effective system size are also relevant for the present experiments. The key point is that the parametric resonance tongues start at a finite excitation am-

**III. SURFACE GRADIENT SPECTRA**

Let us recall that weak wave turbulence predicts algebraic spectra of the surface elevation $h$ in the capillary regime,

$$E_h(\omega) \sim \omega^{-17/6} \text{ and } E_h(k) \sim k^{-15/4},$$  \hspace{1cm} (1)

where one follows from the other one through the dispersion relation of capillary waves, $k_0(\omega) = (\rho \omega^2/\sigma)^{1/3}$. We assume that the surface gradient spectrum follows from Eq. (1) as

$$E_h(k) = k^2 E_h(k) \sim k^{-7/4} \text{ and consequently }$$

$$E_{h_0}(\omega) = E_h(k) \frac{dk}{d\omega} \sim \omega^{-3/2}.$$  \hspace{1cm} (2)

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In weak wave turbulence of capillary waves, higher harmonics grow. We were able to measure the spectrum point band excitation, at very large \( f/2 \), where \( 2f/2 = l, 1, 2, \ldots \), with \( l \) up to 45 still detectable. Dashed-dotted line: fit through response maxima; \( E(f) = f^{-3.0} \). Open circles: area of peaks. Dashed line: fit of area; \( E(f)\Delta f \sim f^{-4.0} \). Inset: peak width \( \Delta f \) as a function of the order \( l \); dashed line: fit \( \Delta f(l) = 2.4 + 2.4l \).

The experiments of Fig. 2 were done at a large driving frequency \( f_0 = 150 \) Hz and a reduced driving amplitude \( \epsilon = 1 \). The container diameter was \( L = 0.14 \) m, and the surface gradient was measured in configuration (b) (see Fig. 1). As the spectrum covers a large dynamical range of energies, a point of concern is the linearity of the surface gradient measurement. Therefore, also a spectrum was measured of standing waves in a container with pinned boundary conditions at small \( \epsilon \), where we found \( E(\omega_0)/E(\omega_2) \approx 10^{-3} \), approximately two orders of magnitude smaller than the corresponding ratio in Fig. 3. Therefore, the harmonics observed in Fig. 3 (and Fig. 2) are genuine and not due to the nonlinearity of the detector.

The spectrum shows the subharmonic surface response at \( f = f_0/2 \), while many harmonics \( f = nf_0/l \) are present, with \( l \) up to 45 can be distinguished. At this driving strength, the observed energy spectrum is far from continuous and cannot be compared directly to the prediction of weak wave turbulence theory, \( E_{\text{sub}}(\omega) \sim \omega^{-3/2} \). However, a few properties of the measured spectrum point toward weak wave turbulence. In weak wave turbulence of capillary waves, higher harmonics \( \omega_{2l} = l\omega_0 \) are formed through the weakly nonlinear interaction of two waves, each with a frequency \( \omega_1 = l\omega_0/2 \). This implies that the response at a frequency with index \( 2l (\omega = 2l\omega_0/2) \) inherits its width from the waves at \( l \), that is, the resonance width must increase proportionally to \( l \). This is in excellent agreement with the experiment, where we have defined the width \( \Delta f \) as the difference of frequencies where the response has dropped to \( 1/5 \) of the maximum value at \( f_0 \). The linear increase in the peak width extends to \( l = 15 \) and should be compared to the result of a numerical simulation of capillary wave turbulence in Ref. 11, which extends to \( l = 6 \).

It is well known that a system that is driven parametrically at a frequency \( f_0 \) responds (sub)harmonically at frequencies \( (n + \frac{1}{2})f_0 \) and at \( nf_0 \). However, in the presence of finite dissipation, the response at \( n > 0 \) will only be observed at very large \( \epsilon \) (in our case \( \epsilon = 10 \)). Only in the limit of small depth, where the additional bottom dissipation is higher for the subharmonic solution, has it been possible to observe the harmonic response. Therefore, the (sub)harmonics \( n > 0 \) observed in our case are the sole consequence of nonlinearity.

The experiments of Fig. 2 were done at a large driving frequency such that the width of the response maxima can be measured readily. At \( f_0 = 150 \) Hz and \( \epsilon = 1 \), the spectral width of the subharmonic response at \( f_0/2 \) is \( \Delta f_{2}^{\text{sub}} \approx 2.4 \) Hz; however, at \( f_0 = 40 \) Hz, where our remaining results were obtained, \( \Delta f_{2}^{\text{sub}} \) is smaller than 0.3 Hz.

The line through the maxima in a log-log plot of the surface response is \( E_{\text{max}}(\omega) \sim \omega^{-3} \) while, in accordance with the behavior of the peak width, the area of the peaks behaves as \( E_{\text{area}}(\omega) \sim \omega^{-4} \). Both exponents are much smaller than the ones predicted by weak wave turbulence theory. In fact, narrow band excitation (such as in this experiment) was analyzed in the framework of weak wave turbulence theory by Fal'kovich and Shafarenko. They concluded that the area under the peaks should scale with the exponent of the continuous case: for the gradient \( E_{\text{area}} \sim \omega^{-3/2} \), clearly this does not agree with our results.

For increasing driving strength \( \epsilon \), the width of the response peaks increases rapidly. Figure 3 shows the frequency spectrum of \( h_x \) measured in a point at \( \epsilon = 3.0 \). The fluid sur-
face in a 0.44 m diameter container was driven with frequency \( f_0 = 40 \) Hz and the surface gradient was measured using configuration (b) in Fig. 1. At this driving amplitude the spectral response at high frequencies has broadened sufficiently so that the spectrum develops a continuous part which is approximately algebraic, \( E_h(f) \sim f^{-\alpha} \), with exponent \( \alpha = 3.6 \), which is not significantly different from the exponent of the peak areas in Fig. 2.

In weak wave turbulence \( \langle \nabla h^2 \rangle \) gauges the degree of nonlinearity. At a driving amplitude \( \epsilon = 1 \), the root-mean-square surface gradient is \( \langle (h_1) \rangle^{1/2} = 0.14 \) which increases to \( \langle (h_1) \rangle^{1/2} = 0.25 \) at \( \epsilon = 3 \). For a Kolmogorov spectrum to develop, the nonlinearity should be weak, and it is a question whether the measured value of 0.25 can be considered weak. The probability density function of \( h_1 \) is shown in the inset of Fig. 3. It displays very wide tails; rare events on the surface extend to \( |h_1| = 1 \). Obviously, those events cannot be considered weak. In Sec. VI we will associate those events with the intermittency of the surface.

### V. SPACE-TIME MEASUREMENTS

A measurement of the surface in a point [using configuration (b) in Fig. 1] comes with superior time response, dynamical range, and signal to noise ratio, but it only provides information about a point. The line measurement uses configuration (c) of Fig. 1; it has a smaller dynamical range while the signal to noise ratio is limited by the fast scanning in combination with the time response of the PSD. This is illustrated in Fig. 3 where we compare the frequency spectrum (averaged over the line) with the frequency spectrum of a point measurement. Still we believe that the line measurement serves our main purpose, namely, a test of the dispersion relation of the disordered surface wrinkles.

A space-time \((x, t)\) picture of the longitudinal gradient field \( |h_z(x,t)| \) and the transverse \( |h_y(x,t)| \) is shown in Fig. 4. The spatial periodicity corresponding to the wavelength \( 2\pi/k_d(\omega_0/2) = 8.4 \) mm can be clearly seen in the longitudinal gradient field, but it is marred by the random emergence of defects. Similar structures in \( \nabla^2 h \) were observed by Wright et al. In contrast, the transverse gradient field does not show a clear periodicity. Below we shall demonstrate that the observed difference between longitudinal and transverse surface gradients is consistent with the isotropy of the surface. In both cases structures are seen that travel across the image with velocities comparable to the group velocity \( v_g = 0.175 \) m/s at \( \omega_0/2 \).

The longitudinal wavenumber-frequency spectrum of \( h_z \) is shown in Fig. 5. Assuming isotropy, we show \( \frac{1}{2}[E_{h_z}(k_z, \omega) + E_{h_z}(-k_z, \omega)] \), but waves running to the left and to the right are shown separately in Fig. 6. The wavenumber values of local maxima of \( E^z(k, \omega) \) were found by fitting Gaussians to sections of the spectrum at the frequency (sub)harmonics \( \omega_0/2, \omega_0/4, \omega_0/8, \ldots, \) of the parametric response. Especially for the high harmonics, these maxima are very broad and precise determination of their \( k \) values is difficult. As shown in the inset of Fig. 5, these maxima follow the dispersion relation, but additional maxima are observed. For example, at \( l = 1 \) a local maximum occurs at the wavenumber \( k_d(\omega_0/2) \), but also at \( k_d(3\omega_0/2) \), while at \( l = 3 \) maxima occur at \( k_d(3\omega_0/2) \), but also at \( k_d(\omega_0/2) \).

![Fig. 4. (Color online) Space-time \((x, t)\) picture of (a) \(|h_z(x,t)|\) and (b) \(|h_y(x,t)|\). The surface in a 0.44 m diameter container is driven at \( f_0 = 40 \) Hz and reduced amplitude \( \epsilon = 3.0 \). The spatial periodicity of the surface is only evident in the longitudinal \(|h_z(x,t)|\).](image)

![Fig. 5. Surface plot: longitudinal wavenumber-frequency spectrum of \( h_z \) measured at a driving frequency \( f_0 = 40 \) Hz and reduced driving amplitude of 3.0 in a container with 0.44 m diameter. We show \( \frac{1}{2}[E_{h_z}(k_z, \omega) + E_{h_z}(-k_z, \omega)] \). The length of the error bars is the width of the Gaussians used to determine these local maxima.](image)
surface ripplages we show positive and negative wavenumbers separately. The wavenumber at the subharmonic resonance, $k_0 = k_d (2\omega_0 / \omega)$, extends to the highest frequency in the longitudinal spectrum.

A striking observation in both Figs. 5 and 6 is that the response of the surface is not a sharp ridge on the $k, \omega$ plane, as is assumed by weak wave turbulence theory. Instead, the maxima in the wavenumber direction are very broad. It also appears that the time response of the surface follows the maxima at the wavenumbers and frequencies of the dispersion relation. Indeed, as will be demonstrated below, this is consistent with isotropic Faraday ripplages.

The longitudinal and transverse wavenumber spectra $E_{\ell T}(k) = \int_0^\infty E_{\ell T}(k, \omega) d\omega$ are shown in Fig. 7. The spectra are approximately algebraic $E(k) \sim k^{-\alpha}$, with exponent $\alpha \approx 3.8$. If the dispersion relation for capillary waves would hold exactly, the exponent $\beta$ of the companion frequency spectrum $E(\omega) \sim \omega^{-\beta}$ would have been $\beta = \frac{2}{3} \alpha + \frac{1}{2} = 2.9$, which differs significantly from the measured one, $\beta \approx 3.6$. Again, also the measured exponent $\alpha$ is very different from the prediction of weak wave turbulence theory ($\alpha = 7/4$).

For an isotropically wrinkled surface, longitudinal and transverse spectra are connected through Eq. (3). Figure 7(b) indeed shows that this relation holds for our measured $E_{\ell T}(k)$. Weak wave turbulence should equally work for longitudinal and transverse spectra; they are related trivially through isotropy, although the appearance of the spectra in Figs. 6(a) and 6(b) is different at first sight.

According to the scenario of weak wave turbulence, the surface elevation should have Gaussian statistics. More precisely, the probability density function of the Fourier amplitudes $h(k, t)$ of the surface elevation is the Rayleigh distribution $P[h(k, t)] = \frac{k}{2 \sigma^2} e^{-(k^2 / 4 \sigma^2)}$ for wavenumbers $k$ in the inertial range. In Fig. 8 we show the PDF of $E_\Delta(k, t) = \int_\Delta h(k, t)^2 dk$, integrated over three adjacent intervals which are indicated in Fig. 8(b). In agreement with weak wave turbulence theory, the statistics of $[h(k, t)]^2$, with $k$, in the inertial range, is close to a Rayleigh distribution. Intermittency shows at large wavenumbers where large energies have an enhanced probability.

For parametrically driven surface waves we have found a pervasive imprint of the narrow band forcing. This imprint fades at stronger driving. It may be contended that the spectrum of the surface only becomes continuous at values of the nonlinearity that are already beyond the regime of weak wave turbulence. In order to destroy the narrow band time response of the waves, we have repeated our experiments using parametric excitation with Gaussian noise in the frequency band $f_0 \in [30, 50]$ Hz. Noise driven parametric surface waves have been studied by Residori et al. In our case, noise gives rise to an intermittent temporal global surface response: the surface randomly alternates between episodes with turbulent waves and the flat state. We have found that with parametric excitation by noise, only the first three peaks...
in the frequency spectrum are discernible, but all our conclusions regarding the spectral exponents remained unaltered.

VI. INTERMITTENCY

Apart from spectra we also measured structure functions of the surface gradient $\nabla h(t)$ in a point. The motivation of these experiments is the recent discovery of intermittency in surface gradient fluctuations by Falcon et al.\textsuperscript{20} In fact, they did not measure the proper surface gradient but the second difference of the surface elevation signal in a point, leading to the increment $h_2(\tau) = h(t + \tau) - 2h(t) + h(t - \tau)$ of the surface elevation signal. This increment was found to be strongly non-Gaussian, with moments $\langle |h_2(\tau)|^p \rangle$ that depend algebraically on the delay time $\tau$, $\langle |h_2(\tau)|^p \rangle \sim \tau^\alpha$. Not only was the second difference $h_2$ chosen because of its resemblance to the surface gradient but also because its moments showed anomalous scaling, contrary to those of the first increment $h_1(\tau) = h(t + \tau) - h(t)$. The absence of anomalous scaling of $\langle |h_1(\tau)|^p \rangle$ was ascribed to the steepness of the spectrum $E_0(\omega)$ and, consequently, the non-differentiability of the signal $h(t)$.

In the present experiment, the surface gradient is measured directly and we have studied the scaling properties of the temporal increments $\Delta h, (\tau) = |h(t + \tau) - h(t)|$ at a driving frequency $f_0 = 40$ Hz and a reduced driving amplitude $\epsilon = 3.0$. The intermittency of random surface waves is vividly illustrated in the high-pass filtered signal in Fig. 9(a) where it shows as intense bursts, alternated with quiet episodes. The high-order structure functions $G_p(\tau) = \langle |\Delta h(\tau)|^p \rangle$ are shown in Fig. 9(b). For time delays shorter than the principal wave period $2/f_0$ and larger than the ones on which the signal becomes smooth, these structure functions have approximate algebraic behavior, $G_p(\tau) \sim \tau^{\zeta_p}$ with the scaling exponents $\zeta_p$ shown in Fig. 9(c). For self-similar, nonintermittent signals, $\zeta_p$ would be proportional to $p$, whereas the measured $\zeta_p/p$ decreases with $p$. Therefore, the scaling exponents are anomalous.

For the order 1 ($p = 1$) structure function of the second-order increments $|h_2(\tau)|$ of the surface elevation, which were thought to resemble the first increments of the surface gradient, Falcon et al.\textsuperscript{20} found an exponent $\zeta_1 = 1.55$, while we find that the true gradient signal gives $\zeta_1 = 0.95$. This exponent is an inertial-range property; at smaller scales the gradient signal $h_s(t)$ becomes smooth, and the corresponding slopes $\zeta_p$ tend to $\zeta_p = p$, as Fig. 9(b) suggests.

In order to compare our results to those of Ref. 20 we plot $\zeta_p/\zeta_1$ in Fig. 9(c). Although it is only in a relative sense, the results of both experiments approximately agree and point to strongly anomalous scaling exponents.

VII. CONCLUSION

We have studied disordered parametrically driven surface waves in the context of the theory of weak wave turbulence. For the first time we have measured the space-time statistics of the surface gradient field. It appears that the dispersion relation, which is pivotal in weak wave turbulence, is only satisfied approximately in our experiments. We emphasize that this is the first time that the dispersion relation for disordered surface waves has been tested; in fact it should be measured in any experiment of disordered waves.

The $k$ spectrum is more diffuse than the $\omega$ spectrum possibly because of the strong spatial disorder induced by localized defects. The exponents of both $k$ and $\omega$ spectra are very different from the prediction of weak wave turbulence theory, although a measurement of the surface gradient rather than its elevation would make these exponents better accessible experimentally.

In contrast to earlier experiments in disordered parametrically driven surface waves, we have measured the surface gradient field. While these earlier experiments\textsuperscript{12,17,18} find agreement with weak wave turbulence, our exponents disagree with its predictions. Spatial resolution may be a possible explanation for this discrepancy; for example, other experiments do not show the sharp harmonics such as in Figs. 2 and 3.

However, many other aspects of our results do point to wave interaction, such as the characteristic increase in the width of the narrow spectral peaks and the emergence of Gaussian statistics of waves with inertial-range wavenumbers.

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3. S. Annenkov and V. Shriria, “Direct numerical simulation of downshift...