Whittaker and Weyl representations for time-domain modes

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Abstract — In the spectral theory of transients one may express the temporal wavefield inside a waveguide in terms of an angular integral representation of global time-domain spectral-mode constituents. The remaining spectral integral may be evaluated asymptotically. Depending on the choice of the large parameter, the evaluation of the integrals along steepest descent paths yields Whittaker- or Weyl expansions for time-domain modes.

1 INTRODUCTION

Within the framework of the spectral theory of transients one may express the temporal wavefield inside a waveguide in terms of an integral representation over the launch-angle (or receiver angle) of local ray-field constituents, or global time-domain spectral-mode constituents. For electromagnetic problems involving short pulses and large distances the remaining spectral integral may efficiently be evaluated asymptotically [1].

Asymptotic expansions of integrals serve multiple purposes. Computation times are usually negligible, and the resulting expressions may provide a cogent description of the underlying physics. The asymptotic analysis may also unearth useful information for the development of quadrature rules with faster error decay [2]. In many cases, the original path of integration may be replaced by an integral along a steepest descent path (SDP), but often this is not the immediate objective. Knowledge about SDPs may be used to generate alternative integral representations, (not necessarily along an SDP) that are amenable to rapid evaluation [3], or it may indicate which leaky-wave modes should be taken into account in the evaluation of the radiation field for specific source and observer positions in dielectric waveguide configurations.

In the asymptotic analysis of a spectral representation of a time-domain mode (TDM), there are two natural choices for the phase, one based only on the kinematics of the problem, and another that includes the dynamics. Although quantitative information about the SDPs is not often required, qualitative information is indispensable. Below, we demonstrate that asymptotic evaluation of the integrals along the SDPs leads to an asymptotic expansion for a Whittaker-type modal field expansion (consisting of a causal and an anti-causal part [4]) in one case, and a causal Weyl-type expansion in the other. There are merits to both approaches.

2 TIME-DOMAIN PLANE-WAVE SYNTHESIS OF MODES

Let us consider a parallel-plate waveguide configuration for an isotropic line source, parallel to the y-axis. The scalar Green’s function, \( G(\mathbf{x}, t) \), satisfies the scalar wave equation

\[
\left( \partial_x^2 + \partial_z^2 - c^2 \partial_t^2 \right) G = -\delta(\mathbf{x} - \mathbf{x}') \delta(t - t'),
\]

in which \( c \) denotes the wave speed, and \( \{x', t'\} \) denote the space-time source coordinates. The \( x- \) and \( z \)-axes point in the directions transverse and longitudinal to the waveguide, respectively. The parallel plates are perfect conductors, located at \( x = 0 \) and \( x = a \), implying that \( G|_{x=0} = G|_{x=a} = 0 \). Without loss of generality, we may assume that \( x - x' > 0 \), \( z > 0 \), \( z' = 0 \), and \( t' = 0 \).

We synthesise the wavefield using a spectral representation for the analytic-signal extension, \( \hat{\mathcal{G}} \), of the space-time Green’s function. Once \( \hat{\mathcal{G}} \) has been determined for an isotropic line source, complex-source pulsed-beam solutions follow upon applying complexification of the source coordinates [5]. Below, we are eventually interested in field solutions due to an isotropic source with a source signature \( \partial_t^n \delta(t - iT) \) i.e., the \( m \)-th derivative of the Rayleigh pulse. Here, \( T > 0 \) is a measure of the pulsedwidth that provides a means to analyse the response to finite pulses of arbitrary width. However, we shall commence by setting \( T = 0 \). The wavefield can be constructed in several alternative ways. We focus on a representation in terms of modal (i.e., global) constituents, which is most appropriate for distances beyond the Fresnel distance \( F = a^2/cT \) of the waveguide.

We employ a transverse spectral field synthesis, in which the spectral variable is the angle of propagation \( \omega \) at the point of observation. The analytic-signal extension, \( \hat{\mathcal{G}} \), of the space-time Green’s function may be expanded in terms of TDMs, \( \hat{G}_T^M \), with...
mode index \( \ell \), according to [1]
\[
\begin{align*}
\hat{G} &= \sum_{\ell=1}^{\infty} \hat{G}_\ell^M = \sum_{\ell=1}^{4} \sum_{j=1}^{4} \hat{g}_{\ell,j}^M,
\end{align*}
\]
where the TDMs have been decomposed into distinct wave species, indexed by \( j \in \{1, 2, 3, 4\} \), and given by
\[
\begin{align*}
\hat{g}_{\ell,m,j}^M &= \int_{\mathcal{W}_c} dw \ A_{\ell,j} \frac{[-i\omega_\ell(w)]^m}{\xi(w)} e^{-i\phi_\ell(w)},
\end{align*}
\]
in which we have introduced a respective amplitude and phase
\[
\begin{align*}
A_{\ell,j} &= \frac{i\Sigma_j}{4\pi a} e^{\pi i \sigma_j (x-x_n)/a}, \quad \phi_\ell = \omega_\ell(w)[t - \zeta(w) z/c],
\end{align*}
\]
and the four species are distinguished through
\[
\begin{align*}
\sigma_j &= \{-1, -1, +1, +1\} \quad \text{for} \quad j = 1, 2, 3, 4,
\end{align*}
\]
\[
\begin{align*}
\zeta_j &= \{+1, -1, +1, -1\}
\end{align*}
\]
In Eqs. (3)–(5), the modal frequency, and the transverse and longitudinal slownesses are given by
\[
\begin{align*}
\omega_\ell = \frac{\pi \ell c}{\xi a}, \quad \xi = \sin w, \quad \text{and} \quad \zeta = \cos w,
\end{align*}
\]
respectively.

3 ALTERNATIVE REPRESENTATIONS FOR TIME-DOMAIN MODAL FIELDS

Due to the symmetry of the spectral mode constituents (plane-wave congruences), it suffices to consider the closed strip \( \mathcal{D}_w = \{ w | 0 \leq \text{Re}(w) \leq \pi \} \) in the complex \( w \)-plane. In view of the presence of the transverse slowness \( \xi = \sin w \) in the denominator of the phase \( \phi_\ell \), the angles \( w = 0, \pi \) are essential singularities of the integrand. The associated directions of propagation are the positive and negative \( z \)-directions, respectively. However, the corresponding transverse wavenumbers, \( k_\ell = \omega_\ell \xi/c = \pi \ell/a \), \( l = 1, 2, \ldots \) are fixed, implying that the associated modal frequencies must tend to infinity.

The integral in Eq. (3) is carried out along the Weyl-type contour of integration, depicted in Figure 1. The Weyl expansion is a causal integral representation involving spectral constituents that either propagate in the forward direction, or are evanescent. We shall investigate two strategies for expanding Eq. (3) asymptotically, and shall comment on their benefits and drawbacks.

3.1 Kinematics-based asymptotic expansion

To analyse the TDM species, we investigate the following generic quantity (cf. Eq. (3))
\[
\int_{\mathcal{W}_c} \frac{d^\Omega}{\Omega_{\ell,m}} = \int_{\mathcal{W}_c} dw \ [\xi(w)]^{-m-1} e^{-i\Omega_{\ell,m}(w)}, \quad (8)
\]
where \( \phi_\ell \) and \( \xi \) have respectively been defined in Eqs. (5) and (7), while \( \Omega \) is a large parameter in terms of which Eq. (8) may be expanded asymptotically. In \( \mathcal{D}_w \) there is a simple, single stationary point, \( w = w_\ell(t) \), defined through \( d_w \phi_\ell |_{w=w_\ell} = 0 \). In view of Eqs. (5) and (7), we have
\[
\begin{align*}
\zeta(t) &= \cos w_\ell = \frac{z}{\xi t},
\end{align*}
\]
\[
\begin{align*}
\phi_{\ell,t} &= \frac{\pi \ell c t \xi t}{a}, \quad \phi_{\ell,t}^{(1)} = 0, \quad \phi_{\ell,t}^{(2)} = \omega_{\ell,t} = \frac{\phi_{\ell,t}}{\xi t},
\end{align*}
\]
where \( \zeta \) denotes the instantaneous longitudinal slowness and \( \phi_{\ell,t}^{(n)} = \frac{d^n \phi_{\ell,t}}{dw^m} |_{w=w_\ell} \). The respective instantaneous transverse slowness, \( \xi_\ell \), and frequency

Figure 1: The contour of integration in the complex \( w \)-plane (Weyl contour).

Figure 2: The SDP associated with Eq. (8), the Whittaker contour (dashed line) and the location of the stationary point in the complex \( w \)-plane for \( a = 22.86 \text{ mm}, \ z = 27 \text{ mm}, \ t = 110 \text{ ps}, \ T = 0 \text{ s}, \ell = 0 \) and \( m = 0 \).
the SDP can be supplemented with an integral Whittaker expansion for a TDM. The integral along the SDP yields an asymptotic expansion of the SDP is a deformed version of a Whittaker contour equivalent to our original Weyl expansion. Instead, the SDP through the stationary point is not equivalent representation that follows upon integrating along the complex $w$-plane. The same parameters were used as in Figure 2.

$$\omega_{\ell,t}$$ are given by

$$\xi_\ell(t) = \sqrt{1 - \xi_\ell^2}, \quad \text{Re}(\xi_\ell) \geq 0, \quad \text{and} \quad \omega_{\ell,t}(t) = \frac{\pi \ell c}{\xi_\ell a}, \quad (11)$$

respectively. For the problem at hand, the SDP through the stationary point can be determined explicitly. It may be parameterised in terms of a real parameter $s$ according to

$$\xi = \sin w = \frac{\chi + \xi \gamma}{\chi^2 + \xi^2}, \quad \zeta = \cos w = \frac{\xi - \chi \gamma}{\chi^2 + \xi^2}, \quad (12)$$

in which

$$\chi = \xi_t - i a \frac{\gamma^2}{\pi \ell c}, \quad \gamma = -i a \frac{\phi_{\ell,t}}{\pi \ell c} \sqrt{s^2 + 2i s \xi_t}, \quad (13)$$

implying that

$$\xi|_{s=0} = \xi_t, \quad \zeta|_{s=0} = \zeta_t, \quad \lim_{s \to \pm \infty} \zeta = \pm 1. \quad (14)$$

From Eq. (14) we infer that $s = 0$ corresponds to the stationary point and that the endpoints of the SDP are the essential singularities at $w = 0$ and $w = \pi$. Since the stationary point we have found is the only one in $0 \leq \text{Re}(w) \leq \pi$, the modal field representation that follows upon integrating along the SDP through the stationary point is not equivalent to our original Weyl expansion. Instead, the SDP is a deformed version of a Whittaker contour (see Figure 2), so application of Laplace’s method to the SDP yields an asymptotic expansion of the Whittaker expansion for a TDM. The integral along the SDP can be supplemented with an integral along the half line from $w = \pi$ to $w = \pi - i \infty$.

Figure 3: The two SDPs associated with Eq. (15), and the location of the relevant stationary points in the complex $w$-plane. The same parameters were used as in Figure 2.

which, in the indicated direction, is a path of steepest ascent (SAP), and an integral from $\pi - i \infty$ to $\pi/2 - i \infty$. The latter integral vanishes, while the integral along the SAP is amenable to asymptotic evaluation about $w = \pi - i \infty$. Along the SAP, $\zeta, i \xi$ and $-i \omega_t$ are negative, indicating exponential decay both in $t$ and in $z$.

The choice of associating the large parameter $\Omega$ with the phase in Eq. (8), is kinematic in its origin. Below, we investigate what happens if we account for the dynamics in the asymptotic analysis.

### 3.2 Alternative asymptotic expansion

The factor $[\xi(w)]^{-m-1}$ may be included in the phase before the introduction of the large parameter $\Omega$, which leads to an alternative generic modal quantity, viz.,

$$\int_{\ell, m, \text{alt}}^\Omega dw e^{-i \phi_{\ell,m}(w)}, \quad (15)$$

in which

$$\phi_{\ell,m}(w) = \phi(w) - i(m+1) \log(\xi). \quad (16)$$

For $\Omega = 1$, Eqs. (8) and (15) are equivalent. The first derivative of $\phi_{\ell,m}$ is found to be

$$d_w \phi_{\ell,m} = \frac{\pi \ell}{a} \frac{z - ct \xi}{\xi^2} - i(m+1) \frac{\zeta}{\xi}. \quad (17)$$

To analyse the stationary points in $D_w$, it is instructive to consider the complex $\zeta$-plane first. Recall that $z > 0$. Via $\xi = \sin w = \sqrt{1 - \xi^2}$ on $D_w$, the strip $D_w$ corresponds to $\text{Re}(\xi) \geq 0$, which is the closure of the upper Riemann sheet of the Riemann surface, associated with $\xi = \xi(\zeta)$. The lines $\text{Re}(w) = 0$ and $\text{Re}(w) = \pi$ map to the branch cut $\zeta \leq 0$. The condition for the stationary points in the complex $\zeta$-plane reads

$$\pi \ell (z - ct \xi) = i(m+1) a \xi. \quad (18)$$

Evaluation of the square of Eq. (18) yields an equation of degree four in $\xi$, with four roots. Below, we examine whether those four roots satisfy Eq. (18).

It is easy to show that $d_w \psi_{\ell,m} \in \mathbb{R}$ for $\text{Re}(\ell) = 0$, and for $\text{Re}(w - \pi) = 0$. Since $\lim_{w \to \pm i \infty} d_w \psi_{\ell,m} = \mp (m+1)$, while $\lim_{w \to \pm 0} d_w \psi_{\ell,m} \to \text{sgn}(ct - z) \infty$, we infer that the number of stationary points on the imaginary $w$-axis must be odd. We arrive at the same conclusion for the line $\text{Re}(w - \pi) = 0$. From Eqs. (7) and (18) it is obvious that there are no roots on $\{w | w \in [0, \pi]\}$. Now, suppose that there is a complex root $w = \omega_t$ in $\{w | \text{Re}(w) \in (0, \pi), \text{Im}(w) \neq 0\}$ that satisfies Eq. (18), with
and one root lies in \( \{ \Re(w) \geq 0, \Im(w) \geq 0 \} \). Figure 3. The integrals along the two SDPs, supplemented with an integral from \( \pi \) to \( \infty \), involving two stationary points is given in Figure 3. The integrals along the two SDPs, supplemented with an integral from \( \pi \) to \( \infty \) that vanishes, are equivalent to the causal Weyl-type expansion. In Figure 4 we have depicted the situation for a finite pulse width \( T = 80 \text{ ps} \), and have also included the SDP associated with the asymptotic expansion of Eq. (8).

**4 DISCUSSION AND CONCLUSIONS**

The integral representations for a TDM, involving integration along the SDPs associated with Eqs. (8) and (15) are Whittaker- and Weyl-type representations. The former consists of causal and anticausal parts, while the latter is causal. However, the causal part of a Whittaker representation may be isolated by starting to record the field at a time \( t_1 \) after the source has effectively ceased to act. For short-pulse fields at considerable longitudinal distances from the source, the condition on \( t_1 \) may be relaxed further. Alternatively, the Whittaker expansion may be supplemented with an integral along an SAP to restore causality.

In the asymptotic evaluation of Eq. (8), the choice of the phase is based on kinematics only, and allows for a physical interpretation as a field constituent generated by moving launch point [1]. Further, all higher-order modes share the same stationary point and SDP, which may both be evaluated in closed form. The significance of this is that the resulting modal series for the total time-domain field is amenable to a highly efficient rational approximation [1]. Although the accuracy of retaining only the leading term in the asymptotic expansion of Eq. (8) is not the same as that in retaining only the leading terms in the asymptotic expansion of Eq. (15), in both cases the error may be estimated.

In the asymptotic evaluation of Eq. (15), the dynamics of the problem is included in the choice of the phase. The stationary points are the roots of a polynomial of degree four, which may still be determined analytically. However, the lucidity of the underlying physics is less transparent.

One may argue that for more interesting configurations, such as the slab waveguide, closed-form expressions are not available, so one would have to resort to numerical evaluation of the corresponding quantities anyway. However, even if one prefers to evaluate the spectral integrals numerically, the asymptotic analysis may prove essential in the development of efficient quadrature rules [2].

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**References**


