Solution to Problem 64-14: Resistance of a ladder network
Bouwkamp, C.J.

Published in:
SIAM Review

DOI:
10.1137/1008019

Published: 01/01/1966

Please check the document version of this publication:

A submitted manuscript is the author's version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.

The final author version and the galley proof are versions of the publication after peer review.

The final published version features the final layout of the paper including the volume, issue and page numbers.

Citation for published version (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Download date: 12. Dec. 2018
Problem 64-14, Resistance of a Ladder Network, by William D. Fryer (Cornell Aeronautical Laboratory).

Find the input resistance $R_n$ to the $n$-section ladder network as shown in the figure. Also, determine $\lim_{n \to \infty} R_n$.

Solution by C. J. Bouwkamp (Technological University, Eindhoven, Netherlands).

Divide all the resistances of the $n$-section ladder network by 2. Then the resulting network has input resistance $R_n/2$ and the $(n+1)$-section ladder network is a series connection of a unit resistor and a network consisting of two resistors in parallel, one of value 1 and the other of value $R_n/2$. Consequently, we must have the recurrence relation

$$R_1 = 2, \quad R_{n+1} = \frac{2 + 2R_n}{2 + R_n}, \quad n = 1, 2, 3, \ldots$$

One method of finding an explicit expression for $R_n$ is to let $R_n = p_n/q_n$, where

$$p_{n+1} = 2p_n + 2q_n, \quad q_{n+1} = p_n + 2q_n, \quad p_1 = 2q_1 = 2.$$

It then follows that $p_n$ and $q_n$ are two linearly independent solutions of the recurrence relation $f_{n+2} - 4f_{n+1} + 4f_n = 0$ whose solutions must be of the form $f_n = A(2 + \sqrt{2})^n + B(2 - \sqrt{2})^n$. Hence,

$$p_n = \frac{1}{2} [(2 + \sqrt{2})^n + (2 - \sqrt{2})^n], \quad q_n = \frac{\sqrt{2}}{4} [(2 + \sqrt{2})^n - (2 - \sqrt{2})^n].$$

Finally,

$$R_n = \frac{(2 + \sqrt{2})^n + (2 - \sqrt{2})^n}{(2 + \sqrt{2})^n - (2 - \sqrt{2})^n}$$

and $\lim_{n \to \infty} R_n = \sqrt{2}$.

Also solved by E. C. Bittner (National Aeronautics and Space Administration), T. S. Engelar, Jr. (Research Institute for Advanced Studies), M. Fox (Sylvania Electronics Systems), J. M. Holt (Collins Radio Corporation), R. Kelisky (IBM Watson Research Center), P. G. Kirmser (Kansas State University), B. K. Larkin (Martin Company), J. D. Lawson (Oak Ridge Institute of Nuclear Studies), W. C. Lynch (Case Institute of Technology), E. Liban and G. Stoodley (Grumman Aircraft Engineering Corporation), I. Navot, two solutions (Israel Institute of Technology, Haifa, Israel), R. Oravec (Price Waterhouse and Company), K. A. Post (Technological University, Eindhoven, Netherlands), J. Richman (Drexel Institute of Technology), S. J. Roberts
Editorial Note. The recurrence relation for \( R_n \) can also be solved by finding the \( n \)th power of the matrix \[
\begin{bmatrix}
2 & 2 \\
1 & 2
\end{bmatrix}
\].

Waligorski also determines \( R_n \) for the network in which the elements of the \( r \)th section, \( 2^{1-r} \) and \( 2^{1-r} \), are replaced by \( \beta \alpha^{1-r} \) and \( \alpha^{1-r} \), \( r = 1, 2, \cdots, n \). The limit is

\[
R_\infty = \frac{1}{2} \left( \beta + 1 - \alpha + \sqrt{(\beta + 1 - \alpha)^2 + 4\alpha\beta} \right)
\]

and is evaluated by means of the periodic continued fraction which is obtained from the recurrence relation.


**Problem 64-16, On a Probability of Overlap**, by Murray S. Klamkin (Ford Scientific Laboratory).

Prove directly or by an immediate application of a theorem in statistics that the conjecture in the following abstract from Mathematical Reviews, March, 1964, p. 589, is valid:

“Oleskiewicz, M. The probability that three independent phenomenon of equal duration will overlap. (Polish, Russian and English summaries) Prace Mat. 64 (1960), 1–7.

The value \( P_3 \) of the probability that 3 stochastically independent phenomenon of equal duration \( t_0 \) which all occur during the time \( t + t_0 \) will overlap is shown by geometrical methods to be equal to \( (3t_0^2 - 2t_0^3)/t^3 \). The author makes the conjecture that a similar formula holds for \( n \) independent events, namely \( P_n = (nt_0^{n-1} - (n - 1)t_0^n)/t^n \).”

Solution by P. C. Hemmer (Norges Tekniske Høgskole, Trondheim, Norway).

We note that an overlap has to start simultaneously with the onset of one of the \( n \) events. The probability of overlap when this one event is assumed to be number 1 gives \( 1/n \) of the desired probability \( P_n \). Denote the whole time by \( (0, t + t_0) \) and let event number 1 occur in \( (\lambda, \lambda + t_0) \), where \( \lambda \) is uniformly distributed in \( (0, t) \). An overlap exists if and only if the other \( n - 1 \) events occur at time \( \lambda \). Independence and uniform distribution guarantee the following probability for this to happen:

\[
P(\lambda) = \begin{cases} 
(\lambda/t)^{n-1} & \text{if } 0 \leq \lambda \leq t_0, \\
(t_0/t)^{n-1} & \text{if } t_0 \leq \lambda \leq t.
\end{cases}
\]