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ON EXACT GROUP EXTENSIONS

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ABSTRACT. We give conditions for the exactness of $\mathbb{R}^d$-extensions.

§0 INTRODUCTION

A nonsingular transformation $(X, B, m, T)$ of a standard probability space is called a fibred system if there is a generating measurable partition $\alpha$ such that $T : a \to Ta$ is invertible, nonsingular for $a \in \alpha$, and a Markov map (or Markov fibred system) if in addition, $Ta \in \sigma(\alpha) \mod m \ \forall \ a \in \alpha$.

Write $\alpha = \{a_s : s \in S\}$ and endow $S^N$ with its canonical (Polish) product topology. Let

$$\Sigma = \{s = (s_1, s_2, \ldots) \in S^N : m(\bigcap_{k=1}^{n} T^{-k}a_{s_k}) > 0 \ \forall \ n \geq 1\},$$

then $\Sigma$ is a closed, shift invariant subset of $S^N$, and there is a measurable map $\phi : \Sigma \to X$ defined by $\{\phi(s_1, s_2, \ldots)\} := \bigcap_{k=1}^{\infty} T^{-k}a_{s_k}$.

The closed support of the probability $m' = m \circ \phi^{-1}$ is $\Sigma$, and $\phi$ is a conjugacy of $(X, B, m, T)$ with $(\Sigma, B(\Sigma), m', \text{shift})$. Thus we may, and sometimes do, assume that $X = \Sigma$, $T$ is the shift, and $\alpha = \{[s] : s \in S\}$.

For $n \geq 1$, there are $m$-nonsingular inverse branches of $T$ denoted $v_a : T^n a \to a$ and with Radon Nikodym derivatives denoted

$$v'_a := \frac{dm \circ v_a}{dm}.$$

Let $(X, B, m, R)$ be a nonsingular transformation of a standard probability space. The Frobenius-Perron operators $P_{R^n} = P_{R^n,m} : L^1(m) \to L^1(m)$ are defined by

$$\int_X P_{R^n} f \cdot g \ dm = \int_X f \cdot g \circ R^n dm$$

and for the locally invertible $(X, B, m, T, \alpha)$ (as above) have the form

$$P_{T^n} f = \sum_{a \in \alpha_0^{n-1}} 1_{T^n a} v'_a \cdot f \circ v_a.$$

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A locally invertible map \((X, \mathcal{B}, m, T, \alpha)\) has:

- the Renyi property if \(\exists C > 1\) such that \(\forall n \geq 1, a \in \alpha_0^{n-1}, m(a) > 0: \frac{|v'_a(x)|}{v'_a(y)} \leq C\) for \(m \times m\)-a.e. \((x, y) \in T^n a \times T^n a\).

It is well known (a proof is recalled in [A-D-U]) that any topologically mixing probability preserving Markov map with the Renyi property is exact in the sense that \(\bigcap_{n \geq 1} T^{-n} \mathcal{B} = \{\emptyset, X\} \mod m\).

Examples include:

- topological Markov shifts equipped with Gibbs measures ([Bo],[Bo-Ru]) and
- uniformly expanding, piecewise onto \(C^2\) interval maps \(T: [0, 1] \to [0, 1]\) satisfying Adler's condition \(\sup_{x \in [0,1]} \frac{|T''(x)|}{T'(x)^2} < \infty\) ([Ad]);

or, more generally,

- Gibbs-Markov maps as in [A-D1].

Now let \(\phi: X \to \mathbb{R}^d\) be measurable and consider the skew product \(T_\phi: X \times \mathbb{R}^d \to X \times \mathbb{R}^d\) defined by \(T_\phi(x, y) := (Tx, y + \phi(x))\) with respect to the (invariant) product measure \(m \times m_{\mathbb{R}^d}\) where \(m_{\mathbb{R}^d}\) denotes Lebesgue measure.

We say that \(\phi\) is aperiodic if \(\gamma(\phi) = z \bar{h}h \circ T\) has no nontrivial solution in \(\gamma \in \mathbb{R}^d, z \in S^1\) and \(h: X \to S^1\) measurable. It is not hard to show that if \(T_\phi\) is ergodic, and \(T\) is weakly mixing, then \(T_\phi\) is weakly mixing iff \(\phi\) is aperiodic.

We're interested in the exactness of \(T_\phi\).

We establish two (partial) results in this direction.

**Theorem 1.**

Suppose that \((X, \mathcal{B}, m, T, \alpha)\) is a probability preserving Markov map with the Renyi property. Let \(N \geq 1\) and \(\phi: X \to \mathbb{R}^d\) be \(\alpha_0^{N-1}\)-measurable (i.e. \(\phi(x) = \phi(\alpha_0^{N-1}(x))\) where \(x \in \alpha_0^{N-1}(x) \in \alpha_0^{N-1}\)).

If \(T_\phi\) is topologically mixing, then \(T_\phi\) is exact.

For the other result, we assume that \((X, \mathcal{B}, m, T, \alpha)\) is an exact probability preserving locally invertible map with the property that for some Banach space \((L, \| \cdot \|_L)\) of functions with \(\| \cdot \|_2 \leq \| \cdot \|_L\), such that \(P_T: L \to L\) and \(\exists M > 0, \theta \in (0, 1)\) such that

\[
\|P_T f - \int_X f dm\|_L \leq M \theta^n \|f\|_L \ \forall \ f \in L.
\]

This property can be obtained as a consequence of the quasi compactness of Doeblin-Fortet operators, see [D-F], [IT-M]).

Given \(\phi: X \to \mathbb{R}^d\) measurable, we define the characteristic function operators \(P_t(f) = P_T(e^{i(t, \phi)} f)\) (\(t \in \mathbb{R}^d\)).

We assume also that \(P_t: L \to L\) (\(t \in \mathbb{R}^d\)) and that \(t \to P_t\) is continuous (\(\mathbb{R}^d \to \text{Hom}(L, L)\)).

It is shown in [Nag] (see also theorem 4.1 of [A-D1]) that

\((i)\) there are constants \(\epsilon > 0, K > 0\) and \(\theta \in (0, 1)\); and continuous functions \(\lambda: B(0, \epsilon) \to B_C(0, 1), g: B(0, \epsilon) \to L\) such that

\[
\|P_t^n h - \lambda(t)^n g(t) \int_X h dm\|_L \leq K \theta^n \|h\|_L \ \forall \ |t| < \epsilon, \ n \geq 1, \ h \in L;
\]

and
(ii) in case \( \phi \) is aperiodic, then \( \forall \ 0 < \delta < M < \infty, \ \exists \ K > 0, \ 0 < \rho < 1 \) such that
\[
\| P^n \gamma h \|_L \leq K \rho^n \ \forall \ h \in L, \ n \geq 1, \ \delta \leq |\gamma| \leq M.
\]

Examples include:
- (see [A-D1], \( (X, B, m, T, \alpha) \) a Gibbs-Markov maps and \( \phi : X \to \mathbb{R}^d \) uniformly Hölder continuous on partition sets. Here \( L \) is a space of Hölder continuous functions \( f : X \to \mathbb{C} \).
- (see [Rou], [Ry]), \( X = [0,1], \ m \text{ Lebesgue measure, } \alpha \text{ a partition of } X \mod m \) into open intervals, and \( T : a \to Ta \) an invertible, \( m \)-nonsingular homeomorphism for each \( a \in \alpha \) with \( \inf |T'| > 1 \) and \( \frac{1}{T} \) of bounded variation on \( X \); and \( \phi : X \to \mathbb{R}^d \) either: of bounded variation on \( X \); or constant on each \( a \in \alpha \).

Set \( \phi_n = \phi + \phi \circ T + \ldots + \phi \circ T^{n-1} \).

**Theorem 2.**

Suppose that
\[
(\diamond) \quad \forall \ \lambda > 1 \ \exists \ n_k \to \infty \text{ such that } \frac{\phi_{n_k}}{\lambda^{n_k}} \to 0 \text{ a.e. as } k \to \infty
\]
and that \( \phi \) is aperiodic;
then \( T_\phi \) is exact.

**Remarks.**

1) Theorem 2 generalises the corresponding theorem on page 443 in [G].
2) The condition \( (\diamond) \) is satisfied if \( m \)-dist \( (\phi) \) is in the domain of attraction of a stable law.
3) The condition \( (\diamond) \) is not satisfied iff \( \exists \ \lambda > 1 \) and \( \epsilon > 0 \) such that \( m(|\phi_n| > \lambda^n) \geq \epsilon \ \forall \ n \geq 1 \) and there are independent processes like this.

§1 **Frobenius-Perron Operators, Exactness and Relative Exactness**

Let \( (X, B, m, R) \) be a nonsingular transformation of a standard probability space. The tail \( \sigma \)-algebra of \( (X, B, m, R) \) is \( T(R) := \bigcap_{n=1}^\infty R^{-n}B \) and the nonsingular transformation \( R \) is called exact if \( \{\emptyset, X\} \mod m \).

**Theorem 1.1** [D-L].

\[
\| P^n R f \|_1 \to \| E(f|T(R)) \|_1 \text{ as } n \to \infty \ \forall \ f \in L^1(m).
\]

In particular (see [L]), \( R \) is exact if \( \| P^n R f \|_1 \to 0 \ \forall \ f \in L^1(m), \int_X f dm = 0. \)

**Proof.**

First note that \( |P_T f| \leq P_T |f| \) whence \( \| P^n R f \|_1 \downarrow \) and \( \exists \lim_{n \to \infty} \| P^n R f \|_1. \) Next, \( \forall \ n \geq 1 \ \exists \ g_n \in L^\infty(B) \) with \( \int_X (P^n R f) g_n dm = \| P^n R f \|_1, \) whence
\[
\| P^n R f \|_1 = \int_X f g_n \circ R^n dm.
\]

By weak * compactness, \( \exists \ n_k \to \infty \) and \( g \in L^\infty(B) \) such that \( g_{n_k} \circ R^{n_k} \to g \) weak * in \( L^\infty(B). \)
It follows that $g \in L^\infty(T(R))$, $\|g\|_\infty \leq 1$ and $\lim_{n \to \infty} \|P_{R^n}f\|_1 = \int_X fgdm$.

Thus
\[
\lim_{n \to \infty} \|P_{R^n}f\|_1 \leq \sup \left\{ \int_X fhdm : h \in L^\infty(T(R)), \|h\|_\infty \leq 1 \right\} = \|E(f|T(R))\|_1.
\]

To show the converse inequality, note that $\exists \ g \in L^\infty(T(R)), \|g\|_\infty = 1$ such that
\[
\|E(f|T(R))\|_1 = \int_X E(f|T(R))gdm = \int_X fgdm
\]
whence $\forall \ n \geq 1$, $\exists \ g_n \in L^\infty(B), g = g_n \circ R^n$ and
\[
\|E(f|T(R))\|_1 = \int_X fgdm = \int_X f g_n \circ R^n dm = \int_X (P_{R^n}f)g_n dm \leq \|P_{R^n}f\|_1.
\]

Let $(X, B, m, R)$ and $(Y, C, \mu, S)$ be nonsingular transformations of standard probability spaces. A factor map is a function $\pi : X \to Y$ satisfying $\pi^{-1}C \subset B$, $\pi \circ T = S \circ \pi$, $m \circ \pi^{-1} = \mu$.

The fibre expectation of the factor map $\pi : X \to Y$ is an operator $f \mapsto E(f|\pi)$, $L^1(X, B, m) \to L^1(Y, C, \mu)$ defined by $\int_Y E(f|\pi)gd\mu = \int_X fg \circ \pi dm$.

The factor map $\pi : X \to Y$ is called relatively exact if $f \in L^1(B), \ E(f|\pi) = 0 \ a.e. \implies \|P_{R^n}f\|_1 \to 0$.

The corollary below appears in [G]. For the convenience of the reader, we supply a (possibly different) proof.

**Proposition 1.2.** Suppose that $\pi : X \to Y$ is relatively exact, then $T(R) = \pi^{-1}T(S) \mod m$.

**Proof.**

Evidently, $\pi^{-1}T(S) \subseteq T(R)$. We show that $\pi^{-1}T(S) \supseteq T(R)$.

By relative exactness and theorem 1.1, if $f \in L^1(B)$ and $E(f|\pi) = 0 \ a.e.$, then $\int_X fgdm = 0 \ \forall \ g \in L^\infty(T(R))$.

Thus if $f \in L^2(B) \ominus L^2(\pi^{-1}C)$, then $E(f|\pi) = 0 \ a.e.$ and so
\[
\int_X fgdm = 0 \ \forall \ g \in L^\infty(T(R)), \implies f \perp L^2(T(R)).
\]

Thus $L^2(B) \ominus L^2(\pi^{-1}C) \subset L^2(B) \ominus L^2(T(R))$ whence $L^2(T(R)) \subset L^2(\pi^{-1}C)$ and $T(R) \subset \pi^{-1}C \mod m$.

To see that in fact $T(R) \subseteq \pi^{-1}T(S) \mod m$, fix $N \geq 1$, then
\[
T(R) = \bigcap_{n \geq 1} R^{-n}B = \bigcap_{n \geq N+1} R^{-n}B
= R^{-N}T(R) \subset R^{-N}\pi^{-1}C = \pi^{-1}S^{-N}C.
\]

Taking the intersection over $N$ shows the claim. $\Box$

**Corollary 1.3 ([G], proposition 1).**

If $S$ is exact and $\pi : X \to Y$ is relatively exact, then $T$ is exact.
For a nonsingular transformation \((X, \mathcal{B}, m, R)\), define the tail relation of \(R\):

\[ \mathcal{T}(R) := \{(x, y) \in X \times X : \exists n \geq 0, R^n x = R^n y\}. \]

Evidently \(\mathcal{T}(R)\) is an equivalence relation and if \((X, \mathcal{B}, m)\) is standard, then \(\mathcal{T}(R) \in \mathcal{B}(X \times X)\).

If \(R\) is locally invertible, then \(\mathcal{T}(R)\) has countable equivalence classes and is nonsingular in the sense that \(m(\mathcal{T}(R)(A)) = 0 \forall A \in \mathcal{B}, m(A) = 0\) where \(\mathcal{T}(R)(A) := \{y \in X : \exists x \in A \ (x, y) \in \mathcal{T}(R)\}\).

A set \(A \in \mathcal{B}(X)\) is invariant under the equivalence relation \(\mathcal{T} \in \mathcal{B}(X \times X)\) if \(\mathcal{T}(A) = A\) and the equivalence relation \(\mathcal{T}\) is called ergodic if \(\mathcal{T}\)-invariant sets have either zero, or full measure.

The collection of invariant sets under \(\mathcal{T}(R)\) is the tail \(\sigma\)-algebra \(\mathcal{T}(R)\) (whence the name "tail relation").

In order to prove theorem 1, it suffices to show that \(\mathcal{T}(T^\phi)\) is ergodic.

The tail relation of \(T^\phi\) is given by

\[ \mathcal{T}(T^\phi) = \{((x, s), (y, t)) \in (X \times G)^2 : \exists n \geq 0, T^n x = T^n y, s - t = \phi_n(y) - \phi_n(x)\} \]

\[ = \{((x, s), (y, t)) \in (X \times G)^2 : (x, y) \in \mathcal{T}(T), \phi(x, y) = s - t\} \]

where \(\phi : \mathcal{T}(T) \to \mathbb{R}^d\) is defined by \(\phi(x, y) := \sum_{n=0}^{\infty} (\phi(T^n x) - \phi(T^n y))\).

We prove that \(\mathcal{T}(T^\phi)\) is ergodic by the method of Schmidt (explained in [S]), by showing that \(\forall t \in \mathbb{R}^d, U\) a neighbourhood of \(t\) and \(A \in \mathcal{B} m(A) > 0, \exists B \in \mathcal{B} B \subset A\) and \(\tau : B \to B\) nonsingular such that \((x, \tau(x)) \in \mathcal{T}(T)\) and \(\phi(x, \tau(x)) \in U \forall x \in B\).

This boils down to showing that

\[ \forall A \in \mathcal{B}_+ g_0 \in \mathbb{R}^d \eta > 0, \exists B \in \mathcal{B}_+ B \subset A, n \geq 1 \]

\[ \text{and } \tau : B \to \tau B \subset A \text{ nonsingular such that} \]

\[ T^n \circ \tau \equiv T^n \text{ and } \|\phi_n \circ \tau - \phi_n - g_0\| < \eta \text{ on } B. \]

The proof of (†) will be written as a sequence of minor claims, §0, §1, . . .

§0 We first claim that there is no loss in generality in assuming that \(N = 1\) (i.e. that \(\phi : X \to \mathbb{R}^d\) is \(\alpha\)-measurable). This is because \((X, \mathcal{B}, m, T, \alpha^{N-1})\) is also a probability preserving Markov map with the Renyi property and inducing the same (shift) topology on \(X\) as \((X, \mathcal{B}, m, T, \alpha)\).

§1 \(\forall s, t \in S, \exists \kappa = \kappa_{s,t} \geq 1\) and \(a = a_{s,t} = [a_1, \ldots a_\kappa], b = b_{s,t} = [b_1, \ldots b_\kappa] \in \alpha_0^{\kappa-1}, a_1 = b_1 = s a_\kappa = b_\kappa = t\) such that \(\|\phi_{\kappa}(b) - \phi_{\kappa}(a) - g_0\| < \eta\).

This follows from topological mixing of \(T^\phi\).

By the Renyi property, \(\exists M > 1\) such that

\[ M^{-1}m(u)m(v) \leq m(u \cap T^{-k}v) \leq Mm(u)m(v) \forall u \in \alpha_0^{k-1}, v \in \alpha_0^{l-1}, [v_1] \subset T[u_k]. \]
Given \( u = [u_1, \ldots, u_n] \in \alpha_0^{n-1} \) with \( u_n = t \), define \( \tau = \tau_u : u \cap T^{-n}a \to u \cap T^{-n}b \) by

\[
\tau(u_1, \ldots, u_n, a_1, \ldots a_\kappa, y) := \tau(u_1, \ldots, u_n, b_1, \ldots b_\kappa, y).
\]

\[ \mu \tau = \tau \mu \] is invertible nonsingular and \( \frac{d\mu \tau}{d\mu} = M^{\pm 4} \frac{m(b)}{m(a)}. \)

**Proof**

\[
\int_{u \cap T^{-n}a \cap c} \frac{dm \circ \tau}{dm} dm = m(u \cap T^{-n}b \cap c)
\]

\[
= M^{\pm 2} \frac{m(b)}{m(a)} m(u) m(b) m(c)
\]

\[
= M^{\pm 4} \frac{m(b)}{m(a)} m(u \cap T^{-n}a \cap c).
\]

\[ \square \]

**§3 Proof of Theorem 2**

We prove theorem 2 via corollary 1.3. To do this, we must consider \( T_\phi \) as a nonsingular transformation with respect to some probability \( P \sim m \times m_{\mathbb{R}^d} \).

Let \( p : \mathbb{R}^d \to \mathbb{R}_+ \) be continuous with \( \int_{\mathbb{R}^d} p(y) dy = 1 \) and define a probability \( P \) on \( X \times \mathbb{R}^d \) by \( dP(x, y) := p(y) dm(x) dy \); then \( (X \times \mathbb{R}^d, B(X \times \mathbb{R}^d), P, T_\phi) \) is a nonsingular transformation with Frobenius-Perron operators given by
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\[ P_{T^\phi} Pf(x, y) = \frac{1}{p(y)} P_{T^\phi} (f \cdot 1 \otimes p)(x, y) \]

where \( P_{T^\phi} := P_{T^\phi,m \times m_{\mathbb{R}^d}} \).

Consider the map \( \pi : X \times \mathbb{R}^d \to X \) defined by \( \pi(x, y) = x \). This is a factor map as it satisfies \( \pi^{-1}B(X) \subset B(X \times \mathbb{R}^d) \), \( \pi \circ T^\phi = T \circ \pi \), \( P \circ \pi^{-1} = m \).

The fibre expectation of \( \pi \) is given by

\[ E(f|\pi)(x) = \int_{\mathbb{R}^d} f(x, y)p(y)dy \quad (f \in L^1(X \times \mathbb{R}^d, B(X \times \mathbb{R}^d), P)). \]

By corollary 1.3 and exactness of \( T \), it suffices to show that \( \pi \) is relatively exact. To do this, we show that

\[ \int_{\mathbb{R}^d} f(x, y)p(y)dy = 0 \text{ a.e.} \implies \int_{X \times \mathbb{R}^d} |P_{T^\phi} Pf|dP = \int_{X \times \mathbb{R}^d} |P_{T^\phi} (f \cdot 1 \otimes p)|dm \to 0 \]

as \( n \to \infty \); equivalently (taking \( F(x, y) := f(x, y)p(y) \)),

\[ \int_{\mathbb{R}^d} F(x, y)dy = 0 \text{ a.e.} \implies \int_{X \times \mathbb{R}^d} |P_{T^\phi} F|dm \to 0 \]

as \( n \to \infty \).

To prove (*), we first claim that

\[ \|P_{T^\phi} (h \otimes f)\|_1 \leq C \lambda^{-\frac{n+d}{2}} \|P_{T^\phi} (h \otimes f)\|_2 + o(1) \]

as \( k \to \infty \) where \( C = 2^{\frac{d}{2}}m(B(0, 1)) \) and \( \frac{\phi_n}{\lambda^k} \to 0 \) a.e.

**Proof** As can be checked,

\[ P_{T^\phi}(h \otimes f)(x, y) = P_{T^\phi}(h(\cdot)f(y - \phi_n(\cdot)))(x) \quad (h \in L^1(m), \ f \in L^1(\mathbb{R}^d)). \]

Denoting \( E(H) := \int_X H dm \) for \( H \in L^1(m) \), we have

\[ \|P_{T^\phi} (h \otimes f)\|_1 = \int_{\mathbb{R}^d} |E(P_{T^\phi} (h(\cdot)f(y - \phi_n(\cdot)))(x))|dy \leq \int_{|y| \leq 2\lambda^k} + \int_{|y| > 2\lambda^k}. \]

By the Cauchy-Schwartz inequality,

\[ \int_{|y| \leq 2\lambda^k} \leq \sqrt{m_{\mathbb{R}^d}(B(0, 2\lambda^k))}\|P_{T^\phi} (h \otimes f)\|_2 = C \lambda^{-\frac{n+d}{2}} \|P_{T^\phi} (h \otimes f)\|_2 \]

whereas
\[
\int_{|y|>2\lambda^{n_k}} \leq \int_{|y|>2\lambda^{n_k}} |E(P_{\tau^{n_k}}(h(\cdot)f(y - \phi_{n_k}(\cdot))1[|\phi_{n_k}(\cdot)| \leq \lambda^{n_k}]|)dy
+ \int_{|y|>2\lambda^{n_k}} |E(P_{\tau^{n_k}}(h(\cdot)f(y - \phi_{n_k}(\cdot))1[|\phi_{n_k}(\cdot)| > \lambda^{n_k}]|)dy = I + II.
\]

Here as \(k \to \infty:\)

\[
II \leq \|f\|_1 E(|h|1[|\phi_{n_k}(\cdot)| > \lambda^{n_k}]) \to 0
\]
since \(\frac{\phi_{n_k}}{\lambda^{n_k}} \to 0\) a.e.; and

\[
I \leq \int_{|y|>2\lambda^{n_k}} E(|h||f(y - \phi_{n_k})|1[|\phi_{n_k}(\cdot)| \leq \lambda^{n_k}])dy = E(|h|1[|\phi_{n_k}| \leq \lambda^{n_k}]) \int_{|y|>2\lambda^{n_k}} |f(y - \phi_{n_k})|dy
\]

\[
\leq E(|h|) \int_{|y|>\lambda^{n_k}} |f(y)|dy \to 0.
\]

Substituting (3), (4) and (5) into (2) proves \(\|\|\).

To complete the proof of \((*)\), let \(F \in L^1(m \times m_{\mathbb{R}^d})\) satisfy \(\int_{\mathbb{R}^d} F(x, y)dy = 0\) for \(m\)-a.e. \(x \in X\) and fix \(\epsilon > 0\). We show that

\[
(*)_{\epsilon}
\limsup_{n \to \infty} \int_{X \times \mathbb{R}^d} |P_{\tau^n} F|d(m \times m_{\mathbb{R}^d}) < \epsilon.
\]

Standard approximation techniques show that \(\forall \epsilon > 0, \exists N \in \mathbb{N}, h_1, \ldots, h_N \in L, g_1, \ldots, g_N \in L^1(\mathbb{R}^d)\) such that \(\int_{\mathbb{R}^d} g_k(y)dy = 0\) \((1 \leq k \leq N)\) and

\[
\|F - \sum_{k=1}^N h_k \otimes g_k\|_{L^1(m \times m_{\mathbb{R}^d})} < \frac{\epsilon}{2}.
\]

Next, it follows from theorems 1.6.3 and 1.6.4 in [Rud] that

\[
\exists f_1, \ldots, f_N \in L^1 \cap L^2\] such that

\[
\bullet [f_k \neq 0] \text{ is compact and bounded away from 0} \quad (1 \leq k \leq N);\]

and

\[
\bullet \|f_k - g_k\|_{L^1(\mathbb{R}^d)} < \frac{\epsilon}{2N\|h_k\|_{L^1(m)}} \quad (1 \leq k \leq N),\]

whence

\[
\left\| \sum_{k=1}^N h_k \otimes f_k - \sum_{k=1}^N h_k \otimes g_k \right\|_{L^1(m \times m_{\mathbb{R}^d})} \leq \sum_{k=1}^N \|h_k\|_{L^1(m)} \cdot \|f_k - g_k\|_{L^1(\mathbb{R}^d)} < \frac{\epsilon}{2},
\]

\[
\left\| F - \sum_{k=1}^N h_k \otimes f_k \right\|_{L^1(m \times m_{\mathbb{R}^d})} < \epsilon
\]
where $h \in L$ and $f \in L^1 \cap L^2$ is such that $\hat{f} \neq 0$ is compact and bounded away from 0.

We claim

If $h \in L$ and $f \in L^1 \cap L^2$ is such that $\hat{f} \neq 0$ is compact and bounded away from 0, then $\exists 0 < \rho < 1$ such that

$$\|P_{T^2}(h \otimes f)\|_2 = O(\rho^n) ~ \text{as} ~ n \to \infty.$$  

Proof

Let $\hat{f} \neq 0 \subset B(0, M) \setminus B(0, \delta)$. By (ii) (above), $\exists K > 0, \ 0 < \rho < 1$ such that

$$|P^n h(x)| \leq K \rho^n \quad \forall x \in X, \ n \geq 1, \ \delta \leq |\gamma| \leq M,$$

whence using the fact that the Fourier transform of $y \mapsto P^n(h \otimes f)(x, y)$ is $\gamma \mapsto \hat{f}(\gamma)P^n h(x)$ and Plancherel's formula, we have

$$\|P_{T^2}(h \otimes f)\|_2^2 = \int_X \left( \int_{\mathbb{R}^d} |P_{T^2}(h \otimes f)(x, y)|^2 dy \right) dm(x)$$

$$= \int_X \left( \int_{\mathbb{R}^d} |\hat{f}(\gamma)|^2 |P^n h(x)|^2 d\gamma \right) dm(x)$$

$$= \int_{\mathbb{R}^d} |\hat{f}(\gamma)|^2 \|P^n h\|_2^2 d\gamma \leq K^2 \rho^{2n} \int_{\mathbb{R}^d} |\hat{f}(\gamma)|^2 d\gamma$$

proving $\text{¶}2$. \qed

To finish the proof of theorem 2, we claim

$\text{¶}3$ if (6) holds for $h \in L$ and $f \in L^1 \cap L^2$, then

$$\|P_{T^2}(h \otimes f)\|_1 \to 0.$$  

Proof

Fix $\lambda > 1$ such that $\lambda \frac{d}{\rho} < 1$. Suppose that $\frac{P^n \hat{f}}{\lambda^n} \to 0$ a.e.. Using (6), we have by $\text{¶}1$,

$$\|P_{T^2}(h \otimes f)\|_1 \leq C \lambda \frac{n^d}{2} \|P_{T^2}(h \otimes f)\|_2 + o(1) = O(\lambda \frac{n^d}{2} \rho^n) + o(1) \to 0$$

as $k \to \infty$; establishing (7) since $\|P_{T^2}(h \otimes f)\|_1 \downarrow$. \qed

This completes the proof of theorem 2.

References


