Towards automated loop-shaping in controller parameter space

Citation for published version (APA):

Aladagli, I., Hamer, den, A. J., Steinbuch, M., & Angelis, G. Z. (2010). Towards automated loop-shaping in controller parameter space. In *Proceedings of the 2010 American Control Conference (ACC 2010)*, 30 June 30 - 2 July 2010, Baltimore, Maryland, USA (pp. 1880-1885). Institute of Electrical and Electronics Engineers.

Document status and date:

Published: 01/01/2010

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy

If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

Towards Automated Loop-Shaping in Controller Parameter Space

Irmak Aladagli*, Arjen den Hamer*, Maarten Steinbuch*, Georgo Angelis**

Abstract—This paper presents a data-based control design method for optimal tuning of the parameters of a fixed structure controller with respect to the given closed-loop (CL) performance specifications. The proposed approach is to translate the CL specifications into controller parameter space and to automate the parameter choice through optimization problems. The developed method ensures CL stability and close to optimal controller design for any user defined controller structure, given the frequency response data set of the uncertain plant.

I. INTRODUCTION

Most practical control design problems require a robust, low-order controller that optimally satisfies a set of closed-loop (CL) performance specifications. In control design for applications whose behavior is well represented with linear models, like high precision motion systems, data-based methods are preferred over methods that require low-order parametric plant models; since in general, it is more accurate to represent a complex linear plant by its frequency response (FR) data rather than an approximated parametric model. Moreover, automated design methods are appreciated as they decrease the amount of engineering costs required for control design and also are more suited for MIMO applications.

A design method and the resulting controller that satisfy the requirements described above, may take either norm-based methods or manual loop-shaping (LS) (for SISO systems) as its base. Each of these two main design branches have inherent advantages and disadvantages. Norm-based design methods are strong in handling uncertainty and CL performance specifications, resulting in an optimal controller. However, they require a parametric model of the plant and they usually deliver high-order controllers. Manual LS on the other hand, does not require a parametric model and the order of the controller is the choice of the designer. However, this method does not systematically account for CL performance specifications or uncertainties. Moreover, its success depends on the skills of the engineer and solutions are per problem.

The shortcomings of the available tools for synthesis of feedback controllers motivate the development of a data-based control design method that can combine the optimality of norm-based methods with the flexibility of LS. This paper presents such a method of controller parameter tuning with the possibility of automated design, inspired by Quantitative Feedback Theory (QFT), for a fixed structure controller.

Traditional QFT, first introduced by Isaac Horowitz around 1960, is a transparent graphical design tool in frequency domain that offers promising solutions to above problems (see [1]). QFT allows design considering the CL performance specifications, can handle parametric or non-parametric uncertainties and the order of the controller is the choice of

the designer. The ability of working with FR data is another motivation to use QFT for controller synthesis, as this makes tedious parametric model building, futile. However, QFT still has the disadvantage that controller design relies on manual LS. Thus a natural next step in QFT research has been trying to automate this last controller design step.

Automating controller design via QFT has been vigorously researched since it was first suggested by [2], trying to bridge the gap between norm-based methods and manual LS. A very nice summary of research on automated synthesis via QFT can be found in [3]. There are several approaches in the literature that describe design automation via QFT. However, most of these methods are inadequate. Some require fixing the poles of the CL system, [4], or, an initial guess on the location of the poles, [5]. Others cause over-design due to approximations of plant templates, [6] and linearization of bounds, [7]. There are also some numerical methods using genetic algorithms ([8]), interval constraint satisfaction techniques as in [3]. The disadvantage of numerical methods is that some of them are not able to guarantee that the solution is optimal or in close proximity to an optimal solution and some of them require great computational power, especially as the number of parameters of the controller increases.

In this research we attempt to quantify the design procedure of a fixed structure controller using FR data of the plant only. The approach is to translate CL specifications into controller parameter space and to automate the parameter choice through optimization problems. Given the FR data set of an uncertain plant, the developed method ensures CL stability and a near-optimal controller with a predefined structure. By translating CL performance specifications into the controller parameter space, we first determine the allowable parameter region and then apply stability and predefined optimization criteria to choose a set of parameters that results in an optimal or close to optimal stable controller. This analytical method can handle both affine and non-affine parameters of the controller parameters, design for uncertain plant sets and design in the absence of a parametric plant model. One can also work with the simpler case of a parametric model.

The outline of this paper is as follows. Section II describes the problem at hand, Section III introduces the approach to this problem. Section IV includes some practical design examples and discusses stability. Section V extends the problem to plants with uncertainty. Finally, Section VI gives conclusions and recommendations for future work.

II. PROBLEM DEFINITION

Given FR data of an uncertain SISO plant $P(s)$, denoted by $\mathcal{P}_i = \mathcal{P}(j\omega_i) + \Delta_i$, with $\omega_i \in \Omega$, we would like to find a stabilizing controller $G(s, \theta)$ that satisfies the Mixed Sensitivity (MS) problem:

*Authors are with the Control Systems Technology Group, Mechanical Engineering Department, Eindhoven University of Technology, P.O. box 513 5600 MB Eindhoven, The Netherlands i.aladagli@tue.nl

**Author is with Philips Applied Technologies, High Tech Campus 7 5656 AE Eindhoven, The Netherlands

$$\left\| \begin{matrix} W_1 S \\ W_2 T \end{matrix} \right\|_{\infty} \leq \gamma, \quad W_1, W_2 \in \mathcal{RH}_{\infty} \quad (1)$$

Where $L := PG$, $S := (1 + L)^{-1}$, and $T := L(1 + L)^{-1}$. Since only FR data for the plant is available on the grid $\omega_i \in \Omega$, clouds $\mathcal{P}_i$ are only known at discrete frequency points ω_i . Therefore, the continuous problem is redefined for the discrete frequency domain. Under the assumptions of smoothness of the functions $W_1 S$ and $W_2 T$, and a dense enough discrete frequency grid Ω (see [9]), the continuous problem in (1) can be approximated by the discrete one stated as, find θ of $G(s, \theta)$, such that:

$$\Theta_{al} = \{ \Theta \in \mathbb{R}^n \mid \max_{\omega_i} \left\| \begin{matrix} W_1(\omega) S(j\omega_i) \\ W_2(\omega) T(j\omega_i) \end{matrix} \right\|_2 \leq \gamma^2, \forall \omega_i \in \Omega \} \quad (2)$$

Eq.(2) does not impose a stability condition on the CL system as opposed to (1). Therefore, some stability criterion should be imposed on the problem. Moreover, (2) only defines an allowable space of all possible controllers that satisfy the given constraints. However, it does not point to any single optimal controller. Thus an optimization criterion is introduced to attain a unique, optimal, stabilizing controller. In the rest of this paper, we will divide the controller synthesis problem into two subproblems:

- 1) *Calculate allowable parameter space, Θ_{al} :* To solve (2) analytically, the controller, $G(j\omega, \theta)$, needs to be stated in terms of its parameters, θ_i . Once (2) is solved for all $\omega_i \in \Omega$, we have the space of all parameters that satisfy the given CL performance specifications. This space is called the allowable parameter space, Θ_{al} .
- 2) *Choose a set of unique parameters θ_i from Θ_{al} :* A unique value for all parameters θ_i of the controller should be chosen from Θ_{al} , such that the resulting controller $G(j\omega, \theta)$ is stabilizing and optimal with respect to some predefined criteria.

Different criteria can be employed for the optimization step, like minimizing the high frequency gain. Since the formal definition of this problem depends on the problem, it will be further explained in the examples. The region of stabilizing allowable parameters is called Θ_{stab} , and is calculated by analyzing -1 crossings of the open loop (OL).

III. APPROACH TO THE PROBLEM

In classical QFT, the first step is to plot the plant templates and bounds on the OL function on to a Nichols chart per frequency. Once these bounds are plotted, the most intuitive approach would be to draw a loop along the constraints and then extract the transfer function (TF) information of this loop. This approach would allow the designer to easily choose the important characteristics of the final system as bandwidth, gain and phase margins, high frequency roll-off, etc. However, not all loops drawn on the Nichols plane are realizable with finite order TF's, since the gain and phase of a TF depend on each other through the Bode gain-phase relationship. Thus the Nichols plane has two dependent variables on its axes and is not suited to freely shape the system. This problem is overcome by assuming a TF with

unknown parameters and translating the problem and the related bounds into the space of these unknown parameters.

In this paper, we translate the CL performance specification bounds in the OL domain into the controller parameter space by solving (2) for all $\omega_i \in \Omega$. In the next sequel, it will be shown that (2) solved for controller parameters per frequency, MS specifications result in quadratic inequalities (QI's). These QI's define bounds in the form of conic sections in controller parameter space and allow the designer or an optimization algorithm to choose from the allowable parameter space according to the needs of the problem.

Lemma 1. *Given a predefined controller structure with unknown parameters defined as follows:*

$$G(s, \theta) := \frac{\sum_{k=0}^m s^k \theta_k}{\sum_{l=0}^n s^l \theta_l} = \frac{N_G(s, \theta)}{D_G(s, \theta)} \quad (3)$$

Then, (2) results in quadratic inequalities per frequency in terms of the controller parameters θ_i .

Proof. Let the plant be represented in its real and imaginary parts as $P(j\omega_i) = a(\omega_i) + jb(\omega_i)$, where a and b are known for every $\omega_i \in \Omega$. Then the inequalities given in the problem definition (2), can be worked into the controller space as follows. Eq.(2) can be rewritten as:

$$W_1^* S^* S W_1 + W_2^* T^* T W_2 \leq \gamma^2, \quad \forall \omega_i \in \Omega \quad (4)$$

Where conjugate transpose is indicated by $(\cdot)^*$ and all items are complex numbers corresponding to the function they are representing at frequency ω_i . For compactness of notation, ω_i is omitted from this point on however, all expressions hereon are per frequency point, unless stated otherwise.

Let us assume, without loss of generalization that weight filters W_1 and W_2 both equal 1 for all frequencies. Then,

$$\left(\frac{1}{1+L} \right)^* \left(\frac{1}{1+L} \right) + \left(\frac{L}{1+L} \right)^* \left(\frac{L}{1+L} \right) \leq \gamma^2 \quad (5)$$

Let us define the following quantities through the definition of the controller structure given in (3):

$$\begin{aligned} N_{RG} &:= \text{Re } N_G = f_1(\theta_k), & N_{IG} &:= \text{Im } N_G = f_2(\theta_k) \\ D_{RG} &:= \text{Re } D_G = f_3(\theta_l), & D_{IG} &:= \text{Im } D_G = f_4(\theta_l) \end{aligned}$$

Where all f_i are real linear functions in terms of their variables. Then the controller in terms of the real and imaginary terms of its numerator and denominator can be written as:

$$\begin{aligned} G &= \frac{(N_{RG} + jN_{IG})(D_{RG} - jD_{IG})}{D_{RG}^2 + D_{IG}^2} \\ &= \frac{(N_{RG}D_{RG} + N_{IG}D_{IG}) + j(N_{IG}D_{RG} - N_{RG}D_{IG})}{D_{RG}^2 + D_{IG}^2} \end{aligned} \quad (6)$$

Inserting (6) into (5) and rearranging gives:

$$\begin{aligned} &(\gamma^2 - 1)(a^2 + b^2)(N_{RG}^2 + N_{IG}^2) + \\ &2\gamma^2 a(N_{RG}D_{RG} + N_{IG}D_{IG}) - 2\gamma^2 b(N_{IG}D_{RG} - N_{RG}D_{IG}) \\ &\geq (1 - \gamma^2)(D_{RG}^2 + D_{IG}^2) \end{aligned} \quad (7)$$

We see that (7) has first and second powers only of the terms N_{RG} , D_{RG} , N_{IG} , D_{IG} . We have already shown

that these terms are linear functions of controller parameters θ_i . Therefore we conclude that (7) is quadratic in controller parameters θ_i . $\square$

Although it is possible to continue using (7) resulting from the problem definition in (2), in most practical cases specifications on sensitivity and complementary sensitivity functions are given separately as follows.

$$|S(j\omega)| \leq m(\omega), \quad \forall \omega \in \Omega_S \quad (8)$$

$$|T(j\omega)| \leq t(\omega), \quad \forall \omega \in \Omega_T \quad (9)$$

These two equations are a more constrained version of the general MS problem and can be converted into the general MS problem by loosing information on the two independent relations as follows. Let $\gamma^2 = m^2 + t^2$, then,

$$|S|^2 + |T|^2 \leq m^2 + t^2 = \gamma^2$$

In the rest of the paper, we will use (8) and (9) as it is more practical to give CL performance specifications in this manner.

Let real and imaginary parts of the controller be defined as $G(j\omega_i) = c(\omega_i) + jd(\omega_i)$. Given the sensitivity specification

$$|S| := \left| \frac{1}{1+L} \right| \leq m \quad (10)$$

$$0 \leq (1+L)^*(1+L) - m^{-2}$$

We will now start the computation of bounds on the controller parameters as a function of the given CL specification. The above specification in terms of real and imaginary parts of L is as follows:

$$(\text{Re } L)^2 + (\text{Im } L)^2 + 2\text{Re } L + 1 - m^{-2} \geq 0 \quad (11)$$

In terms of a,b,c,d:

$$(a^2 + b^2)(c^2 + d^2) + 2ac - 2bd + 1 - m^{-2} \geq 0 \quad (12)$$

Doing the same for a specification on complementary sensitivity TF,

$$|T| := \left| \frac{L}{1+L} \right| \leq t \quad (13)$$

$$0 \leq |1+L|^2 - t^{-2}|L|^2$$

Let $\tau = (1 - t^{-2})$. Eq. (13) in terms of real and imaginary parts of L is then,

$$0 \leq 1 + 2\text{Re } L + \tau((\text{Re } L)^2 + (\text{Im } L)^2) \quad (14)$$

In terms of a,b,c,d:

$$\tau(a^2 + b^2)(c^2 + d^2) + 2ac - 2bd + 1 \geq 0 \quad (15)$$

Hence, substituting parameters of the controller into c and d and solving (12) and (15), the CL specifications are converted to bounds in the parameter space.

The final step before choosing the actual set of parameters is to determine stabilizing and destabilizing parameter regions. The method for this is given in the next proposal.

Lemma 2. When CL poles switch between $\mathbb{C}^+$ and $\mathbb{C}^-$, the corresponding sensitivity function goes to infinity. Thus computing parameter bounds $\{\theta \mid |S| \rightarrow \infty\}$ corresponds

Fig. 1. Two mass-spring-damper system is used as the example plant (Fig. edited from common data base).

Fig. 2. Bode diagrams of the functions L, S and T for the given plant

to the boundary between the stabilizing and unstabilizing controller parameter spaces.

Proof. Solving (8) for a value of $|S| = m \rightarrow \infty$ shrinks the conic section to the point where the OL function switches between the stable and unstable regions. Using this proposition one can calculate the bound of stability in the parameter region. This bound ensures the crossover occurs along the shrunk conics and a coarse gridding of the parameter space is enough to test for stable and unstable regions. $\square$

IV. DESIGN EXAMPLES

Next, the outlined method is illustrated by two examples. Both examples assume a controller of two parameters, as it is computationally and graphically more convenient to depict the results. However, it is possible to extend the suggested method to higher order controller design.

Let us design a controller for a two mass-spring-damper system shown in the Fig. 1. The numerical values are $m_1 = 1$ kg and $m_2 = 1.2$ kg for the masses, $k_1 = 500$ N/m and $k_2 = 250$ N/m for the springs and $z_1 = z_2 = 0.01$ for the damping ratios. The corresponding plant TF is as follows and the Bode diagram is given in Fig. 2.

$$P(s) = \frac{X_2(s)}{X_0(s)} = \frac{0.16s^2 + 285s + 125000}{1.2s^4 + 1.3s^3 + 1150s^2 + 285s + 125000} \quad (16)$$

This plant evaluated on a discrete frequency set Ω , also used to plot the corresponding Bode diagrams, will be used in all of the examples given in this section.

Fig. 3. Ellipses that bound k_p - k_d space. Outside of the ellipses is the allowable space. Purple markers show the locations the ellipses shrink when $|S| = m \rightarrow \infty$. Green and red markers show stable and unstable regions respectively. Solid lines for complimentary sensitivity, dashed lines for sensitivity function specifications.

A. Controllers Affine in Parameters

Let us design a controller that is affine in its parameters and that satisfies the CL specifications given as

$$|S(j\omega)| \leq 6dB, \quad |T(j\omega)| \leq 5dB, \quad \forall \omega \in \Omega \quad (17)$$

Controllers affine in parameters are controllers whose poles are fixed but zeros are variable. A PID or a PD controller are examples to such controllers. These controllers, that have their parameters in their numerators, always result in quadratic polynomials in their parameter, as can be seen in Lemma 1. Let us use this theory on controller parameter space for a PID controller, defined as:

$$G(s) = k_p + s k_d + \frac{k_i}{s}$$

In frequency domain this is equivalent to,

$$G(j\omega) = c + jd = k_p + j\omega k_d - j\frac{k_i}{\omega}$$

So $c = k_p$ and $d = \omega k_d - k_i/\omega$. For a sensitivity specification, putting c and d into (12),

$$(a^2 + b^2) \left(k_p^2 + \omega^2 k_d^2 + \frac{k_i^2}{\omega^2} \right) + 2(a^2 + b^2) k_d k_i - 2ak_p - 2b\omega k_d + 1 - m^{-2} \geq 0 \quad (18)$$

Which is a QI in 3 dimensions. Let us choose $k_i = 0$ to have only two affine controller parameters to make the explanation and visualization easier. Then the equations become:

$$(a^2 + b^2) (k_p^2 + \omega^2 k_d^2) + 2ak_p - 2b\omega k_d + 1 - m^{-2} \geq 0$$

$$\tau (a^2 + b^2) (k_p^2 + \omega^2 k_d^2) + 2ak_p - 2b\omega k_d + 1 \geq 0 \quad (19)$$

Solving these equations for all frequencies in the discrete frequency set gives conic sections. These conic sections are the boundaries separating allowable parameter space from the inallowable parameter space. The choice of exact parameters from the allowable parameter space depends on the problem at hand. Resulting conic sections that define the allowable parameter space is shown in Fig. 3.

Fig. 4. Bode diagrams of the system after implementing the PD controller, zoom in on the right.

Next step is to employ the stability criterion. This is done numerically by gridding the parameter space and testing for stability per grid point. More information on how to perform this gridding, and performing the stability test for the data-based plant model can be found in [9]. The allowable side of the bound is calculated by testing if the center point of the conic section satisfies the related inequality or not. This center point is already calculated to define the conic section so this step does not bring extra computational burden.

Next, a stabilizing controller from the allowable parameter space is chosen. In this example, staying close to the bounds, to ensure there is no over-design, is the only concern. Chosen values are $k_p = 5.1426e6$ and $k_d = 1.9335e4$. The resulting performance is seen in Fig. 4. This figure shows the maximum value of the complementary sensitivity function is 5dB, just as given in the required specifications.

B. Controllers Non-Affine in Parameters

In this section we aim to demonstrate that non-affine parameters also result in QI's, see Lemma 1. Non-affine controller parameters are the denominator variables. Let us design a lead-lag controller for the CL specifications, given in (20).

$$G(s) = \frac{z + s}{p + s}, \quad |S(j\omega)| \leq 5dB, \quad \forall \omega \in \Omega \quad (20)$$

for the plant given in (16). Then,

$$G(j\omega) = c + jd = \frac{zp + \omega^2}{p^2 + \omega^2} + j\frac{(p - z)\omega}{p^2 + \omega^2} \quad (21)$$

Let us define,

$$c = \frac{zp + \omega^2}{p^2 + \omega^2} = \frac{n_c}{den} \quad \text{and} \quad d = \frac{(p - z)\omega}{p^2 + \omega^2} = \frac{n_d}{den}$$

For a sensitivity specification, putting c and d into (12):

$$(a^2 + b^2) \frac{n_c^2 + n_d^2}{den^2} + 2a\frac{n_c}{den} - 2b\frac{n_d}{den} + 1 - m^{-2} \geq 0$$

Evaluating and regrouping and simplifying ,

$$(a^2 + b^2) (\omega^2 + z^2) + 2a n_c - 2b n_d + (1 - m^{-2}) den \geq 0$$

Expanding and collecting terms,

$$p^2 (1 - m^{-2}) + z^2 (a^2 + b^2) - 2b\omega p + 2b\omega z + 2azp + \omega^2 (1 - m^{-2} + a^2 + b^2 + 2a) \geq 0 \quad (22)$$

Fig. 5. Parameter space corresponding to the controller and CL specifications given in (20). Solid lines mean one should stay outside and dotted lines mean one should stay inside the curves. Green markers show the stable and red markers show the unstable region.

This last equation shows the coefficients of the QI for the sensitivity specification.

Although it is not needed for this given problem, a complementary sensitivity specification would result in the following equations. Putting in c and d in (15),

$$T(a^2 + b^2) \left(\frac{n_c^2 + n_d^2}{den^2} \right) + 2a \frac{n_c}{den} - 2b \frac{n_d}{den} + 1 \geq 0$$

Simplifying and rearranging,

$$p^2 + z^2 T(a^2 + b^2) - 2b\omega p + 2b\omega z + 2azp + \omega^2 (T(a^2 + b^2) + 2a + 1) \geq 0 \quad (23)$$

This last equation shows the coefficients of the QI for the complementary sensitivity specification. Solving (22) for all frequencies in the discrete frequency set result in the conic sections shown in Fig. 5. Stable and unstable parameter regions are calculated as explained in the previous section. Fig. 5 shows that the solution of this problem results in hyperbolas and ellipses. Although the figure seems quite complicated, it is fairly easy to choose a stabilizing parameter set from the allowable region. Let us choose $p = 436$ and $z = 24$, which is almost on the border of one of the hyperbolas. The corresponding Bode plots are shown in Fig. 6. It is seen that the designed controller satisfies the design specifications.

V. INCORPORATING NON-PARAMETRIC PLANT UNCERTAINTY

In the case of plant uncertainty, the approach remains almost unchanged. Uncertainty is accounted for by defining a plant set that includes all possible plants. The plant set and frequency range to be analyzed are discretized. Thus, for a given CL specification at a given frequency, there is more than one plant. Each of these plants map to a QI, and each QI maps in to a conic section. These conic sections are calculated for all CL specifications at all frequencies for all plants in the plant set, which results in a large number of conic sections.

To decrease the number of bounding conic sections, an additional step is introduced. This step relies on the observation that bounds for all the plants in the plant set remain closely packed together at a given frequency. Thus the bounds at a

Fig. 6. Bode plot of the system with the designed lead-lag controller.

Fig. 7. Ellipses that correspond to 20% plant uncertainty at 51 rad/s for a sensitivity specification given in (17).

given frequency are approximated by a single over-bounding curve. The design procedure then goes on as in the single plant case. The stability criteria is employed to define the parameter space that result in a stable system. Controller parameters are chosen according to an optimization criteria, like minimizing high frequency gain, from the resulting allowable parameter space.

Next, the calculation of the over-bounding curve is explained per the type of conic section. For the extent of this paper, only types of conic sections to be considered are ellipses and hyperbolas, CL specifications with a two parameters controller result in these curves. However, a similar approach can be applied to other conic sections in higher dimensions as well.

For more than one ellipses at a given frequency, the over-bounding curve is the minimum volume ellipse (ellipsoid for higher dimensions) covering the union of all ellipses corresponding to the bounds associated to the plants in the plant set. Such an ellipsoid is called the Löwner-John ellipsoid, and can be calculated by an efficient optimization algorithm described in [10]. An example of this case is shown in Fig. 7. This approximation brings some conservatism but it ensures that all criteria are satisfied.

For several hyperbolas at a given frequency, the over-

Fig. 8. Hyperbolas that correspond to 20% plant uncertainty at 10 rad/s for the sensitivity specification given in (20).

bounding curve is chosen to be the two intersecting lines. The intersection point is taken to be the average of all centers and the slopes are chosen to be equivalent to the slopes of the asymptotes that define the most constrained region. An example of this case is shown in Fig. 8. This approximation is not ensured to satisfy all constraints, and still introduces some conservatism. However, it can be used under the assumption that the centers of the hyperbola are close enough, which is the case in the following example with parametric uncertainty.

Above figures are plotted for the following example. In this example non-parametric uncertainty in multiplicative form is used to describe plant uncertainty. However, the nature of the uncertainty is not important since the first step is to build a finite order discrete plant set that represents all possible plant uncertainties in any case. Given a nominal plant $P_o = a_o + jb_o$, the uncertain plant set can be described as follows.

$$\begin{aligned} \mathcal{P} &= \{P_o(1 + \Delta) \text{ s.t. } |\Delta| \leq r\} \\ &= \{P_o + P_o|\Delta|e^{j\phi} \text{ s.t. } \phi \in [0^\circ, 360^\circ]\} \end{aligned} \quad (24)$$

For the plants $P = a + jb$, $P \in \mathcal{P}$, boundary of the plant template occurs at $\Delta = r$. Then the following relation holds on the plant template boundary:

$$(a - a_o)^2 + (b - b_o)^2 = r^2 \quad (25)$$

Expanding,

$$[a, b] \in \{a, b \text{ s.t. } a^2 + b^2 - 2a_o a - 2b_o b + a_o^2 + b_o^2 - r^2 = 0\}$$

Solving the QI for all a, b that belong to the above set for all $\omega \in \Omega$ gives all the bounds that should be satisfied to satisfy the requirements given in the problem definition. At a single frequency, this results in multiple overlapping bounds. These overlapping bounds can be further approximated and the controller design can be carried out as explained before.

In the case of bounds for a given uncertain plant at a given frequency result in ellipsoids, these bounds can be approximated by the minimum volume ellipsoids. As can be seen in [10], these ellipsoid can be calculated analytically and very efficiently. This approximation does not introduce much conservatism in the case the the uncertainty is limited, as in most cases. In the case of bounds resulting in hyperbola, these hyperbolas may be approximated by the over bounding

asymptotes that divide the controller parameter space by intersecting lines. Examples to both cases are given in Fig. 7, where there is 20% uncertainty for the plant at every frequency $\omega \in \Omega$.

VI. CONCLUSION

A number of control problems relate to highly linear plants lacking parametric models. This makes it attractive to use FR data to design a stabilizing controller that optimally satisfies given CL performance specifications for uncertain plants. In this research such a data-based design method for fixed structure controllers is proposed. The proposed method uses bounds in the parameter space of a controller with predefined structure and QFT techniques to arrive at an (close to optimal) solution. This also enables the designer to choose the order of the controller to be implemented on the practical setup. Moreover, the optimality criteria makes it possible to automatically search for controller parameters.

In this paper, it is proved that CL specifications result in QI's in terms of both the numerator and denominator parameters of a controller with predefined structure. This is a significant improvement of the existing automated LS approaches as it renders fixing the poles of the controller, convexifying the bounds, or sequentially iterating over numerator and denominator parameters, obsolete. QI's are easily embedded in computer algorithms and consume very little computation time, especially in comparison to numerical methods.

The described feedback control design procedure can also be used for plants with uncertainty. In this case bounds at a given frequency are calculated for all plants in the discrete plant set and are approximated by one outer bound, after which design goes just like the case without uncertainty.

Although this paper only shows application in two parameter space, theories developed hold equally well for higher dimension spaces. Further research is going on to apply these theories to high dimension controller parameter space and to develop automated parameter choice strategies.

REFERENCES

- [1] I. M. Horowitz, *A Gourmet QFT Dinner*. QFT Publications, 1993.
- [2] A. Gera and I. Horowitz, "Optimization of the loop transfer function," *International Journal of Control*, vol. 31, pp. 389–398, 1980.
- [3] P. S. V. Nataraj and M. M. Deshpande, "Automated synthesis of fixed structure qft controller using interval constraint satisfaction techniques," in *17th IFAC World Congress*, 2008.
- [4] Y. Chait, Q. Chen, and C. Hollot, "Automatic loop-shaping of qft controllers via linear programming," *Transactions of the ASME Journal of Dynamic Systems, Measurement and Control*, vol. 121, pp. 351–357, 1999.
- [5] V. Croulard, E. Godoy, and J. Boichot, "Qft controller optimization for automatic design," in *Proc. 39th IEEE Conference on Decision and Control*, vol. 5, 2000, pp. 4735–4740.
- [6] D. Thompson and O. Nwokah, "Analytical loop shaping methods in quantitative feedback theory," *Transactions of the ASME Journal of Dynamic Systems, Measurement and Control*, vol. 116, pp. 169–177, 1994.
- [7] G. Bryant and G. Halikias, "Optimal loop-shaping for systems with large parameter uncertainty via linear programming," *International Journal of Control*, vol. 62(3), pp. 557–568, 1995.
- [8] W.-H. Chen, D. Ballance, W. Feng, and Y. Li, "Genetic algorithm enabled computer-automated design of qft control systems," in *Proc. IEEE International Symposium on Computer Aided Control System Design*, 22–27 Aug. 1999, pp. 492–497.
- [9] A. den Hamer, S. Weiland, M. Steinbuch, and G. Angelis, "Stability and causality constraints on frequency response coefficients applied for non-parametric $\mathcal{H}_2$ and $\mathcal{H}_\infty$ synthesis," in *IEEE CDC*, 2008, pp. 3670–3675.
- [10] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.