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Cramér-Rao bounds for rate-constrained distributed time-delay estimation

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Abstract—This paper investigates time-delay estimation for acoustic source localization in a distributed microphone array. The microphones are assumed to be part of a wireless sensor network, with a constraint on the number of bits that may be exchanged between the sensors. Consequently, at the fusion center, time-delay estimation needs to be performed using quantized signals. In this paper, the relation between the communication bit-rate and the Cramér-Rao lower bound (CRLB) on the variance of the time-delay estimation error is explored. The minimum bit-rate required to ensure that the CRLB is attained is also derived.

I. INTRODUCTION

IN source localization problems, signals received at two or more sensors are combined to obtain an estimate of the location of a sound source. This is accomplished by estimating the differential delay between the sensor signals, which is referred to as the time delay (TD) estimation problem [1]–[3]. Based on the TD estimate, a bearing estimate is subsequently obtained.

Performance bounds on TD estimation when all the sensor signals are available at the processing unit have been extensively studied in the past [4]–[8]. In distributed sensor networks however, not all sensor signals may be available at the processing unit, and wireless signal transmission is necessary.

This paper considers the case where the TD estimation is performed using signals from sensors that are connected wirelessly. As wireless transmission is power intensive, it is important to transmit as few bits as possible. Quantizers designed specifically for the TD estimation task have been studied in [9], [10]. Such quantizers typically require knowledge of e.g., the joint probability distribution of the sensor signals. In this paper, TD estimation is considered in the absence of any prior information. Furthermore, it is assumed that the sensor signals received at the fusion center are used not only for TD estimation but also for other tasks such as signal estimation through beamforming. Thus, in the absence of prior information, and in this general setting, the sensors simply quantize their data so as to minimize the reconstruction error, and TD estimation is performed at the fusion center using the quantized signals.

Within the above framework, using results from classical rate-distortion theory, this paper studies the relation between

the communication bit-rate and the resulting accuracy of TD estimation by analyzing the Cramér-Rao lower bound (CRLB) on the variance of the estimation error as a function of the bit-rate.

For a given observation duration, the CRLB is tight, i.e., the bound is attainable, only if the signal-to-noise ratio (SNR) exceeds a certain threshold [4], [5]. In the rate-constrained case, it is intuitive that the CRLB is attained only beyond certain bit-rates. For a given observation time and SNR, an expression that establishes a condition on the minimum bit-rate required to ensure attainability of the CRLB is also derived in this paper.

The remainder of the paper is organized as follows. Section II introduces the signal model employed, and the relevant rate-distortion relations used in the analysis. The CRLB in the rate-constrained case is obtained in Section III. The minimum bit-rate that renders the CRLB tight is derived in Section IV. Conclusions are summarized in Section V.

II. SIGNAL MODEL

The discussion in this paper is limited to the case of two microphones. The following signal model holds for the TD estimation problem:

$$\begin{aligned} x_1(t) &= s(t) + n_1(t) \\ x_2(t) &= s(t - \tau) + n_2(t), \end{aligned} \quad (1)$$

where $x_1(t)$ and $x_2(t)$ are the received microphone signals, $s(t)$ is the source signal, τ is the time delay, and $n_1(t)$ and $n_2(t)$ are the noise processes at the two sensors. The signals $s(t)$, $n_1(t)$, and $n_2(t)$ are assumed to be zero-mean stationary Gaussian random processes with power spectral densities $\Phi_s(\omega)$, $\Phi_{n_1}(\omega)$, and $\Phi_{n_2}(\omega)$, respectively.

It is assumed that the first node is the fusion center and that the second node transmits its signal $x_2(t)$ at a rate R to the first node, where the TD estimate is obtained from the two signals. When the signal $x_2(t)$ is quantized at a rate R , the relation between R and the resulting quantization error can be expressed by the following parametric rate-distortion relation [11]:

$$\begin{aligned} R(\lambda) &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \max \left(0, \log_2 \frac{\Phi_{x_2}(\omega)}{\lambda} \right) d\omega, \\ D(\lambda) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \min(\lambda, \Phi_{x_2}(\omega)) d\omega. \end{aligned} \quad (2)$$

The distortion here is the mean-squared error (MSE) between $x_2(t)$ and its reconstruction. Equation (2) provides the relation

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between the number of bits R used to represent the signal, and the resulting distortion D in the reconstructed signal. As the relation between R and D cannot be obtained in closed form, it is expressed in terms of a parameter λ . Inserting a particular value of λ in (2) results in certain rate R , and a corresponding distortion D . An R - D curve is obtained as λ traverses the interval $[0, \text{ess sup } \Phi_{x_2}(\omega)]$, where ess sup is the essential supremum.

Let $x_2^R(t)$ denote the signal transmitted by the second node, at rate R . It is assumed that the channel is lossless. The TD estimate is obtained using signals $x_1(t)$ and $x_2^R(t)$. The signal $x_2^R(t)$ can be described using the *forward channel* representation [11] as shown in Fig. 1. It is obtained by first bandlimiting $x_2(n)$ with a filter with frequency response

$$B(\omega) = \max\left(0, \frac{\Phi_{x_2}(\omega) - \lambda}{\Phi_{x_2}(\omega)}\right), \quad (3)$$

and then adding independent zero-mean Gaussian noise with PSD given by

$$\Phi_z(\omega) = \max\left(0, \lambda \frac{\Phi_{x_2}(\omega) - \lambda}{\Phi_{x_2}(\omega)}\right). \quad (4)$$

Note that both $B(\omega)$ and $\Phi_z(\omega)$ are real-valued.

Fig. 1. The forward channel representation.

III. CRAMÉR-RAO LOWER BOUND FOR RATE-CONSTRAINED TD ESTIMATION

The CRLB for the TD estimate can be expressed using the magnitude squared coherence (MSC) between the signals $x_1(t)$ and $x_2^R(t)$ [2]. The MSC in this case is given by

$$C(\omega) = \frac{|\Phi_{x_1 x_2^R}(\omega)|^2}{\Phi_{x_1}(\omega) \Phi_{x_2^R}(\omega)}, \quad (5)$$

where $\Phi_{x_1 x_2^R}(\omega)$ is the cross PSD of the sensor signals $x_1(t)$ and $x_2^R(t)$, and $\Phi_{x_1}(\omega)$ and $\Phi_{x_2^R}(\omega)$ are the PSDs of $x_1(t)$ and $x_2^R(t)$, respectively. The CRLB is then given by [3]

$$\text{CRLB}(\tau) = \left[\frac{T}{2\pi} \int_{-\infty}^{\infty} \omega^2 \frac{C(\omega)}{1 - C(\omega)} d\omega \right]^{-1}, \quad (6)$$

where T is the observation time in seconds.

The cross PSD $\Phi_{x_1 x_2^R}(\omega)$ is given by

$$\Phi_{x_1 x_2^R}(\omega) = B(\omega) \Phi_s(\omega), \quad (7)$$

and the PSD of $x_2^R(t)$ is given by

$$\Phi_{x_2^R}(\omega) = B^2(\omega) \Phi_{x_2}(\omega) + \Phi_z(\omega). \quad (8)$$

Note that $B(\omega)$ is a real entity. Using (7) and (8) in (5), the MSC between the signals $x_1(t)$ and $x_2^R(t)$ can be written as

$$C(\omega) = \frac{B^2(\omega) \Phi_s^2(\omega)}{\Phi_{x_1}(\omega) (B^2(\omega) \Phi_{x_2}(\omega) + \Phi_z(\omega))}, \quad (9)$$

The CRLB under the rate-constraint follows by inserting (9) into (6):

$$\text{CRLB}(\tau) = \left[\frac{T}{2\pi} \int_{-\infty}^{\infty} \omega^2 \frac{\Phi_s(\omega)}{\Phi_{n_1}(\omega)} \frac{\Phi_s(\omega)}{\Phi_{n_2}(\omega)} \times \frac{1}{\left(1 + \frac{\Phi_s(\omega)}{\Phi_{n_1}(\omega)} + \frac{\Phi_s(\omega)}{\Phi_{n_2}(\omega)}\right) + \frac{\Phi_{x_1}(\omega)}{\Phi_{n_1}(\omega) \Phi_{n_2}(\omega)} \frac{\Phi_z(\omega)}{B^2(\omega)}} d\omega \right]^{-1} \quad (10)$$

The bound in (10) differs from the well known expression for the CRLB without rate-constraints in the term

$$\frac{\Phi_{x_1}(\omega)}{\Phi_{n_1}(\omega) \Phi_{n_2}(\omega)} \frac{\Phi_z(\omega)}{B^2(\omega)}$$

in the denominator of the integrand. Inserting $\lambda = 0$ (i.e., zero distortion and thus infinite bit rate) into (3) and (4) yields $B(\omega) = 1$ and $\Phi_z(\omega) = 0$ so that the additional term in (10) vanishes. At finite bit-rates, the additional term results in an increase in the lower bound on the variance of the estimation error.

Equation (10) can be simplified under the following assumptions. The signal and noise PSDs are assumed to be flat in a band $\Delta\omega$ rad/s around a center frequency ω_0 rad/s, and zero otherwise, and the SNR is assumed to be the same at both sensors:

$$\Phi_s(\omega) = \begin{cases} \Phi_s & |\omega_0 - \omega| \leq \Delta\omega/2, \\ 0 & |\omega_0 - \omega| > \Delta\omega/2 \end{cases} \quad (11)$$

and for $i = 1, 2$,

$$\Phi_{n_i}(\omega) = \begin{cases} \Phi_{n_i} & |\omega_0 - \omega| \leq \Delta\omega/2, \\ 0 & |\omega_0 - \omega| > \Delta\omega/2. \end{cases} \quad (12)$$

Consequently,

$$\Phi_{x_1}(\omega) = \Phi_{x_2}(\omega) = \Phi_x = \Phi_s + \Phi_{n_i}. \quad (13)$$

Assuming $\lambda < \Phi_x$ in (2) results in

$$\lambda = \Phi_x 2^{-2\pi R/\Delta\omega}. \quad (14)$$

The CRLB then reduces to

$$\text{CRLB}(\tau) = \frac{1}{\xi} \frac{1}{2\omega_0^2 \left(\frac{\Delta\omega T}{2\pi}\right) \left[1 + \frac{1}{12} \left(\frac{\Delta\omega}{\omega_0}\right)^2\right]}, \quad (15)$$

where

$$\xi = \frac{\rho^2}{(1 + \rho)^2 \frac{1}{1 - 2^{-2\pi R/\Delta\omega}} - \rho^2}, \quad (16)$$

and

$$\rho = \frac{\Phi_s}{\Phi_n} \quad (17)$$

is the SNR. The relation between $\text{CRLB}(\tau)$ and the rate R can now be analyzed using (15)-(17).

Fig. 2. $\sqrt{\text{CRLB}(\tau)}$ as a function of the bit-rate, for a source signal with center frequency 100 Hz and bandwidth 50 Hz as the observation time T is varied. The SNR is 20 dB. Note that the scale of the x-axis is logarithmic.

Fig. 2 shows the relation between the CRLB and the bit-rate for a source signal with center frequency 100 Hz and bandwidth 50 Hz at an SNR of 20 dB, for different values of the observation time T . For a given value of T , the CRLB drops sharply as the bit-rate increases, up to around 200 bits per second. At high bit-rates, because of the relatively high SNR, increasing the observation time has little effect on the CRLB. At low bit-rates however, larger values of T result in a lower CRLB values at a given bit-rate. Depending on the constraints of the application, e.g., the center frequency, bandwidth, and the observation duration, and for a certain desired accuracy, an appropriate bit-rate may be chosen using (15) and (16).

Fig. 3 plots the CRLB as a function of the bit-rate for a source signal with center frequency 100 Hz, bandwidths 50 Hz (solid) and 10 Hz (dashed), and observation time $T = 2$ s, for an SNR of 20 dB. At low bit-rates, a bandwidth of 10 Hz results in a lower CRLB than a bandwidth of 50 Hz, as seen from the figure. In the absence of rate constraints, i.e., at infinite bit-rates, it is well known (also evident in Fig. 3 at high bit-rates) that using a larger bandwidth results in a lower CRLB, but it is not so at low rates. This is because there is a trade-off between requiring more bits to represent the greater amount of information in a large bandwidth signal on one hand, and on the other hand the greater precision in TD estimation using a large bandwidth signal. Beyond a certain threshold bit-rate, which is approx. 185 bits per second in Fig. 3, a bandwidth of 50 Hz results in a lower CRLB.

Using (15) and (16), the CRLB at a rate R can be expressed as the sum of the CRLB at infinite bit-rate and a term that

Fig. 3. $\sqrt{\text{CRLB}(\tau)}$ as a function of the bit-rate, for a source signal with center frequency 100 Hz, and bandwidth 50 Hz (solid) and 10 Hz (dashed), and observation time $T = 2$ s. The SNR is 20 dB. Note that the scale of the x-axis is logarithmic.

corresponds to the additional error due to the rate constraint:

$$\text{CRLB}(\tau) = \text{CRLB}(\tau)|_{R=\infty} + \epsilon^2(R), \quad (18)$$

where

$$\epsilon^2(R) = \left(\frac{1+\rho}{\rho} \right)^2 \left[\frac{2^{-2\pi R/\Delta\omega}}{1 - 2^{-2\pi R/\Delta\omega}} \right] \times \frac{1}{2\omega_0^2 \left(\frac{\Delta\omega T}{2\pi} \right) \left[1 + \frac{1}{12} \left(\frac{\Delta\omega}{\omega_0} \right)^2 \right]}. \quad (19)$$

Fig. 4 plots the additional term $\epsilon^2(R)$ as a function of R (solid line), for a center frequency of 100 Hz, bandwidth 50 Hz, $T = 2$ s, and an SNR of 10 dB. The CRLB is also plotted for comparison (dashed line). At low bit-rates, the quantization error $\epsilon^2(R)$ is the dominating term, while at high bit-rates the SNR is the limiting factor.

IV. ATTAINABILITY OF THE CRLB

Applying the modified Ziv-Zakai bound derived in [4] to the rate-constrained case results in the following condition for the CRLB to be attained:

$$\xi \geq \xi_{\text{th}} = \frac{6}{\pi^2 \frac{\Delta\omega T}{2\pi}} \left(\frac{\omega_0}{\Delta\omega} \right)^2 \left[\phi^{-1} \left(\frac{1}{24} \left(\frac{\Delta\omega}{\omega_0} \right)^2 \right) \right]^2, \quad (20)$$

where

$$\phi(y) = \frac{1}{2\pi} \int_y^\infty e^{-\frac{t^2}{2}} dt. \quad (21)$$

Note that for (20) to be satisfied for a finite bit-rate R , it has to first hold at infinite bit-rate, i.e., a necessary condition for $\xi \geq \xi_{\text{th}}$ is $\xi|_{\Phi_s=0} \geq \xi_{\text{th}}$. This follows directly from the definition of ξ in (16). It is also intuitive that the CRLB cannot be attained at a finite bit-rate R if it is not attained at infinite rate.

Fig. 4. $\sqrt{\text{CRLB}(\tau)}$ (dashed) and $\sqrt{\epsilon^2(R)}$ (solid) as a function of R , for a source signal with center frequency 100 Hz and bandwidth 50 Hz, $T = 2$ s, and SNR of 10 dB. Note that the scale of the x-axis is logarithmic.

Using (16) and (20), the minimum bit-rate that ensures attainability of the CRLB for a given T , ρ , ω_0 , and $\Delta\omega$ can be obtained as

$$R_{\min} \geq -\frac{\Delta\omega}{2\pi} \log_2 \left(1 - \frac{\xi_{\text{th}}}{1 + \xi_{\text{th}}} \left(\frac{1 + \rho}{\rho} \right)^2 \right) \quad (22)$$

Fig. 5 plots $R_{\min}$ as a function of the bandwidth, for a source signal with center frequency 100 Hz, SNR 20 dB, and for different observation times. For a bandwidth of 10 Hz and $T = 2$ s, the CRLB is tight starting from a bit-rate of around 70 bits per second. As the time-bandwidth product $\Delta\omega T$ increases, the CRLB is attained at lower bit rates.

Fig. 5. Minimum bit-rate at which the CRLB is attained, for different bandwidths. The source has a center frequency of 100 Hz and the SNR is 20 dB.

V. CONCLUSIONS

In a rate-constrained wireless sensor network, TD estimation needs to be performed using one or more quantized sensor

signals. In this paper, results from rate-distortion theory and traditional time delay estimation have been applied to derive a relation between the quantization bit-rate and the CRLB on the variance of the TD estimation error. This relation can be used to determine the minimum bit-rate necessary to achieve a certain desired estimation accuracy, depending on the constraints of the application, e.g., the center frequency, bandwidth, and the observation duration.

The analysis reveals an interesting finding. In the absence of rate constraints, it is well known that for a given SNR, center frequency and observation time, signals with large bandwidths result in a lower CRLB than those with small bandwidths. In the rate-constrained case however, at low bit-rates, a smaller bandwidth results in a lower CRLB than a larger bandwidth signal. This behavior is observed until a certain threshold bit-rate is reached, after which using a larger bandwidth provides better results. Finally, the modified Ziv-Zakai bound has been applied to derive an expression for the minimum rate at which the CRLB is tight.

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