Stokes-Dirichlet/Neuman problems and complex analysis

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Stokes-Dirichlet/Neuman problems
and complex analysis

by

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Stokes-Dirichlet/Neuman Problems and Complex Analysis

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Abstract

On a bounded and simply connected open set $G \subset \mathbb{R}^2 \cong \mathbb{C}$, with a sufficiently smooth boundary $\partial G$, the following boundary value problem for a pair $\{\varphi, \chi\}$ of analytic functions is studied:

\[
\begin{cases}
\varphi, \chi : G \to \mathbb{C}, & \text{both analytic,} \\
\left[ z\varphi \pm \varphi + \chi \right] \bigg|_{\partial G} = G \in L_2(\partial G),
\end{cases}
\]  

(0.1)

Multiplication by $i$ transforms the + version into the − version.

Necessary and sufficient conditions on $G$ for solvability and also results on the behaviour of the solution near $\partial G$ are found.

The original motivation for this study is to provide a sound mathematical link between 2D Stokes boundary value problems and 2D free boundary evolution equations of Hopper type, cf. [H], with ’arbitrary Hamiltonian’, cf. [G]. During this, the interesting (and for the author unexpected) fact came up that both the Dirichlet and the Neumann Problem for the 2D-Stokes equations can be reduced to the problem (0.1). Full details of all this are in the underlying note. A brief overview now follows.

On $G \subset \mathbb{R}^2 \cong \mathbb{C}$, the stationary behaviour of a pressure-velocity flow pair $\{p, v\}$, where $p : G \to \mathbb{R}$ and $v : G \to \mathbb{R}^2$, can often be modelled by Stokes’ equations

\[
\begin{cases}
\nabla \cdot \mathbf{T} = 0 \\
\nabla \cdot v = 0
\end{cases}
\]

with stress matrix

\[\mathbf{T} = -p \mathbf{I} + \left[ \frac{dv}{dx} \right] + \left[ \frac{dv}{dy} \right]^\top.\]

(0.2)

Only Cartesian coordinates will be employed!

It is classical folklore, scattered in the literature, that there exists a bi-harmonic potential pair $\psi, \phi : G \to \mathbb{R}$, (the stream function and Airy function, respectively), such that, cf. (1.3),

\[v = \nabla \times (\psi \mathbf{e}_3), \quad \mathbf{T} = 2 \left[ (D^2 \phi) - (\Delta \phi) \mathbf{I} \right].\]

(0.3)

Consistency in $\mathbf{T}$ requires that $\phi$ and $\psi$ are related: For $z = x + iy \in G$ one necessarily has, cf. Appendix B,

\[\phi(x) + i\psi(x) = z\varphi(z) + \chi(z), \quad \text{with analytic} \quad \varphi, \chi : G \to \mathbb{C}.\]

(0.4)
Also this is classical folklore. For a strongly related approach in the field of ‘elasticity’ cf. [E] and [M] Ch 4. In the Appendices to this note full details are presented on \( \psi, \phi, \varphi, \chi \) and on the kinematic expressions derived from them. For a full set of the latter see (1.5).

By means of the analytic potentials \( \varphi, \chi \) we investigate boundary value problems for Stokes’ equations with respective boundary conditions:

**Stokes-Dirichlet:** \( v \bigg|_{\partial G} \in L^2(\partial G) \), \n
**Stokes-Neumann:** \( T_{1/2} \bigg|_{\partial G} \in H^{-1}(\partial G) \).

(0.5)

As it turns out both problems can be reduced to (0.1). By means of a conformal mapping the problem (0.1) is then transformed to an integral operator equation on the unit circle.

**Contents**

1. Generalities on Stokes’ Equations in \( \mathbb{R}^2 \): Gives an overview of solutions of Stokes’ equations in terms of potentials. Without taking boundary conditions into consideration.

2. Boundary Value Problems and their Uniqueness: Formulation of the Dirichlet and Neumann problem for Stokes’ equations. The consistency of the boundary conditions get a physical interpretation. Reformulation as (0.1), together with uniqueness conditions.

3. A Basic Existence Result: By means of a conformal mapping (0.1) is transformed to a problem on the unit disk. The previous uniqueness result together with a version of the ‘Fredholm Alternative’ leads to unique solvability. Some properties of the solution near the boundary are studied.

4. Results on Stokes Boundary Value Problems: The obtained results are transformed back from the unit disk to the original domain. A special class of solutions related to [H],[G] is introduced. Finally, some ‘non-physical’ boundary value problems are considered.

A. APPENDIX. Complex Analysis revisited: Contains all results on analytic functions formulated in the way we need them.

B. APPENDIX. Details on Stokes’ equations: Contains full proofs of all results with potentials as presented in section 1.

- Acknowledgements
- References

For convenience nothing new is claimed here! JdG November 2010
1 Generalities on Stokes’ Equations in $\mathbb{R}^2$

On a bounded simply connected open domain $G \subset \mathbb{R}^2$, $0 \in G$, we consider the set of Stokes equations

\[
\begin{align*}
  \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} - \frac{\partial p}{\partial x} &= 0, \\
  \frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} - \frac{\partial p}{\partial y} &= 0, \\
  \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} &= 0.
\end{align*}
\]

(1.1)

Alternative formulations are

\[
\begin{align*}
  \Delta v - \nabla p &= 0, \\
  \nabla \cdot v &= 0, \\
  \Delta \phi &= 0, \\
  \Delta \psi &= 0, \\
  \partial_i T_{ij} &= 0, \\
  \partial_i v_i &= 0.
\end{align*}
\]

(1.2)

with

\[
T = -p I + \left[ \frac{dv}{dx} \right] + \left[ \frac{dv}{dz} \right]^\top \quad \text{and} \quad T_{ij} = -p \delta_{ij} + \partial_j v_i + \partial_i v_j.
\]

The boundary $\partial G$ of $G$ is supposed to admit a positively oriented arclength parametrization $s \mapsto x(s)$, $0 \leq s < L$ with bounded (generalized) derivative $s \mapsto \dot{x}(s)$. Besides the unit tangent vector $s \mapsto t(x(s)) = \dot{x}(s) = \text{kol}[\dot{x}(s), \dot{y}(s)]$ we also need the outside normal $s \mapsto n(x(s)) = \text{kol}[\dot{y}(s), -\dot{x}(s)]$.

The next theorem contains some classical results regarding the general solution of Stokes’ equations without regarding boundary conditions.

**Theorem 1.1 (Classical results)**

- If $x \mapsto p(x)$, $\nabla p(x)$ solves (1.1), (1.2) on $G$, then there exist a 'stream function' $x \mapsto \psi(x)$ and an 'Airy function' $x \mapsto \phi(x)$ on $G$, with $\Delta \Delta \phi = 0$, $\Delta \Delta \psi = 0$, such that

\[
v = \begin{bmatrix} \partial_y \psi \\ -\partial_x \psi \end{bmatrix}, \quad p = \Delta \phi, \quad T = 2 \begin{bmatrix} -\partial_y \partial_y \phi & \partial_x \partial_y \phi \\ \partial_x \partial_y \phi & -\partial_x \partial_x \phi \end{bmatrix},
\]

(1.3)

and the function $z = x + iy \mapsto \Delta \phi(x) + i \Delta \psi(x)$ being analytic. Here $\psi$ is unique up to a constant and $\phi$ is unique up to a polynomial of 1st degree.

- The pair of biharmonic functions $\phi, \psi$ cannot be chosen arbitrarily. There has to exist a pair of analytic functions $z \mapsto \varphi(z), \chi(z)$ on $G$, with

\[
\phi(x) + i \psi(x) = \bar{z} \varphi(z) + \chi(z), \quad z = x + iy \in G,
\]

(1.4)

- All solutions of Stokes’ equations have such holomorphic representation.

- Let $s \mapsto z(s) \in \overline{G}$ be a curve with arclength parametrization $s$. Differentiation along such a curve is denoted $\frac{d}{ds}$. We write $\frac{dz}{ds} = \dot{z}$. The ordered pair $\{n, \dot{x}\} = \{-i \dot{z}, \dot{z}\}$ is
meant to be a positively oriented orthonormal system in $\mathbb{R}^2$. We have
\[
v_1 + iv_2 = -\phi + z\overline{\phi} + \chi'
\]
\[
\mathbf{v} \cdot \mathbf{n} = \frac{d}{ds} \text{Im}(z\phi + \chi) = -4 \text{Im} \phi'
\]
\[
\mathbf{v} \cdot \dot{x} = \frac{d}{ds} \text{Re}(z\phi + \chi) - 2 \text{Re}(\overline{\phi'}\dot{z})
\]
\[
T \cdot \mathbf{n} = 2i \frac{d}{ds} (\phi + z\overline{\phi} + \chi')
\]
\[
T \cdot \dot{x} = 2 \frac{d}{ds} \{z\overline{\phi'} + \chi - 4 \text{Re} \phi\}
\]

\text{(1.5)}

- If the pair $\{\phi, \chi\}$ is replaced by the pair $\{\phi + \alpha, \chi + \alpha z + \beta\}$, with $\alpha, \beta \in \mathbb{C}$, the same solution is represented.

The holomorphic representation of a solution by $\{\phi, \chi\}$ is unique if one additionally requires that for some fixed $a \in G$ one has $\phi(a) = \chi(a) = 0$. We usually take $a = 0$.

- In this way the 'Euclidean motion' solution
\[
p(x) = E, \quad v(x) = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} + B \begin{bmatrix} 0 \\ 1 \end{bmatrix} + C \begin{bmatrix} -y \\ x \end{bmatrix}, \quad A, B, C, E \in \mathbb{R}.
\]
\text{(1.6)}

has the unique holomorphic representation
\[
\phi(z) = \frac{1}{4}(E - 2iC)z \quad \chi(z) = (A - iB)z.
\]
\text{(1.7)}

Proof For a detailed mathematical proof of those classical results + some addenda see Appendix B.

\section{2 Boundary Value Problems and their Uniqueness}

The Stokes-Dirichlet problem is formulated as follows
\[
\begin{cases}
\Delta v - \nabla p = 0, \quad x \in G \\
\nabla \cdot v(x) = 0, \quad x \in \mathbb{R} \\
v(x) = g(x), \quad x \in \partial G \\
p(0) = B, \quad B \in \mathbb{R}.
\end{cases}
\]
\text{(2.1)}

On the prescribed boundary velocity field $s \mapsto \frac{d}{ds}g(x(s)) = V_1(s)\mathbf{n}(x(s)) + V_2(s)\mathbf{t}(x(s)) \in \mathbb{R}^2$ we put

\text{Condition on $g$:} \quad \bullet \quad \int_0^L V_1(s) \, ds = 0
\text{(2.2)}

This condition is necessary in order to be consistent with $\nabla \cdot \mathbf{v}(x) = 0, \quad x \in \mathbb{G}$.

Keep in mind that $V_1, V_2$ are not the cartesian components of $g$. 


Theorem 2.1 (Uniqueness of the Stokes-Dirichlet problem)

Consider the Stokes-Dirichlet problem (2.1). Suppose \( 0 \in G \).

- If \( g = 0, \ B = 0 \), then \( v(x) = 0, \ p(x) = 0, \ x \in G \).
- For given \( g \in L^2(\partial G; \mathbb{R}^2) \), \( B \in \mathbb{R} \) there is at most one solution pair \( \{v, p\} \) with (unique) holomorphic representation \( \{\varphi, \chi\} \), if one, in addition to \( \varphi(0) = \chi(0) = 0 \), requires.

\[
\text{Re} \varphi'(0) = \frac{1}{4} B \in \mathbb{R}. \quad (2.3)
\]

Proof

- On \( \partial G \) we suppose

\[
v = \begin{bmatrix} \partial_y \psi \\ -\partial_x \psi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

So we have to investigate the set of solutions of

\[
\Delta \Delta \psi(x) = 0, \quad x \in G, \quad \nabla \psi(x) = 0, \quad x \in \partial G.
\]

It follows that \( \frac{\partial}{\partial n} \psi = \frac{\partial}{\partial t} \psi = 0 \) at \( \partial G \). So \( \psi = C \in \mathbb{R} \) is constant at \( \partial G \). We take \( \psi = 0 \) at \( \partial G \).

With Green II

\[
0 = \int_G \psi(x) \Delta \Delta \psi(x) \, dx = \int_G \psi \frac{\partial}{\partial n} \Delta \psi \, ds - \int_G (\frac{\partial}{\partial n} \psi) \Delta \psi \, ds + \int_G |\Delta \psi|^2 \, dx =
\]

\[
= C \int_G \Delta \Delta \psi \, dx + \int_G |\Delta \psi|^2 \, dx.
\]

it now follows that \( \Delta \psi = 0 \). Hence, the stream function \( \psi = C \). So the velocity \( v = 0 \). The 'consistency conditions' (B.2) tell us that the Airy function \( \phi \) has to satisfy \( \partial_x \partial_y \phi = 0 \) and \( \partial_x \partial_x \phi - \partial_y \partial_y \phi = 0 \). Therefore it has the form \( \phi(x) = \frac{1}{2} B x^\top x + b^\top x + c \). So the pressure \( p = \Delta \phi \) can only be a constant. The condition \( p(0) = 0 \) forces this constant to be 0.

- If there are 2 solutions they differ by the zero solution just found. \( \blacksquare \)

Now we come to the **Stokes-Neumann problem**, which is formulated as follows

\[
\begin{align*}
\nabla \cdot T(x) &= 0, \quad x \in G \\
\nabla \cdot \nu(x) &= 0, \quad x \in G \\
T(x) \cdot n(x) &= f(x), \quad x \in \partial G
\end{align*}
\]

(2.4)

On the prescribed **boundary stress field** \( x \mapsto f(x) \in \mathbb{R}^2 \) we put

\[
\text{Conditions on } f : \quad \begin{align*}
f(x(s)) &= \frac{d}{ds} \{K_1(s)n(x(s)) + K_2(s)\tilde{f}(x(s))\}, \\
\int_{\partial G} K_1(s) \, ds &= 0.
\end{align*}
\]

(2.5)
These nicely correspond to equilibrium of forces and momenta, respectively,
\[ \int_{\partial G} f(\bar{x}(s)) \, ds = 0, \quad \int_{\partial G} \bar{x}(s) \times f(\bar{x}(s)) \, ds = 0. \]
Indeed, if we denote the force at \( \bar{x}(s) \in \partial G \) by \( \alpha(s)\bar{n}(\bar{x}(s)) + \beta(s)\bar{t}(\bar{x}(s)) \), the condition of equilibrium of forces says \( \int_{\partial G} \alpha \bar{n} + \beta \bar{t} \, ds = 0 \). Therefore we can write
\[ \alpha(s)\bar{n}(\bar{x}(s)) + \beta(s)\bar{t}(\bar{x}(s)) = \frac{d}{ds} \{ K_1(s)\bar{n}(\bar{x}(s)) + K_2(s)\bar{t}(\bar{x}(s)) \}. \]
Further, the condition of equilibrium of momenta says \( \int_{\partial G} \bar{x} \times \frac{d}{ds} \{ K_1\bar{n} + K_2\bar{t} \} \, ds = 0 \).
This means
\[ 0 = \int_{\partial G} \frac{d}{ds} \{ \bar{x} \times (K_1\bar{n} + K_2\bar{t}) \} \, ds = \int_{\partial G} \bar{t} \times \{ K_1\bar{n} + K_2\bar{t} \} \, ds. \]
Which says \( \text{e}_3 \int_{\partial G} K_1 \, ds = 0 \). \(^1\)

To (2.5) we could add the **optional condition**
\[ \int_{\partial G} \{ K_1(s)\bar{n}(\bar{x}(s)) + K_2(s)\bar{t}(\bar{x}(s)) \} \, ds = 0, \tag{2.6} \]
because adding a constant vectorfield to \( K_1\bar{n} + K_2\bar{t} \) does not alter \( f \). We don’t. For subtleties regarding this possibility, see the end of this section.

**Example:** The special choice \( K_1 = 0, \ K_2 = \kappa = \text{constant} \), models surface tension at the boundary. Then \( f = -\kappa \bar{n} \). Keep in mind that \( \bar{n} \) is the outside normal!

**Theorem 2.2 (Uniqueness of the Stokes-Neumann problem)**
Consider the Stokes-Neumann problem (2.4). Suppose \( \bar{0} \in G \).
- If \( f = 0 \), the set of solutions is given by the Euclidean motions (1.6) with \( p = E = 0 \).
- For any given \( f \in L_2(\partial G; \mathbb{R}^2) \) and any given \( \varphi(0) = \chi_0 \in \mathbb{R}^2 \), there is at most one solution with (unique) holomorphic representation \( \{ \varphi, \chi \} \) if one, in addition to \( \varphi(0) = \chi(0) = 0 \), requires
  \[ \text{Im} \varphi'(0) = \mu \in \mathbb{R}, \quad \chi'(0) = v_0 \in \mathbb{C}. \] \( \tag{2.7} \)

**Proof**
- On \( \partial G \) we suppose
  \[ \mathcal{T} \cdot \bar{n} = -2 \frac{d}{ds} \begin{bmatrix} \partial_y \phi \\ -\partial_x \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]
So we have to investigate the set of solutions of
  \[ \Delta \Delta \phi(x) = 0, \quad x \in G, \quad \nabla \phi(x) = a = \text{constant}, \quad x \in \partial G. \]

\(^1\)JdG thanks Dr. A.A.F. van de Ven for clearing up this point
Consider \( \tilde{\phi}(x) = \phi(x) - a^\top x \), which satisfies
\[
\Delta \Delta \tilde{\phi}(x) = 0, \quad x \in \mathbb{G}, \quad \nabla \tilde{\phi}(x) = 0, \quad x \in \partial \mathbb{G}.
\]
This implies \( \frac{d}{ds} \tilde{\phi}(x(s)) = 0 \), at \( x(s) \in \partial \mathbb{G} \). Hence \( \tilde{\phi}(x) = \alpha = \text{constant} \), at \( x(s) \in \partial \mathbb{G} \).

Introduce \( \hat{\phi}(x) = \phi(x) - a^\top x - \alpha \), which satisfies
\[
\Delta \Delta \hat{\phi}(x) = 0, \quad x \in \mathbb{G}, \quad \partial_n \hat{\phi}(x) = 0, \quad \hat{\phi}(x) = 0, \quad x \in \partial \mathbb{G}.
\]

From \( 0 = \int_\mathbb{G} \hat{\phi}(x) \Delta \Delta \hat{\phi}(x) \, dx \) and Green II it now follows that \( \hat{\phi} = 0 \) and therefore the Airy function is of the form \( \phi(x) = a^\top x + \alpha \). The ‘consistency conditions’ (B.2) tell us that the stream function \( \psi \) has to satisfy \( \partial_x \partial_y \psi = 0 \) and \( \partial_x \partial_x \psi - \partial_y \partial_y \psi = 0 \). Therefore it has the form \( \psi(x) = \frac{1}{2} C x^\top x + b^\top x + c \).

As a consequence the homogeneous Stokes-Neumann problem is solved by all Euclidean motion solutions (1.6), represented by (1.7) with \( E = 0 \).

- If there are 2 solutions they differ by a solution represented by (2.7) which is reduced to 0 because of \( \text{Im } \varphi'(0) = 0, \chi'(0) = 0 \).

\[\text{Lemma 2.3}\]
Let \( \varphi, \chi : \mathbb{G} \to \mathbb{C} \) be analytic with \( \varphi(0) = \chi(0) = 0 \).
Suppose that \( z \mapsto \varphi(z) \) and \( z \mapsto \overline{z} \varphi'(z) + \chi'(z) \) both extend to a continuous function on \( \overline{\mathbb{G}} \).

- If \( \text{Re } \varphi'(0) = 0 \) and for all \( s \)
  \[
  z(s) \varphi'(z(s)) - \varphi(z(s)) + \overline{\chi'(z(s))} = C, \quad z(s) \in \partial \mathbb{G},
  \]
with \( C \in \mathbb{C} \) a constant.
Then \( \varphi(z) = 0 \), identically on \( \mathbb{G} \) and \( \chi(z) = \overline{Cz} \).

- If \( \text{Im } \varphi'(0) = 0 \) and for all \( s \)
  \[
  z(s) \overline{\varphi'(z(s))} + \varphi(z(s)) + \overline{\chi'(z(s))} = D, \quad z(s) \in \partial \mathbb{G},
  \]
with \( D \in \mathbb{C} \) a constant.
Then \( \varphi(z) = 0 \), identically on \( \mathbb{G} \), and \( \chi(z) = \overline{Dz} \).

\[\text{Proof}\]
- First suppose \( C = 0 \) and consider the pair \( \{\varphi, \chi\} \) as a holomorphic representation of the solution of Stokes’ equations. Then, according to Theorem 2.1, \( v_1 + iv_2 \) and \( p \) vanish identically on \( \mathbb{G} \). Therefore \( z \overline{\varphi'} - \varphi + \overline{\chi'} = 0 \), identically on \( \mathbb{G} \). Taking the derivative \( \frac{\partial}{\partial z} \) leads to \( \text{Im } \varphi' = 0 \) on \( \mathbb{G} \). So \( \varphi(z) = Az \), with \( A \in \mathbb{R} \). Because \( \text{Re } \varphi'(0) = 0 \) we necessarily have \( A = 0 \). Then from (2.8) also \( \chi' \) has to be 0. Hence \( \chi \) is constant. With the condition \( \chi(0) = 0 \) it follows that \( \chi = 0 \) on \( \mathbb{G} \).
Finally, if $C \neq 0$, the only solution pair can be $\varphi(z) = 0$, $\chi(z) = \overline{C}z$ on $\mathcal{G}$.

- Two proofs are presented.

First take $C = iD$ in (2.8) and multiply both sides by $-i$. We get back (2.9), with $\varphi$, $\chi$ replaced by $i\varphi$, $i\chi$. Now the first result can be applied.

For the second proof consider the pair $\{\varphi, \chi\}$ as a holomorphic representation of the solution of Stokes’ equations. We find at $\partial \mathcal{G}$

$$T u(s) = 2i \frac{d}{ds} \left( z(s)\overline{\varphi'(z(s))} + \varphi(z(s)) + \overline{\chi'(z(s))} \right) = 2i \frac{d}{ds} D = 0.$$  

According to the uniqueness result in Theorem (2.2) we necessarily have $\varphi(z) = -\frac{i}{2}Cz$, $\chi(z) = (A - iB)z$, $A, B, C \in \mathbb{R}$. Then $\operatorname{Im} \varphi'(0) = 0$ implies $C = 0$. Finally, with (2.9), $A - iB = E$. ■

Concluding this section we look at the Stokes-Neumann problem in terms of $\varphi, \chi$.
So we want to find analytic $\varphi, \chi : \mathcal{G} \rightarrow \mathbb{C}$, such that at the boundary $\partial \mathcal{G}$

$$T u(s) = 2i \frac{d}{ds} \left( z(s)\overline{\varphi'(z(s))} + \varphi(z(s)) + \overline{\chi'(z(s))} \right) = -i \frac{d}{ds} \{K(s)\dot{z}(s)\}. \quad (2.10)$$

Here $K(s) = K_1(s) + iK_2(s)$, cf. (2.5).

**Note** that (2.10) does not alter if $\varphi$ is replaced by $\varphi - \frac{i}{2}Cz + C_1$ and $\chi$ by $\chi + (A - iB)z + C_2$, with constants $A, B, C \in \mathbb{R}$ and $C_1, C_2 \in \mathbb{C}$.

Now in identity (2.10) we 'cancel' the $i \frac{d}{ds}$ and with Lemma 2.10 we acquire uniqueness for the system

$$\begin{cases}
  z(s)\overline{\varphi'(z(s))} + \varphi(z(s)) + \overline{\chi'(z(s))} = -\frac{1}{2}K(s)\dot{z}(s), & z(s) \in \partial \mathcal{G}, \\
  \varphi(0) = \chi(0) = 0, & \operatorname{Im} \varphi'(0) = 0.
\end{cases} \quad (2.11)$$

There is a subtlety here! 2 If we add a constant $E \in \mathbb{C}$ to the right hand side in (2.11) the (unique if it exists) solution $\chi(z)$ becomes $\chi(z) + \overline{E}z$, a uniform rectilinear motion is added to the solution of Stokes’ equations. It we kept to the 'optional' condition (2.6), it would forbid adding such $E$ and leads us into consistency troubles. A requirement of type $\chi'(a) = 0$ at a suitable point $a \in \mathcal{G}$ could possibly 'save' the optional condition.

At this point however we are quite content with the achieved uniqueness for problem (2.11).

\[\text{This sublety arose and was cleared up in a 'discussion on the constants' with Nasrin Arab.}\]
3 A Basic Existence Result

On a simply connected open domain \( \mathcal{G} \), \( 0 \in \mathcal{G} \) with 'sufficiently smooth' boundary \( \partial \mathcal{G} \) and prescribed \( F = F_1 + iF_2 : \partial \mathcal{G} \to \mathbb{C} \) we want to show the existence of analytic \( \varphi, \chi : \mathcal{G} \to \mathbb{C} \)

\[
\begin{aligned}
z(s)\varphi'(z(s)) + \varphi(z(s)) + \overline{\chi'(z(s))} &= F(s)\dot{z}(s), \quad z(s) \in \partial \mathcal{G}, \\
\varphi(0) = \chi(0) = 0, \quad \text{Im} \varphi'(0) = 0.
\end{aligned}
\]

(3.1)

In this equation, instead of \(+\varphi(z(s))\) also \(-\varphi(z(s))\) can be taken. As we have seen, this is just a matter of redefining the unknown functions by a factor \( i \). We keep to the \(+\) sign in this section.

Multiply both sides of (3.1) by \( \dot{z} \), then

\[
\frac{d}{ds} \left(z(s)\varphi(z(s)) + \overline{\chi(z(s))}\right) + 2 \text{Im} \{ (\varphi(z(s))\dot{z}(s)) \} = F(s).
\]

(3.2)

Integration along \( \partial \mathcal{G} \) of the real part of this identity leads to the necessary condition

\[
\int_{\partial \mathcal{G}} F_1(s) \, ds = 0,
\]

for solvability. This nicely corresponds to the conditions (2.5), casu quo (2.2).

At this point the unique \textbf{conformal bijection}

\[
\Omega : \mathbb{D} \to \mathcal{G}, \quad \zeta \mapsto \Omega(\zeta), \quad \Omega(0) = 0, \quad \Omega'(0) > 0,
\]

(3.3)

is introduced from the open unit disk \( \mathbb{D} \) in the \( \zeta \)-plane into the complex \( z = x + iy \)-plane. Note that, if \( \partial \mathcal{G} \) happens to be a Jordan curve with a Hölder continuous derivative, then \( \Omega \) extends to a bijective \( \mathcal{C}^{1,\alpha} \)-map \( \Omega : \overline{\mathbb{D}} \to \overline{\mathcal{G}} \), cf. [P] Thm 3.6, p49.

Corresponding to the usual parametrisation \( \theta \to e^{i\theta}, \quad 0 \leq \theta < 2\pi \) of \( \partial \mathbb{D} = S^1 \) we define \( \theta \mapsto s(\theta) \) by \( z(s(\theta)) = \Omega(e^{i\theta}) \).

Finally the new unknown functions

\[
\Phi(\zeta) = \varphi(\Omega(\zeta)), \quad \chi(\zeta) = \chi(\Omega(\zeta)),
\]

(3.4)

are introduced. Then, with

\[
\partial_\theta \Phi(e^{i\theta}) = \Phi'(e^{i\theta}) i e^{i\theta} = \varphi'(\Omega(e^{i\theta})) \partial_\theta \Omega(e^{i\theta}) = \varphi'(\Omega(e^{i\theta})) \Omega'(e^{i\theta}) i e^{i\theta},
\]

(3.1) can be rewritten, along \( \partial \mathbb{D} \), as

\[
\begin{aligned}
\Omega(\zeta)(\partial_\theta \Phi(\zeta)) + (\overline{\partial_\theta \Omega(\zeta)} \Phi(\zeta) + \partial_\theta \overline{\chi(\zeta)}) = |\partial_\theta \Omega(\zeta)| F(s(\theta)), \quad \zeta = e^{i\theta}, \\
\Phi(0) = \chi(0) = 0, \quad \text{Im} \Phi'(0) = 0.
\end{aligned}
\]

(3.5)

The first line can be rewritten

\[
\partial_\theta \left[ \Omega(\zeta) \Phi(\zeta) + \overline{\chi(\zeta)} \right] + 2 i \text{Im} \left[ (\overline{\partial_\theta \Omega(\zeta)} \Phi(\zeta)) \right] = |\partial_\theta \Omega(\zeta)| F(s(\theta)), \quad \zeta = e^{i\theta}.
\]

(3.6)

Integration of the real part of this identity leads once more to the necessary condition

\[
\int_0^{2\pi} F_1(s(\theta)) \frac{ds(\theta)}{d\theta} \, d\theta = 0,
\]

for solvability.

We start the investigation of (3.5) with a Lemma
Lemma 3.1
Let $f : \mathbb{D} \to \mathbb{C}$ be analytic with $f(0) = 0$.

Split in real and imaginary parts $f(\zeta) = f_1(\zeta) + i f_2(\zeta)$.

We have
1. $\theta \mapsto f_1(e^{i\theta}) \in L_2(S^1; \mathbb{R})$ if and only if $\theta \mapsto f_2(e^{i\theta}) \in L_2(S^1; \mathbb{R})$.
2. The mapping $J : L_2(S^1; \mathbb{R}; \{1\}^\perp) \to L_2(S^1; \mathbb{R}; \{1\}^\perp)$, $f_1 \mapsto Jf_1 = f_2$, is orthogonal and $JJ^* = -J = J^{-1}$, $J^2 = -1$, $J \cos n\theta = \sin n\theta$, $J \sin n\theta = -\cos n\theta$, $n \in \mathbb{N}$.
3. The operator $J$ is represented by the principal value integral

$$Jf_1(\theta) = f_2(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cot \left( \frac{1}{2}(\theta - \theta_1) \right) f_1(\theta_1) \, d\theta_1. \quad (3.7)$$

4. $\partial_\theta J = J \partial_\theta$, $\partial_\theta f_1(e^{i\theta}) + i \partial_\theta f_2(e^{i\theta}) = i(\zeta \partial_\zeta f)(e^{i\theta})$.
5. Product formula for $f, g : \mathbb{D} \to \mathbb{C}$, both $\mathbb{C}$-analytic

$$J(f_1 g_1) = J((Jf_1)(Jg_1)) + (Jf_1)g_1 + f_1(Jg_1).$$

Proof See Appendix A sub 11. ■

We now come to the main theorem of this section

Theorem 3.2 (Basic Existence Result)
Let $F_1, F_2 : \partial G \to \mathbb{R}$ be given.

Suppose the conformal mapping $\Omega : \mathbb{D} \to G \subset \mathbb{C}$ to be such that

a. $\theta \mapsto |\partial_\theta \Omega(e^{i\theta})| F_1(s(\theta)) \in L_2(S^1; \mathbb{R}; \{1\}^\perp)$.

b. $\theta \mapsto |\partial_\theta \Omega(e^{i\theta})| F_2(s(\theta)) \in L_2(S^1; \mathbb{R})$.

c. $\theta \mapsto |\partial_\theta \Omega(e^{i\theta})|$ and $\theta \mapsto |\partial_\theta \Omega(e^{i\theta})|^{-1}$ are bounded on $S^1$.

d. $\theta \mapsto |\partial_\theta \partial_\theta \Omega(e^{i\theta})|$ is bounded on $S^1$.

Then there exist unique $\Phi, \chi : \partial \mathbb{D} \to \mathbb{C}$, with properties

- $\theta \mapsto \Phi(e^{i\theta}) \in L_2(S; \mathbb{C})$, $\theta \mapsto \chi(e^{i\theta}) \in L_2(S; \mathbb{C})$,
- $\Phi, \chi$ extend to $\Phi, \chi : \mathbb{D} \to \mathbb{C}$, which are analytic on $\mathbb{D}$.

and which satisfy

$$\begin{cases}
\Omega(\zeta)(\partial_\zeta \Phi(\zeta)) + (\partial_\zeta \Omega(\zeta))\Phi(\zeta) + \partial_\zeta \chi(\zeta) = |\partial_\theta \Omega(\zeta)| F(s(\theta)), & \zeta = e^{i\theta}, \\
\Phi(0) = \chi(0) = 0, & \text{Im} \Phi'(0) = 0.
\end{cases} \quad (3.9)$$

If, instead of condition d., we require the Hölder condition
e. \( \theta \mapsto \Omega(e^{i\theta}) \in C^{1,\alpha}(S^1) \), for some \( 0 < \alpha < 1 \),
the theorem holds as well.

\textbf{Proof} We proceed in 6 steps.

I. Split (3.5), (3.6) in real and imaginary parts at \( \partial \mathcal{G} \)

\[
\begin{align*}
\frac{\partial}{\partial \theta} \text{Re} \left[ \frac{\Omega}{2} \Phi \right] + \partial_\theta X_1 &= |\Omega'| F_1 \\
-\partial_\theta \text{Im} \left[ \frac{\Omega}{2} \Phi \right] + 2 \text{Im} \left[ (\partial_\theta \Omega) \Phi \right] - \partial_\theta X_2 &= |\Omega'| F_2
\end{align*}
\]

(3.10)

By the way, note that the pair \( X = 0, \Phi = -i\Omega \) satisfies this set of equations if \( F_1 = F_2 = 0 \). However it does NOT satisfy our condition \( \text{Im} \Phi'(0) = 0 \).

We now eliminate \( X_2 \) by applying \( J \) to the 1st line and add it to the 2nd.

\[
\begin{align*}
\frac{\partial}{\partial \theta} \text{Re} \left[ \frac{\Omega}{2} \Phi \right] + \partial_\theta X_1 &= |\Omega'| F_1 \\
\partial_\theta \{ J \text{Re} \left[ \frac{\Omega}{2} \Phi \right] - \text{Im} \left[ \frac{\Omega}{2} \Phi \right] \} + 2 \text{Im} \left[ (\partial_\theta \Omega) \Phi \right] &= J \left( |\Omega'| F_1 \right) + |\Omega'| F_2
\end{align*}
\]

(3.11)

From now on the factors \( \Omega_1, \Omega_2, \partial_\theta \Omega_1 = \hat{\Omega}_1, \partial_\theta \Omega_2 = \hat{\Omega}_2 \), are to be considered as multiplication operators. Because of the analytic extendibility requirement we put, cf. Lemma 3.1, \( \Phi = \Phi_1 + i \Phi_1 \), etc. Thus the 2nd equation becomes an operator equation for \( \Phi_1 \) only.

Using the product formula of Lemma 3.1, which gives us

\[
J((J\Omega_1)(J\Phi_1)) = J(\Omega_1 \Phi_1) - (J\Omega_1) \Phi_1 - \Omega_1 (J\Phi_1),
\]

(3.12)

combined with the 2nd line of (3.11), we find the operator equation

\[
\partial_\theta \left( [J\Omega_1 - \Omega_1 J] \Phi_1 \right) + [\hat{\Omega}_1 J - \hat{\Omega}_2] \Phi_1 = \frac{1}{2} \left[ J(|\Omega'| F_1) + |\Omega'| F_2 \right].
\]

(3.13)

So we have to study the operators on the left hand side of (3.13).

II. First notice that the operator

\[
L : L_2(S^1; \mathbb{R} ; \{1, \sin \theta \}^\perp) \to L_2(S^1; \mathbb{R} ) : \Phi_1 \mapsto L\Phi_1 = [\hat{\Omega}_1 J - \hat{\Omega}_2] \Phi_1,
\]

is a bijection. Indeed, on \( S^1 \) investigate

\[
[\hat{\Omega}_1 J - \hat{\Omega}_2] \Phi_1 = \text{Im} \{ \hat{\Omega} \Phi \} = \text{Re} \{ -\hat{\Omega} \Phi \} = R \in L_2(S^1).
\]

Divide by \( |\hat{\Omega}|^2 \), then on \( S^1 \),

\[
\text{Re} \frac{\Phi}{i\Omega} = \frac{R}{|\hat{\Omega}|^2} = S(\theta) + \overline{S(\theta)},
\]

where \( S \) is uniquely written as the \textit{complex} Fourier expansion (of a \( \mathbb{R} \)-valued function)

\[
S(\theta) = \sum_{\ell=0}^{\infty} s_\ell e^{i\ell \theta}, \quad \text{with} \quad s_\ell \in \mathbb{C}, \ s_0 \in \mathbb{R}.
\]
After analytic extension into $D$ we write

$$- \text{Re} \frac{\Phi(\zeta)}{\zeta \Omega'(\zeta)} = S(\zeta) + S^i(\zeta), \text{ for } \zeta = e^{i\theta},$$

from which $\Phi(\zeta) = -2\zeta \Omega'(\zeta)S(\zeta) + i\alpha \zeta \Omega'(\zeta)$ for $|\zeta| < 1$ and $\alpha \in \mathbb{R}$, follows. Since $\Phi'(0) \in \mathbb{R}$ is required, only $\alpha = 0$ is acceptable. The $L_2$-properties follow from the (supposed) boundedness of $\Omega'$ and $(\Omega')^{-1}$ on $S^1$.

**III.** Together with (3.7) the operator

$$K : \mathbb{L}_2(S^1; \mathbb{R} : \{1, \sin \theta\}^+) \rightarrow \mathbb{L}_2(S^1; \mathbb{R}) : \Phi_1 \mapsto K\Phi_1 = \partial_{\theta} \left( [L \Omega_1 - \Omega_1 J] \Phi_1 \right),$$

can be written, with some trigonometry,

$$K\Phi_1(\theta) = -\frac{1}{2\pi} \partial_{\theta} \int_{-\pi}^{\pi} \cot \left( \frac{\theta - \theta_1}{2} \right) \left( \Omega_1(e^{i\theta}) - \Omega_1(e^{i\theta_1}) \right) \Phi_1(\theta_1) \, d\theta_1 =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(\theta - \theta_1)}{1 - \cos(\theta - \theta_1)} \left[ \Omega_1(e^{i\theta}) - \Omega_1(e^{i\theta_1}) \right] - \partial_{\theta} \Omega_1(e^{i\theta}) \right] \Phi_1(\theta_1) \, d\theta_1 \quad (3.14)$$

Then condition d., together with L'Hôpital's rule, imply that $K$ is Hilbert-Schmidt. If there were $\Phi_1 \in \mathbb{L}_2(S^1; \mathbb{R} : \{1, \sin \theta\}^+), \Phi_1 \neq 0$, with $(K + L)\Phi_1 = 0$, we could introduce $X_1 = -\text{Re} \left[ \overline{\Omega} \Phi \right] + \gamma$, with constant $\gamma = \text{Re} \int_{-\pi}^{\pi} \left[ \overline{\Omega}(e^{i\theta}) \Phi(e^{i\theta}) \right] \, d\theta$. Note that such $\Phi_1$ is necessarily continuous!!

The nonzero pair $\{ \Phi_1 + iJ\Phi_1, X_1 + iJX_1 \}$ then leads to a non-zero solution pair $\{ \varphi, \chi \}$ of (2.11), with $K = 0$, which contradicts the uniqueness result of Lemma 2.3. So $K + L$ is injective.

Since $K + L$ is a compact perturbation of the bijection $L$, which has index 0, the problem $(K + L)\Phi_1 = R$ is uniquely solvable for any $R \in \mathbb{L}_2(S^1; \mathbb{R})$. For the 'index theory' see, e.g., [GGK].

**IV.** Substitute the found $\Phi_1$ with $J\Phi_1$ in the first equation of (3.11). Its righthand side $-\frac{1}{2} |\partial_{\theta} \Omega|/K_1$ can be written as a derivative. With the requirement $X'(0) = 0$, it leads to a unique $X$.

**V.** Split the operator $K = K_\varepsilon + K_{\pi - \varepsilon}$, $0 < \varepsilon < \pi$. On the square $[-\pi, \pi] \times [-\pi, \pi]$, and inside the strip $|\theta - \theta_1| < \varepsilon$, the kernel of $K_\varepsilon$ takes the values of the kernel of $K$. Outside this strip it is taken to be 0. So

$$K_\varepsilon \Phi_1(\theta) = \frac{1}{2\pi} \int_{\max\{\pi, \theta - \varepsilon\}}^{\min\{\pi, \theta + \varepsilon\}} K(\theta, \theta_1) \Phi_1(\theta_1) \, d\theta_1,$$

with $K$ the kernel of (3.14).
Note that the 'remains' $K_{\pi-\varepsilon}$ is Hilbert-Schmidt.
We now show that for some $C > 0$ we have $\|K_\varepsilon\| \leq C\varepsilon^{\min\{\alpha, \frac{1}{2}\}}$.
The Mean Value Theorem applied to
\[
x \mapsto \Omega_1(e^{ix}) + \frac{\Omega_1(e^{i \theta_1}) - \Omega_1(e^{i \theta})}{\sin(\theta - \theta_1)} \sin(x - \theta_1), \quad \text{on interval } [\theta_1, \theta] \text{ or } [\theta, \theta_1],
\]
provides us with
\[
\frac{\Omega_1(e^{i \theta}) - \Omega_1(e^{i \theta_1})}{\sin(\theta - \theta_1)} = \frac{\partial_\theta \Omega_1(e^{i \xi})}{\cos(\xi - \theta_1)}, \quad \text{for some } \xi \text{ in between } \theta, \theta_1.
\]
We now split $K_\varepsilon$ in a 'bounded kernel part' and a 'singular kernel part'
\[
K_\varepsilon = K_{\varepsilon,B} + K_{\varepsilon,S}.
\]
For some $\xi$ in between $\theta, \theta_1$,
\[
K_{\varepsilon,B} \Phi_1(\theta) = \int_{\max\{-\pi, \theta - \varepsilon\}}^{\min\{\pi, \theta + \varepsilon\}} \sin(\theta - \theta_1) \frac{1 - \cos(\xi - \theta_1)}{1 - \cos(\theta - \theta_1)} \frac{\partial_\theta \Omega_1(e^{i \xi})}{\cos(\xi - \theta_1)} \Phi_1(\theta_1) \, d\theta_1.
\]
and
\[
K_{\varepsilon,S} \Phi_1(\theta) = \int_{\max\{-\pi, \theta - \varepsilon\}}^{\min\{\pi, \theta + \varepsilon\}} \sin(\theta - \theta_1) \left[ \partial_\theta \Omega_1(e^{i \xi}) - \partial_\theta \Omega_1(e^{i \theta}) \right] \Phi_1(\theta_1) \, d\theta_1.
\]
Since the kernel of $K_{\varepsilon,B}$ is bounded we find $\|K_{\varepsilon,B}\| < C_1 \sqrt{\varepsilon}$, for some $C_1 > 0$.
Next, by means of the required Hölder condition the kernel of $K_{\varepsilon,S}$ is estimated
\[
\frac{|\sin(\theta - \theta_1)|}{1 - \cos(\theta - \theta_1)} |\partial_\theta \Omega_1(e^{i \xi}) - \partial_\theta \Omega_1(e^{i \theta})| \leq C_2 \frac{|\xi - \theta_1|}{|\theta - \theta_1|} \leq C_2 |\theta_1 - \theta|^\alpha - 1,
\]
on $[-\pi, \pi]$. It now follows
\[
|K_{\varepsilon,S} \Phi_1(\theta)|^2 \leq C_3 \int_{\max\{-\pi, \theta - \varepsilon\}}^{\min\{\pi, \theta + \varepsilon\}} |\theta - \theta_2|^{\alpha - 1} \, d\theta_2 \cdot \int_{\max\{-\pi, \theta - \varepsilon\}}^{\min\{\pi, \theta + \varepsilon\}} |\theta - \theta_1|^{\alpha - 1} |\Phi_1(\theta_1)|^2 \, d\theta_1.
\]
The first integral is is a function of $\theta$ bounded by $\leq \frac{2}{\alpha} \varepsilon^\alpha$.
Finally, after a change of variables,
\[
\int_{-\pi}^{\pi} |K_{\varepsilon,S} \Phi_1(\theta)|^2 \, d\theta \leq C_3 \left( \frac{2}{\alpha} \varepsilon^\alpha \right)^2 \int_{-\pi}^{\pi} |\Phi_1(\theta)|^2 \, d\theta,
\]
which says
\[
\|K_{\varepsilon,S}\| \leq \sqrt{C_3} \frac{2}{\alpha} \varepsilon^\alpha.
\]
(3.13) can now be written

\[ K_{\pi - \varepsilon} \Phi_1 + (K_{\varepsilon} + L) \Phi_1 = \frac{1}{2} \left[ J(\Omega' | F_1) + |\Omega'| F_2 \right]. \tag{3.15} \]

For \( \varepsilon \) sufficiently small the second operator is still a bijection. The operator \( K + L \) is a compact perturbation of this bijection. Therefore the argument of \( \textbf{III.} \) applies again. \( \blacksquare \)

**Notation**

- For given \( \Theta : \overline{D} \to \mathbb{C} \) we introduce the restriction to a circle \( \Theta \bigg|_{r} : \partial D \to \mathbb{C} : \theta \mapsto \Theta(re^{i\theta}), \quad 0 < r \leq 1. \)

- For \( g \in L_2(S; \mathbb{C}) \) the (complex) Fourier expansion \( g(\theta) = \sum_{\ell = -\infty}^{\infty} g_{\ell} e^{i\ell \theta} \) is split in a positive and negative part, respectively,
  \[ g^+(\theta) = \sum_{\ell = 1}^{\infty} g_{\ell} e^{i\ell \theta} \quad \text{and} \quad g^-(\theta) = \sum_{k = 0}^{\infty} g_{-k} e^{-ik\theta}. \]

The previous Theorem implies \( \Phi \bigg|_{r} \to \Phi \bigg|_{1} \), \( \mathcal{X} \bigg|_{r} \to \mathcal{X} \bigg|_{1} \) in \( L_2(S; \mathbb{C}) \) as \( r \uparrow 1 \).

It follows, since \( \theta \mapsto \Omega(e^{i\theta}) \) is supposed to be continuously differentiable,

- \( \left[ \overline{\Omega} \Phi + \mathcal{X} \right] \bigg|_{r} \to \left[ \overline{\Omega} \Phi + \mathcal{X} \right] \bigg|_{1} \), in \( L_2(S; \mathbb{C}) \), as \( r \uparrow 1 \), \( (3.16) \)
- \( \partial_\theta \left[ \overline{\Omega} \Phi + \mathcal{X} \right] \bigg|_{r} \to \partial_\theta \left[ \overline{\Omega} \Phi + \mathcal{X} \right] \bigg|_{1} \), in \( H^{-1}(S; \mathbb{C}) \), as \( r \uparrow 1 \),

However, since \( \partial_\theta \left[ \overline{\Omega} \Phi + \mathcal{X} \right] \bigg|_{1} \in L_2(S; \mathbb{C}) \), cf. (3.10), we expect the latter convergence also to be in \( L_2(S; \mathbb{C}) \). There is a simple proof for this if we assume some extra smoothness on \( \Omega \).

**Theorem 3.3 (Behaviour near the Boundary 1)**

a. Assume that the sequence of Fourier coefficients \( \{n \mapsto 2n\Omega_n\} \in \ell_1(\mathbb{N}) \), then the solution \( \Phi, \mathcal{X} \) of Theorem 3.2 enjoys the properties

\[ \partial_\theta \left[ \overline{\Omega} \Phi + \mathcal{X} \right] \bigg|_{r} \to \partial_\theta \left[ \overline{\Omega} \Phi + \mathcal{X} \right] \bigg|_{1} \), in \( L_2(S; \mathbb{C}) \), as \( r \uparrow 1 \), \( (3.17) \)

\[ \partial_\theta \left[ \overline{\Omega} \Phi \right] \bigg|_{r} \to \partial_\theta \left[ \overline{\Omega} \Phi \right] \bigg|_{1} \), in \( L_2(S; \mathbb{C}) \), as \( r \uparrow 1 \). \( (3.18) \)

b. Condition a. is satisfied if \( \{\theta \mapsto \Omega(e^{i\theta})\} \in H^{3+\alpha}(S; \mathbb{C}) \cap C^{1+\alpha}(S^1) \), with \( \alpha > 0 \). E.g. if \( \{\theta \mapsto \Omega(e^{i\theta})\} \in C^2(S^1) \).
Proof

• The Fourier expansion of \(-i \partial_\theta [\bar{\Omega} \Phi + \mathcal{X}]\big|_r\) for \(0 < r \leq 1\), reads

\[
- i \partial_\theta \left[ \left( \sum_{n=1}^{\infty} r^n \bar{\Omega}_n e^{-in\theta} \right) \left( \sum_{m=1}^{\infty} r^m \Phi_m e^{im\theta} \right) + \left( \sum_{k=1}^{\infty} r^k \mathcal{X}_k e^{ik\theta} \right) \right] =
\]

\[
= \sum_{k=1}^{\infty} k \left\{ r^k \mathcal{X}_k + \sum_{m-n=k, n \geq 1, m \geq 1} r^{n+m} \bar{\Omega}_n \Phi_m \right\} e^{ik\theta} - \sum_{\ell=0}^{\infty} \ell \left\{ \sum_{n-m=\ell, n \geq 1, m \geq 1} r^{n+m} \bar{\Omega}_n \Phi_m \right\} e^{-i\ell \theta} =
\]

\[
= \sum_{k=1}^{\infty} k \left\{ r^k \mathcal{X}_k + \sum_{n=1}^{\infty} r^{2n+k} \bar{\Omega}_n \Phi_{n+k} \right\} e^{ik\theta} - \sum_{\ell=0}^{\infty} \ell \left\{ \sum_{m=1}^{\infty} r^{2m+\ell} \bar{\Omega}_{m+\ell} \Phi_m \right\} e^{-i\ell \theta}.
\]

From the previous we know that, for \(r = 1\), the coefficient sequences \(k\{\cdot\}\) and \(\ell\{\cdot\}\) are both in \(\ell^2(\mathbb{N})\). Because of analyticity this is also true for \(0 < r < 1\). We have to show that no 'discontinuity' occurs at \(r = 1\).

The positive and negative parts of the coefficient sequences of

\[- i \partial_\theta \left\{ [\bar{\Omega} \Phi + \mathcal{X}] \big|_1 - [\bar{\Omega} \Phi + \mathcal{X}] \big|_r \right\},
\]

are, respectively,

\[
k \mapsto k \left\{ (1 - r^k) \mathcal{X}_k + \sum_{n=1}^{\infty} (1 - r^{2n+k}) \bar{\Omega}_n \Phi_{n+k} \right\}, \quad \ell \mapsto - \ell \left\{ \sum_{m=1}^{\infty} (1 - r^{2m+\ell}) \bar{\Omega}_{m+\ell} \Phi_m \right\}.
\]

We have to show that both tend to 0 in \(\ell^2(\mathbb{N})\), as \(r \uparrow 1\).

We use the identity

\[
(1 - r^k) \frac{1 - r^{2n+k}}{1 - r^k} = (1 - r^k) \left\{ 1 + \frac{r^k}{1 + r + \cdots + r^{k-1}} \left( 1 + r + \cdots + r^{2n-1} \right) \right\},
\]

and the fact that

\[
\frac{r^k}{1 + r + \cdots + r^{k-1}} \uparrow \frac{1}{k} \quad \text{as} \quad r \uparrow 1.
\]

• The 'positive' sequence can be split

\[
k \mapsto (1 - r^k) \left\{ k \{ \mathcal{X}_k + \sum_{n=1}^{\infty} \bar{\Omega}_n \Phi_{n+k} \} + k \frac{r^k}{1 + r + \cdots + r^{k-1}} \sum_{n=1}^{\infty} (1 + r + \cdots + r^{2n-1}) \bar{\Omega}_n \Phi_{n+k} \right\}.
\]

The sequence \(k \mapsto k \{ \mathcal{X}_k + \sum_{n=1}^{\infty} \bar{\Omega}_n \Phi_{n+k} \}\) is \(\ell_2\) because of (3.10). We are ready if we can show that the operators

\[
\{ \Phi_k \} \mapsto \left\{ \sum_{n=1}^{\infty} (1 + r + \cdots + r^{2n-1}) \bar{\Omega}_n \Phi_{n+k} \right\},
\]

(3.20)
are uniformly bounded (as $\ell_2$-operators) on the interval $0 < r \leq 1$. If it happens that \( \{ n \mapsto 2n\Omega_n \} \in \ell_1(\mathbb{N}) \) we estimate
\[
\sum_{k=1}^{\infty} \left| \sum_{n=1}^{\infty} 2n\Omega_n \Phi_{n+k} \right|^2 \leq \sum_{k=1}^{\infty} \left\{ \left| \sum_{m=1}^{\infty} 2m|\Omega_m| \right| \left\{ \sum_{n=1}^{\infty} 2n|\Omega_n| \right\} |\Phi_{n+k}|^2 \right\} \leq \left( \sum_{m=1}^{\infty} 2m|\Omega_m| \right)^2 \sum_{k=1}^{\infty} |\Phi_k|^2.
\]
It follows that the 'positive' sequence tends to 0 if \( r \uparrow 1 \).

- The 'negative' sequence \( \ell \mapsto -\ell \left\{ \sum_{m=1}^{\infty} (1 - r^{2m+\ell})\Omega_{m+\ell}\Phi_m \right\} \) can be written
\[
\ell \mapsto -(1 - r^\ell) \left\{ \ell \left\{ \sum_{m=1}^{\infty} \Omega_{m+\ell}\Phi_m \right\} + \frac{r^\ell}{1 + r + \cdots + r^{\ell-1}} \sum_{m=1}^{\infty} (1 + r + \cdots + r^{2m-1})\Omega_{m+\ell}\Phi_m \right\} \] (3.21)

With a similar estimate as before it turns out that also this $\ell_2(\mathbb{N})$ sequence tends to 0 if \( r \uparrow 1 \).

- For the last statement in the theorem note that the coefficients $X_k$ do not occur in the 'negative' sequence.

The natural question arises whether the results of the previous theorem could also be obtained if only \( \{ \theta \mapsto \Omega(e^{i\theta}) \} \in \mathcal{C}^{1;\alpha}(\mathbb{S}^1) \), with $\alpha > 0$, is assumed. I got half way by invoking a theorem on Fourier multipliers which map periodic Hölder spaces into themselves.  

**Theorem 3.4 (Behaviour near the Boundary 2)**

Assume that \( \{ \theta \mapsto \Omega(e^{i\theta}) \} \in \mathcal{C}^{1;\alpha}(\mathbb{S}^1) \), with $\alpha > 0$, then
\[
\partial_\theta [\overline{\Omega\Phi + X}]^+ \bigg|_r \rightarrow \partial_\theta [\overline{\Omega\Phi + X}]^+ \bigg|_1, \text{ in } L_2(\mathbb{S};\mathbb{C}), \text{ as } r \uparrow 1, \tag{3.22}
\]

**Proof**

We ‘only’ have to show that the the operators (3.20) are still uniformly bounded (as $\ell_2$-operators) on the interval $0 < r \leq 1$ under the weaker condition.

Consider the 'multiplication operator expression'
\[
\left( \sum_{n=1}^{\infty} (1 + r + \cdots + r^{2n-1})\Omega_n e^{-i\theta n} \right) \left( \sum_{m=1}^{\infty} \Phi_m e^{im\theta} \right) =
\]
\[
= \sum_{k=1}^{\infty} \left\{ \sum_{m-n=k, n \geq 1, m \geq 1} (1 + r + \cdots + r^{2n-1})\Omega_n \Phi_m \right\} e^{ik\theta} +
\]

\(^3\) JdG thanks Dr. G. Prokert for advice and references.
\[ + \sum_{\ell=0}^{\infty} \left\{ \sum_{n-m=\ell, n \geq 1, m \geq 1} (1 + r + \ldots + r^{2n-1}) \Omega_n \Phi_m \right\} e^{-i\ell \theta} = \]

\[ = \sum_{k=1}^{\infty} \left\{ \sum_{n=1}^{\infty} (1 + r + \ldots + r^{2n-1}) \Omega_n \Phi_{n+k} \right\} e^{ik \theta} + \sum_{\ell=0}^{\infty} \left\{ \sum_{m=1}^{\infty} (1 + r + \ldots + r^{2(m+\ell)-1}) \Omega_{m+\ell} \Phi_m \right\} e^{-i\ell \theta}. \]

If the very first sum in this expression represents a bounded function, uniformly in \(0 < r \leq 1\), we are ready. According to our assumption, the function

\[ \{ \theta \mapsto \sum_{n=1}^{\infty} n \Omega_n e^{-in \theta} \} \in \mathcal{C}^\alpha(S^1). \]

This remains so if the respective Fourier coefficients are multiplied by \(\frac{1 + r + \ldots + r^{2n-1}}{n}\), because they satisfy the conditions (1.2)-(1.3) in [AB].

**Additional Remark** As for the 'negative' part \(\partial_\theta [\Omega \Phi + X]^{-} \big|_r = \partial_\theta [\Omega \Phi]^{-} \big|_r\), we should be able to prove, cf. (3.21), that from \(\{ \ell \mapsto \sum_{m=1}^{\infty} \Omega_{m+\ell} m \Phi_m \} \in \ell_2\) it follows that also \(\{ \ell \mapsto \sum_{m=1}^{\infty} \Omega_{m+\ell} (1 + r + \ldots + r^{2m-1}) \Phi_m \} \in \ell_2\), and uniformly bounded, for \(0 < r \leq 1\). Let us see how far we get. The second sum in (3.21) can be split

\[ \sum_{m=1}^{\infty} (1 + r + \ldots + r^{2(m+\ell)-1}) \Omega_{m+\ell} \Phi_m - (1 + r + \ldots + r^{2\ell-1}) \sum_{m=1}^{\infty} \Omega_{m+\ell} r^{2m} \Phi_m. \]  

(3.23)

The first term presents no trouble. It is 'multiplication by a bounded function', as in the previous proof. For the second term we would like to show uniform boundedness for

\[ \ell \sum_{m=1}^{\infty} \Omega_{m+\ell} r^{2m} \Phi_m = \sum_{m=1}^{\infty} (m + \ell) \Omega_{m+\ell} r^{2m} \Phi_m - \sum_{m=1}^{\infty} \Omega_{m+\ell} r^{2m} m \Phi_m. \]

Here the first term comes from multiplication by \(\Omega\), which is supposed to be continuous on \(\bar{D}\). The second term finally confronts us with the question whether from \(\{ \ell \mapsto \sum_{m=1}^{\infty} \Omega_{m+\ell} m \Phi_m \} \in \ell_2\) it follows that

\[ \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall 0 < r \leq 1 : \sum_{\ell=N}^{\infty} \left| \sum_{m=1}^{\infty} \Omega_{m+\ell} r^{2m} m \Phi_m \right|^2 < \varepsilon. \]  

(3.24)

I could not prove this!
4 Results on Stokes Boundary Value Problems

In this section we formulate our results for simply connected domains $G \subset \mathbb{R}^2 \sim \mathbb{C}$ with boundary $\partial G$ and $0 \in G$. The boundary is supposed to be an arclength parametrized Jordan curve with a Hölder continuous and positively oriented tangent vector $s \mapsto \dot{x}(s) = \dot{z}(s)$.

Let, as before, $\Omega : \mathbb{D} \rightarrow G$ denote the unique conformal mapping with $\Omega(0) = 0$ and $\Omega'(0) > 0$. Again $\theta \mapsto s(\theta)$ is defined by $\Omega(e^{i\theta}) = s(\theta)$, $0 \leq \theta < 2\pi$.

The following two theorems are immediate consequences of the preceding sections. Looking at the smoothness assumptions of the preceding theorems, it is clear that the $\mathbb{H}^2$-condition on the boundary $\partial G$ in the next theorem can be somewhat relaxed.

**Theorem 4.1 (Stokes-Dirichlet)**

Consider the Stokes-Dirichlet problem (2.1) with boundary $\{s \mapsto x(s)\} \in \mathbb{H}^2(\partial G)$.

The prescribed boundary velocity field is given by

$$s \mapsto g(x(s)) = V_1(s)n(x(s)) + V_2(s)t(x(s)) = -i(V_1(s) + iV_2(s))\dot{z}(s) = -iV(s)\dot{z}(s) \in L_2(\partial G),$$

where $\int_{\partial G} V_1(s) \, ds = 0$.

There exist unique **analytic** $\varphi, \chi : G \rightarrow \mathbb{C}$, with $\varphi(0) = \chi(0) = \text{Re}\varphi'(0) = 0$, and $\varphi|_{\partial G}, \chi|_{\partial G} \in L_2(\partial G)$, such that

$$z(s)\overline{\varphi'(z(s))} - \varphi(z(s)) + \overline{\chi'(z(s))} = -iV(s)\dot{z}(s), \quad z(s) \in \partial G.$$

We have

- $\varphi(\Omega(re^{i\theta})) \rightarrow \varphi|_{\partial G}(s(\theta))$ and $\chi(\Omega(re^{i\theta})) \rightarrow \chi|_{\partial G}(s(\theta))$, in $L_2(S)$-sense, as $r \uparrow 1$.

- $\left[ v_1(z) + iv_2(z) \right]_{z = \Omega(re^{i\theta})} = \left[ z\varphi'(z) - \varphi(z) + \overline{\chi'(z)} \right]_{z = \Omega(re^{i\theta})} \rightarrow g(x(s(\theta)))$, in $L_2(S)$-sense, as $r \uparrow 1$.

- The normal stress at $\partial G$ is well defined (as a $\mathbb{H}^{-1}$-limit) and given by

$$ (T \cdot n)(x(s)) = 2i \frac{d}{ds}g(x(s)) + 4i \frac{d}{ds}\varphi(z(s)) \in \mathbb{H}^{-1}(\partial G). $$
Theorem 4.2 (Stokes-Neumann)
Consider the Stokes-Neumann problem (2.4) with boundary \( \{ s \mapsto \bar{x}(s) \} \in \mathbb{H}^2(\partial G) \).
The prescribed boundary stress field
\[
s \mapsto f(\bar{x}(s)) = T(\bar{x}(s)) \cdot \mathbf{n}(\bar{x}(s)) = 2i \frac{d}{ds} \left( z(s)\varphi'(z(s)) + \varphi(z(s)) + \chi'(z(s)) \right) =
\]
\[= -i \frac{d}{ds} \{ K(s) \dot{z}(s) \} \in \mathbb{H}^{-1}(\partial G), \]
whith \( s \mapsto K(s) = K_1(s) + iK_2(s) \in \mathbb{L}_2(\partial G) \), and \( \int_{\partial G} K_1(s) \, ds = 0 \).
There exist unique analytic \( \varphi, \chi : \mathbb{G} \rightarrow \mathbb{C} \), with \( \varphi(0) = \chi(0) = \text{Im} \varphi'(0) = 0 \), and \( \varphi\big|_{\partial G}, \chi\big|_{\partial G} \in \mathbb{L}_2(\partial G) \), such that
\[
z(s)\varphi'(z(s)) + \varphi(z(s)) + \chi'(z(s)) = -\frac{1}{2} K(s) \dot{z}(s), \quad z(s) \in \partial G.
\]
We have
\begin{itemize}
  \item \( \varphi(\Omega(re^{i\theta})) \rightarrow \varphi\big|_{\partial G}(s(\theta)) \) and \( \chi(\Omega(re^{i\theta})) \rightarrow \chi\big|_{\partial G}(s(\theta)) \),
  \quad in \( \mathbb{L}_2(\mathbb{S}) \)-sense, as \( r \uparrow 1 \).
  \item \( \left[ z\varphi'(z) + \varphi(z) + \chi'(z) \right]_{z=\Omega(re^{i\theta})} \rightarrow g(x(s(\theta))), \quad \text{in} \ \mathbb{L}_2(\mathbb{S}) \)-sense, as \( r \uparrow 1 \).
  \item \( \left( T \cdot \mathbf{n}(z) \right)_{z=\Omega(re^{i\theta})} \rightarrow -i \frac{d}{ds} \{ K(s) \dot{z}(s) \big|_{s=s(\theta)} \} \quad \text{in} \ \mathbb{H}^{-1}(\mathbb{S}) \)-sense, as \( r \uparrow 1 \).
  \item The velocity field at \( \partial G \) is well defined (as a \( \mathbb{L}_2 \)-limit) and given by
  \[
v_1(z(s)) + iv_2(z(s)) = -\frac{1}{2} K(s) \dot{z}(s) - 2\varphi(z(s)) \in \mathbb{L}_2(\partial G).
\]
\end{itemize}

Of special interest in the context of free boundary value problems are solutions of the Stokes-Neumann problems with \( K_1 = 0 \). In [H], taking \( K_1 = 0, K_2 = \kappa = \text{constant} \), (surface tension), Hopper derives an ingenious equation for the time evolution of the domain \( \mathbb{G} \). This \textit{Hopper equation} is a non-linear time evolution equation for the conformal map \( \Omega(\cdot t) : \mathbb{D} \rightarrow \mathbb{G} \). In a series of papers, following [H], Hopper shows that his equation has several classes of exact solutions \( \zeta \mapsto \Omega(\zeta, t) \), which are polynomial or rational in \( \zeta \). For more of those see also [K].
In [G] it has been shown that already \( K_1 = 0, K_2 = K_2(\Omega, t) \) is enough for this phenomenon to happen. Reason enough for looking at the structure of the solution if \( K_1 = 0 \). Then the analytic \( \varphi \) and \( \chi \) are in a special relation to each other:
Suppose \( \frac{d}{ds} \text{Re} \left( z(s) \varphi(z(s)) + \chi(z(s)) \right) \bigg|_{z(s) \in \partial G} = 0 \) and \( \chi : G \to \mathbb{C} \) being given, then

\[
\text{Re} \left\{ \frac{\varphi'}{z} \right\} \bigg|_{\partial G} = \frac{C - \text{Re} \chi}{zz} \bigg|_{\partial G}, \quad \text{with } C \in \mathbb{R} \text{ any constant. Hence, cf. (A.9),}
\]

\[
\varphi(\Omega(\zeta)) = \frac{\Omega(\zeta)}{2\pi} \int_0^{2\pi} \frac{C - \text{Re} \chi(\Omega(e^{i\theta}))}{|\Omega(e^{i\theta})|^2} \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} \, d\theta, \quad |\zeta| < 1. \quad (4.1)
\]

It is straightforward that \( \varphi(0) = 0, \text{ Im} \varphi'(0) = 0 \), in this case.

Suppose \( \frac{d}{ds} \text{Re} \left( z(s) \varphi(z(s)) + \chi(z(s)) \right) \bigg|_{z(s) \in \partial G} = 0 \) and \( \varphi : G \to \mathbb{C} \) being given, then

\[
\text{Re} \{ \chi \} \bigg|_{\partial G} = C - \text{Re} \left( \frac{z\varphi}{\zeta} \right) \bigg|_{\partial G}, \quad \text{with } C \in \mathbb{R}. \text{ Hence, cf. (A.9),}
\]

\[
\chi(\Omega(\zeta)) = \frac{1}{2\pi} \int_0^{2\pi} \text{Re} \left[ C - \Omega(e^{i\theta}) \varphi(\Omega(e^{i\theta})) \right] \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} \, d\theta, \quad |\zeta| < 1. \quad (4.2)
\]

Take \( C = \frac{1}{2\pi} \int_0^{2\pi} \left[ \Omega(e^{i\theta}) \varphi(\Omega(e^{i\theta})) \right] \, d\theta \), then \( \chi(0) = 0. \)

We conclude with a theorem on some unusual (non physical?) boundary value problems for Stokes’ equations. The proof is based on the fact that an analytic function on \( G \) is, up to a constant, fixed by its real (or imaginary) part at the boundary \( \partial G \), on the simple connectedness assumption on \( G \) and on table (1.5).

**Theorem 4.3**

Let \( G \subset \mathbb{R}^2 \) be bounded and simply connected.

Suppose \( \partial G \) has a \( H^1 \) arclength parametrization.

For any of the function pairs \( \{ p, \, v \cdot n \} \), \( \{ p, \, v \cdot \hat{n} \} \), \( \{ \text{rot} \, v, \, v \cdot n \} \), \( \{ \text{rot} \, v, \, v \cdot \hat{n} \} \), prescribed at the boundary and all in \( L_2(\partial G) \), there is a unique pressure-velocity flow pair \( \{ p, v \} \), which solves Stokes’ equations. From within, the boundary values are approached in \( L_2 \)-sense in the way described before.
A APPENDIX: Complex Analysis revisited

1. We identify $\mathbb{R}^2$ and $\mathbb{C}$ by means of the bijection

$$x = \begin{bmatrix} x \\ y \end{bmatrix} \mapsto z = x + iy.$$ 

2. Multiplication by $i$, or by any fixed complex number, complex conjugation, taking real or imaginary parts

$$z \mapsto iz, \quad z \mapsto \bar{z}, \quad z \mapsto \text{Re } z, \quad z \mapsto \text{Im } z,$$

will often be considered as $\mathbb{R}$-linear mappings in $\mathbb{R}^2$.

3. Functions

$$F : \mathbb{C} \to \mathbb{C} : z = x + iy \mapsto F(z) = F(x + iy) = \text{Re } F(z) + i \text{ Im } F(z),$$

possibly local and not necessarily analytic, are identified with, or correspond to

$$F : \mathbb{R}^2 \to \mathbb{R}^2 : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} F_1(x, y) \\ F_2(x, y) \end{bmatrix} = \begin{bmatrix} \text{Re } F(x + iy) \\ \text{Im } F(x + iy) \end{bmatrix},$$

and vice versa. Such functions will sometimes be considered as vector fields. In a context of cartesian coordinates no confusion arises.

4. We have the usual (commuting) vector partial differentiation operators

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y), \quad \text{hence } \partial_x = \partial_z + \partial_{\bar{z}}, \quad \partial_y = i(\partial_z - \partial_{\bar{z}}) \quad (A.1)$$

Note that for the componentwise Laplacian acting on $F$, we have

$$\Delta F = 4\partial_{z} \partial_{\bar{z}} F. \quad (A.2)$$

It follows that if one has $\partial_{z} F = 0$ or/and $\partial_{\bar{z}} F = 0$, then, componentwise, $\Delta F = 0$.

Which says that $F$ is a stack of 2 harmonic functions.

Of importance is also the complex representation of Euler operator

$$x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} = z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial z}. \quad (A.3)$$

5. If $\partial_z F = 0$ we say that $F$ (= $F$) is analytic. If $\partial_{\bar{z}} F = 0$ we say that $F$ (= $F$) is anti-analytic.
This nicely corresponds to the respective Cauchy-Riemann and anti-Cauchy-Riemann relations

\[
\text{C.R.: } \begin{cases}
\frac{\partial_x}{\partial x} \text{Re } F - \frac{\partial_y}{\partial y} \text{Im } F = 0, \\
\frac{\partial_y}{\partial x} \text{Re } F + \frac{\partial_x}{\partial y} \text{Im } F = 0,
\end{cases} \quad \text{a.C.R.: } \begin{cases}
\frac{\partial_x}{\partial y} \text{Re } F + \frac{\partial_y}{\partial x} \text{Im } F = 0, \\
\frac{\partial_y}{\partial y} \text{Re } F - \frac{\partial_x}{\partial x} \text{Im } F = 0.
\end{cases} \tag{A.4}
\]

Note that analyticity of \( z \mapsto F(z) \) implies anti-analyticity of \( z \mapsto \overline{F(z)} \) and vice versa.

6. If a stack \( \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} F_1(x, y) \\ F_2(x, y) \end{bmatrix} \) of two harmonic functions corresponds to an analytic function \( z \mapsto F(z) \), we say that \( F_2 \) is a harmonic conjugate of \( F_1 \). From (A.4) it is clear that a harmonic conjugate is unique up to a constant.

If on a simply connected domain \( G \subset \mathbb{R}^2 \), with \( 0 \in G \), a harmonic function \( x \mapsto F_1(x) \in \mathbb{R} \) is given, a harmonic conjugate is constructed by

\[
x \mapsto F_2(x) = \int_0^x \{-\partial_y F_1(x(s)) \dot{x} + \partial_x F_1(x(s)) \dot{y}\} \, ds. \tag{A.5}
\]

The result does not depend on the path of integration \( s \mapsto x(s) \), since the vectorfield

\[
x \mapsto \left[ \begin{array}{c}
-\partial_y F_1(x) \\
\partial_x F_1(x)
\end{array} \right]
\]

is obviously conservative.

7. If on a connected domain \( G \subset \mathbb{R}^2 \), with \( 0 \in G \), a stack \( x \mapsto \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} \) is harmonic, i.e. \( \Delta F = 0 \), it corresponds to an analytic function \( z \mapsto F(z) \) on \( G \) if one of the C.R.-relations is satisfied all over \( G \) and the other C.R.-relation is satisfied at one point, say \( z = 0 \). Indeed, suppose the second C.R.-relation is satisfied all over \( G \). Then

\[
\partial_x(\partial_x F_1 - \partial_y F_2) = -\partial_y(\partial_y F_1 + \partial_x F_2) = 0 \quad \text{and} \quad \partial_y(\partial_x F_1 - \partial_y F_2) = \partial_x(\partial_y F_1 + \partial_x F_2) = 0.
\]

Therefore \( \partial_x F_1 - \partial_y F_2 = \text{constant} = 0 \).

8. Next we gather some useful expressions for the commutation relations between \( \partial_x, \partial_y, \Delta \) and the projections \( \text{Re, Im} \). All to be applied to smooth \( \mathbb{C} \)-valued functions on domains in \( \mathbb{C} \).

\[
\begin{align*}
\frac{\partial_x}{\partial x} \text{Re } = & \quad \text{Re } \frac{\partial_x}{\partial x} = \text{Re } (\partial_x + \partial_\overline{z})  \\
\frac{\partial_y}{\partial y} \text{Re } = & \quad \text{Re } \frac{\partial_y}{\partial y} = -\text{Im } (\partial_z - \partial_\overline{z})  \\
\Delta \text{Re } = & \quad \text{Re } \Delta = 4 \text{Re } \frac{\partial_x}{\partial z}  \\
\frac{\partial_x}{\partial x} \text{Im } = & \quad \text{Im } \frac{\partial_x}{\partial x} = \text{Im } (\partial_x + \partial_\overline{z})  \\
\frac{\partial_y}{\partial y} \text{Im } = & \quad \text{Im } \frac{\partial_y}{\partial y} = \text{Re } (\partial_z - \partial_\overline{z})  \\
\Delta \text{Im } = & \quad \text{Im } \Delta = 4 \text{Im } \frac{\partial_x}{\partial \overline{z}} \tag{A.6}
\end{align*}
\]

9. On a simply connected domain \( G \subset \mathbb{R}^2 \), with \( 0 \in G \) we consider a biharmonic function \( x \mapsto \phi(x) \). This means \( \Delta^2 \phi = 0 \). The claim is that there exist analytic \( \varphi, \chi : G \to \mathbb{C} \), such that

\[
\phi(x) = \text{Re } (\bar{z} \varphi(z) + \chi(z)), \quad z = x + iy. \tag{A.7}
\]

To show this, note first that \( \Delta \phi \) is harmonic on \( G \). So there is an analytic \( \psi \) on \( G \) such that \( \Delta \phi = \text{Re } \psi \). Introduce the analytic function \( z \mapsto \varphi(z) = \frac{1}{4} \int_0^z \psi(\zeta) \, d\zeta \). Then \( 4 \varphi'(z) = \psi(z) \).
We now have $\Delta \left( \phi(x) - \text{Re} \left( \overline{z} \varphi(z) \right) \right) = 0$. So $\phi - \text{Re} \left( \overline{z} \varphi \right)$ is harmonic on $G$ and there exists analytic $\chi$ on $G$ such that

$$\phi(x) - \text{Re} \left( \overline{z} \varphi(z) \right) = \text{Re} \chi(z), \quad z = x + iy.$$  

This proves the claim.

**11.** Let $L^2(S^1)$ denote the standard real Hilbert space on the unit circle $S^1 \subset \mathbb{C}$. Let $\tilde{f}_1 \in L^2(S^1)$. For $\tilde{f}_1$ we will employ the Fourier expansion convention

$$\tilde{f}_1(\theta) = a_0 + \sum_{n=1}^{\infty} \{a_n \cos(n\theta) - b_n \sin(n\theta)\}.$$  

Extend $\tilde{f}_1$ to a harmonic function $f_1$ on the unit disk $D \subset \mathbb{C}$ by solving the Dirichlet problem. Let $f_2$, the harmonic conjugate of $f_1$, be fixed by taking $f_2(0) = 0$. Let $\tilde{f}_2$ denote the limit to the boundary $S^1$ of $D$. Then

$$\tilde{f}_2(\theta) = \sum_{n=1}^{\infty} \{b_n \cos(n\theta) + a_n \sin(n\theta)\}.$$  

All this can be seen by taking real and imaginary parts from the power series expansion of $f_1 + if_2$ up to the boundary $S^1$

$$\tilde{f}_1(\theta) + i\tilde{f}_2(\theta) = f_1(e^{i\theta}) + if_2(e^{i\theta}) = \sum_{n=0}^{\infty} (a_n + ib_n)e^{in\theta}, \quad b_0 = 0.$$  

Let further $L^2(S^1; \mathbb{R}; \perp \{1\})$ denote the linear subspace of all $\tilde{g} \in L^2(S^1)$ with $\int_0^{2\pi} \tilde{g}(\theta) \, d\theta = 0$.

The operator

$$J : L^2(S^1; \mathbb{R}; \perp \{1\}) : \tilde{f}_1 \mapsto J\tilde{f}_1 = \tilde{f}_2,$$

is orthogonal and skew-symmetric:

$$J^* = -J = J^{-1}, \quad J^2 = -1. \quad (A.8)$$

Note that $J \{ \text{Re} (a_n + ib_n)e^{in\theta} \} = \text{Re} \{ -i(a_n + ib_n)e^{in\theta} \}$.

- The operator $N : L^2(S^1; \mathbb{R}; \perp \{1\}) \to L^2(S^1; \mathbb{R}; \perp \{1\})$ is defined by

$$Nf_1 = \sum_{n=1}^{\infty} n \{b_n \cos(n\theta) + a_n \sin(n\theta)\}.$$  

We have $N^* = N$, $J \partial_{\theta} = \partial_{\theta} J = N$ and therefore $\partial_{\theta} = -NJ$.

- For analytic functions $z \mapsto f(z)$ on the unit disk $D$ we will consider a splitting in real Fourier series on $S^1$. We put

$$f(e^{i\theta}) = \sum_{n=1}^{\infty} (a_n + ib_n)e^{in\theta} = f_1(e^{i\theta}) + if_2(e^{i\theta}) = f_1(e^{i\theta}) + iJf_1(e^{i\theta}).$$

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• Proof of Lemma 1.4
The operator $J$ defined by
\[ J\{a_n \cos(n\theta) - b_n \sin(n\theta)\} = b_n \cos(n\theta) + a_n \sin(n\theta), \quad n = 1, 2, 3, \ldots, \]
can be represented as
\[ Jf_1(\theta) = \lim_{r \uparrow 1} \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} r^n \sin(n(\theta - \theta_1)) f_1(\theta_1) \, d\theta_1, \]
as can easily be checked term by term. Calculate
\[ \sum_{n=1}^{\infty} r^n \sin(n\alpha) = \text{Im} \sum_{n=1}^{\infty} (re^{i\alpha})^n = \frac{r \sin(\alpha)}{1 + r^2 - 2r \cos(\alpha)} = \frac{2r \sin(\frac{1}{2} \alpha) \cos(\frac{1}{2} \alpha)}{(1 - r)^2 + 4r \sin^2(\frac{1}{2} \alpha)} \xrightarrow{r \uparrow 1} \frac{1}{2} \cot(\frac{1}{2} \alpha). \]
Therefore
\[ Jf_1(\theta) = \lim_{r \uparrow 1} \frac{1}{\pi} \int_{-\pi}^{\pi} 2r \sin\left(\frac{1}{2}(\theta - \theta_1)\right) \cos\left(\frac{1}{2}(\theta - \theta_1)\right) f_1(\theta_1) \, d\theta_1, \]
Since the kernel is $2\pi$-periodic and odd in $(\theta - \theta_1)$, the result follows. □

12. Corollary For analytic $F : \mathbb{D} \to \mathbb{C}$, $\text{Im} F'(0) = 0$, we have the presentation
\[ F(\zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Re} F(e^{i\theta}) \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} \, d\theta, \quad |\zeta| < 1. \quad (A.9) \]
Note that taking the real part leads to the Poisson formula.

\section*{B APPENDIX: Details on Stokes’ equations}

\textbf{Proof of Theorem 1.1}

• Suppose that the pair $\nu, p$ is a solution on some domain $\mathcal{G}$. Since $\nabla \cdot \nu = 0$, there exists a 'stream function' $\psi$ such that $\nu = \left[ \begin{array}{c} \partial_y \psi \\ -\partial_x \psi \end{array} \right]$, where $\psi$ is fixed up to a constant.

Similarly, since $\nabla \cdot T = 0$, it follow that, for suitable functions $f, g$ we are allowed to write
\[ T = 2 \left[ \begin{array}{cc} \partial_y f & \partial_y g \\ -\partial_x f & -\partial_x g \end{array} \right]. \]
Because of symmetry $\partial_x f + \partial_y g = 0$. Hence $\left[ \begin{array}{c} f \\ g \end{array} \right] = \left[ \begin{array}{c} -\partial_y \phi \\ \partial_x \phi \end{array} \right]$, for suitable $\phi$, the 'Airy function'. It follows that we are allowed to write
\[ T = 2 \left[ \begin{array}{cc} -\partial_y \partial_y \phi & \partial_x \partial_y \phi \\ \partial_x \partial_y \phi & -\partial_x \partial_x \phi \end{array} \right]. \]
Note that $\phi$ is fixed up to a polynomial of 1st degree.
Because of bi-harmonicity there are analytic functions \( f_1, f_2, g_1, g_2 \) on \( \mathbb{C} \) such that, cf. (A.7),

\[
\phi = \text{Re}(\overline{z} f_1 + g_1), \quad \psi = \text{Im}(\overline{z} f_2 + g_2),
\]

From the C.R.-relations and (A.6) we get

\[
\begin{align*}
\partial_x \phi &= \partial_y \psi \quad \Rightarrow \quad \text{Re} f_1'' = \text{Re} f_2'', \\
\partial_y \phi &= -\partial_x \psi \quad \Rightarrow \quad -\text{Im} f_1'' = -\text{Im} f_2''.
\end{align*}
\]

Next, consistency of the stress matrix requires

\[
\mathcal{T} = 2 \begin{bmatrix} -\partial_y \partial_y \phi & \partial_x \partial_y \phi \\ \partial_x \partial_y \phi & -\partial_x \partial_x \phi \end{bmatrix} = \begin{bmatrix} -\Delta \phi + 2\partial_x \partial_y \psi & -\partial_x \partial_x \psi + \partial_y \partial_y \psi \\ -\partial_x \partial_x \psi + \partial_y \partial_y \psi & -\Delta \phi - 2\partial_x \partial_y \psi \end{bmatrix}.
\]

This requires

\[
\partial_x \partial_x \phi - \partial_y \partial_y \phi = 2\partial_x \partial_y \psi, \quad 2\partial_x \partial_y \phi = -\partial_x \partial_x \psi + \partial_y \partial_y \psi. \tag{B.2}
\]

Calculate, cf. (A.6),

\[
\begin{align*}
\partial_x \phi &= \text{Re}(\overline{z} f_1' + g_1' + f_1) \quad \partial_x \psi &= \text{Im}(\overline{z} f_2' + g_2' + f_1) \\
\partial_y \phi &= -\text{Im}(\overline{z} f_1' + g_1' - f_1) \quad \partial_y \psi &= \text{Re}(\overline{z} f_2' + g_2' - f_1) \\
\partial_x \partial_y \phi &= -\text{Im}(\overline{z} f_1'' + g_1'' - f_1') \quad \partial_x \partial_y \psi &= \text{Re}(\overline{z} f_2'' + g_2'' + f_2') \\
\partial_y \partial_x \phi &= \text{Re}(\overline{z} f_1'' + g_1'' + f_1') \quad \partial_y \partial_x \psi &= \text{Im}(\overline{z} f_2'' + g_2'' - f_2') \\
\partial_y \partial_y \phi &= -\text{Re}(\overline{z} f_1'' + g_1'' - f_1') \quad \partial_y \partial_y \psi &= -\text{Im}(\overline{z} f_2'' + g_2'' - f_2'). \tag{B.3}
\end{align*}
\]

Substitution of (B.3) in (B.2) leads, together with (B.1) to \( g_1'' = g_2'' \).

We find

\[
\psi(x, y) = \text{Im}\{\overline{z} f_2(z) + g_2(z)\}, \quad \phi(x, y) = \text{Re}\{\overline{z} (f_2(z) + \alpha z + \beta) + g_2(z) + \gamma z + \delta\}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}.
\]

Define \( \varphi(z) = f_2(z) + (\text{Re}\alpha) z \) and \( \chi(z) = g_2(z) \), then

\[
\psi(x, y) = \text{Im}\{\overline{z} \varphi(z) + \chi(z)\}, \quad \phi(x, y) = \text{Re}\{\overline{z} \varphi(z) + \chi(z)\} + \text{Re}\{\beta \overline{z} + \gamma z + \delta\}.
\]

If we just throw away the second term in the expression for \( \phi \), the stress matrix \( \mathcal{T} \) is not altered. The only freedom left is a constant added to \( \varphi \). We are left with

\[
\psi(x, y) = \text{Im}\{\overline{z} \varphi(z) + \chi(z)\}, \quad \phi(x, y) = \text{Re}\{\overline{z} \varphi(z) + \chi(z)\}. \tag{B.4}
\]
Finally we check the formulae for the kinematic and dynamic quantities, cf. (B.3),

\[
\begin{align*}
\nu + i\nu_2 &= \frac{\partial_y\psi - i\partial_x\psi}{\partial_y} = \partial_y\text{Im}(z\varphi + \chi) = \\
&= \Re((\partial_z - \partial_z)z\varphi + \chi) - i \Im((\partial_z + \partial_z)z\varphi + \chi) = \\
&= \Re(z\varphi' + \chi' - \varphi) - i \Im(z\varphi' + \chi' + \varphi) = \\
&= z\varphi' + \chi' - \varphi = -\varphi + z\varphi' + \chi'. \\
\partial_x\nu_2 - \partial_y\nu_1 &= \text{Im}(\partial_x - i\partial_y)(\nu_1 + i\nu_2) = 2\text{Im}\,(\partial_z(-\varphi + z\varphi' + \chi')) = -4\Im\,\varphi'. \\
T_{11} + T_{22} &= -2p = -2\Delta = -2\Delta \Re(z\varphi + \chi) = \\
&= -8\Re\,\partial_z\partial_x(z\varphi + \chi) = -8\Re\,\varphi'. \\
T_{22} - T_{11} + 2iT_{12} &= -2\partial_x\partial_x\varphi + 2\partial_y\partial_y\varphi + 4i2\partial_x\partial_y\varphi = \\
&= -2\Re(z\varphi' + \chi'' + 2\varphi') - 2\Re(z\varphi' + \chi'' + 2\varphi') - 4i\Im(z\varphi' + \chi'') = \\
&= -4\Re(z\varphi' + \chi'' + 4i\Im(z\varphi' + \chi'')} = -4(z\varphi' + \chi''). \\
\nu\cdot n &= \Re\{(\nu_1 - i\nu_2)\cdot -i\dot{z}\} = \text{Im}\{(\nu_1 - i\nu_2)\dot{z}\} = \\
&= \text{Im}\{(-\varphi + z\varphi' + \chi')\dot{z}\} = \text{Im}\{(d/ds)(z\varphi + \chi) - \varphi^{\#} - \varphi\dot{z}\} = \\
&= \frac{d}{ds}\text{Im}(z\varphi + \chi). \\
T\cdot n &= 2\begin{bmatrix}
-\partial_y\partial_y\varphi & \partial_x\partial_y\varphi \\
\partial_y\partial_y\varphi & -\partial_x\partial_x\varphi
\end{bmatrix}\begin{bmatrix}
\dot{y} \\
-\dot{x}
\end{bmatrix} = -2\frac{d}{ds}\begin{bmatrix}
-\partial_y\varphi \\
-\partial_x\varphi
\end{bmatrix} = \\
&= -2\frac{d}{ds}\begin{bmatrix}
\partial_x\Re(z\varphi + \chi) \\
-\partial_x\Re(z\varphi + \chi)
\end{bmatrix} = \\
&= 2\frac{d}{ds}\begin{bmatrix}
\text{Im}(z\varphi' + \chi' - \varphi) + i\Re(z\varphi' + \chi + \varphi) \\
i\Re(z\varphi' + \chi + \varphi)
\end{bmatrix} = \\
&= 2i\frac{d}{ds}\{z\varphi' + \chi + \varphi\}. \\
T\cdot \hat{z} &= 2\frac{d}{ds}\{z\varphi' + \chi + 4\Re\varphi\}.
\end{align*}
\]

If we put \(\varphi_1(z) = \varphi(z) + A\) and \(\chi_1(z) = \chi(z) + A\varphi + C\), with \(A, C \in \mathbb{C}\) we still find the same expressions for \(\nu_1, \nu_2, p\). Note also that the corresponding altered stream function \(\psi_1(z) = \psi(z) + \text{Im}(zA + A\varphi + B) = \psi(z) + \text{Im} B\) and the Airy function \(\phi_1(z) = \phi(z) + \Re(zA + A\varphi + B)\) show, respectively, an added constant and an added 1st degree polynomial which don’t alter the velocity and the stress tensor.

**Conclusion** If for some fixed \(g\) in the fluid domain we additionally require \(\varphi(a) = \lambda(a) = 0\), there is precisely one pair \(\{\varphi, \lambda\}\) that describes a solution of the Stokes equations.
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