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The three-dimensional structure of an electromagnetically generated dipolar vortex in a shallow fluid layer

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Many experiments have been performed in electromagnetically driven shallow fluid layers to study quasi-two-dimensional (Q2D) turbulence, the shallowness of the layer commonly is assumed to ensure Q2D dynamics. In this paper, however, we demonstrate that shallow fluid flows exhibit complex three-dimensional (3D) structures. For this purpose we study one of the elementary vortex structures in Q2D turbulence, the dipolar vortex, in a shallow fluid layer. The flow evolution is studied both experimentally and by numerical simulations. Experimentally, stereoscopic particle image velocimetry is used to measure instantaneously all three components of the velocity field in a horizontal plane, and 3D numerical simulations provide the full 3D velocity and vorticity fields over the entire flow domain. It is found that significant and complex 3D structures and vertical motions occur throughout the flow evolution, i.e., during and after the forcing phase. We conclude that the bottom friction is not the main mechanism leading to three-dimensionality of the flow but rather the impermeability of the boundaries. It is further shown that free-surface deformations, i.e., gravity waves, are of minor importance too as a mechanism to generate 3D motion. Furthermore, it is demonstrated that the observations are not due to three-dimensionality introduced by the forcing mechanism but intrinsically due to the flow dynamics itself. The flow evolution is analyzed with respect to its quasi-two-dimensionality by adopting the ratio of “horizontal” to “vertical” kinetic energies, the normalized horizontal divergence, and a measure of the relaxation to a Poiseuille-like profile. An important observation is that, although the relative magnitude of the vertical velocity as compared to the horizontal flow components decreases for decreasing fluid depth, the vertical profile of the horizontal flow relaxes rather slowly to a Poiseuille-like profile, i.e., not faster than the bottom friction time scale. © 2008 American Institute of Physics. [DOI: 10.1063/1.3005452]

I. INTRODUCTION

In order to validate theory on two-dimensional (2D) turbulence, many investigators have performed laboratory experiments in, e.g., rotating fluids,¹ stratified fluids,² and shallow (nonstratified) fluid layers.^{3–7} In the latter case, it is commonly assumed that these shallow flows behave in a 2D fashion when the vertical length scale $\mathcal{H}$ is much smaller than the horizontal length scale $\mathcal{L}$. The rationale behind this thin-layer configuration is that although vertical three-dimensional (3D) motions are present, their magnitude, assumed to be proportional to $\mathcal{H}/\mathcal{L}$, is much smaller than the dominant (2D) horizontal flow speeds. Moreover, the effect of the bottom friction can be parametrized by adding a linear friction term $-\alpha \mathbf{v}_H$ to the 2D Navier–Stokes equation (usually referred to as “Rayleigh friction”) under the assumption of a Poiseuille-like profile in the vertical direction. Here α represents the bottom friction coefficient and $\mathbf{v}_H$ the local depth-averaged horizontal fluid velocity (see, e.g., Ref. 8). This thin-layer configuration has been applied in flowing soap-film experiments^{9–12} and shallow fluid experiments.^{3–7} In some experiments a single fluid layer is utilized, see Refs. 3–7, while in the investigations of Boffetta *et al.*,¹³ Shats *et al.*,¹⁴ Rivera and Ecke,¹⁵ or Tabeling and co-workers^{16,17} a stable two-layer stratification has been used. Although a

slightly different flow forcing was applied in each of these experimental studies, these studies all use the two-layer system to provide an additional mechanism to inhibit vertical motions and to minimize the influence of bottom friction, which is predominantly present in the lower layer.

Just as the ordinary Reynolds number $\text{Re}(=\mathcal{U}\mathcal{L}/\nu)$ expresses the relative importance of lateral diffusion with respect to nonlinear processes, one can define an alternative Reynolds number $\text{Re}_\alpha(=\mathcal{U}/\alpha\mathcal{L})$ based on the influence of the bottom friction.⁵ Here, $\mathcal{U}$ denotes a characteristic velocity, $\mathcal{L}$ a typical horizontal length scale, and ν the viscosity, while the bottom friction coefficient α is given by $\alpha=\nu(\pi/2\mathcal{H})^2$ under the assumption of a Poiseuille-like flow in the vertical. The ratio $\text{Re}_\alpha/\text{Re}$ expresses the relative importance of the bottom friction to the horizontal diffusion. In the case $\text{Re}_\alpha/\text{Re}\ll 1$ [or equivalently $(2\mathcal{H}/\pi\mathcal{L})^2\ll 1$], the flow is considered to be dominated by bottom friction effects.

The linear friction term is nonselective with respect to length scales, therefore it also prevents the pile-up of energy at length scales corresponding to the box size (and is for this reason often used in numerical studies of 2D turbulence^{18,19}). As opposed to this linear friction, the “internal” viscous dissipation predominantly removes energy at the smallest scales. However, in the review paper of Danilov and Gurarie²⁰ it is argued that the influence of the bottom friction

can be parametrized only in a qualitative sense in the form of Rayleigh damping.

Apart from the influence of the bottom boundary layer, the flow forcing mechanism could also act as an additional source of vertical motion. In some experiments a magnetic field is used to force an electrolyte fluid through which an electric current flows. It is then the Lorentz force, resulting from the interaction of the current density and the magnetic field, which drives the flow. However, this magnetic field decays over a limited vertical distance and the forcing will therefore vary with height. In addition to its horizontal components, the Lorentz force also has a vertical component, whose effect on the flow generation has yet to be determined. The consequences of the spatial nonuniformities in the electromagnetic forcing and the vertical component of the Lorentz force are especially of interest for electromagnetically forced shallow flows.

The above described assumptions on the behavior of a flow in a shallow fluid layer have never been verified accurately. Notable exceptions are the numerical investigation by Satijn *et al.*²¹ concerning a decaying monopole in shallow fluid layers or the experimental study of a dipolar vortex by Lin *et al.*²²

The present paper reports on a detailed study of the vertical motions developing in shallow-layer flows. For this purpose, we study one of the elementary vortex structures in 2D turbulence, the dipolar vortex, in a nonrotating homogeneous shallow fluid layer. Stereoscopic particle image velocimetry (SPIV) has been used for an experimental investigation of the flow. To our knowledge, this is the first time that all three velocity components have been measured simultaneously in a horizontal plane inside a shallow fluid. Additionally, 3D numerical simulations are carried out, which easily allow inclusion of different boundary and initial conditions, and which provide the full 3D velocity and vorticity fields over the entire flow domain. Also the three-dimensionality resulting from the electromagnetic forcing itself is investigated.

We report on the significant three-dimensionality of the shallow fluid flow and the remarkably complex and non-trivial 3D structure of the dipole, both during and after the forcing stage. In order to quantify deviations from 2D or quasi-2D (Q2D) flow behavior, we analyze the flow evolution by adopting the ratio of “horizontal” and “vertical” kinetic energies (being the kinetic energy associated with the horizontal and vertical fluid velocity components, respectively) and the normalized horizontal divergence. Besides, the deviation from a Poiseuille-like profile is investigated for decreasing fluid depth.

This paper is organized as follows. Section II discusses the experimental setup and the measurement technique. In Sec. III the numerical method is briefly discussed, together with the implementation of the electromagnetic forcing. The experimental and numerical results of the dipole evolution during and after the forcing are then presented in Sec. IV. Additional simulations are discussed in which different boundary and initial conditions were applied in order to examine the influence of bottom friction, electromagnetic forcing, and initialization on the 3D motions. Based on the laboratory experiments and numerical simulations with no-slip

FIG. 1. (Color online) Schematic of the experimental tank showing the x, y, z -coordinate system. Left: top view. Right: cross section. The electrical current is denoted by I , the fluid depth with H , and B represents the magnetic field produced by the magnet.

bottom, we evaluate three quantities that could be used as criteria for the degree of two-dimensionality of the flow. Finally, in Sec. V the results are discussed and the conclusions are summarized.

II. EXPERIMENTAL SETUP

The laboratory experiments have been performed in a homogeneous shallow layer of electrolyte, in which the motion is generated by electromagnetic forcing. A disk-shaped magnet is placed underneath the fluid layer and an approximately uniform electrical current is running through the fluid, between two electrode plates mounted along opposite side walls. The interaction of the current density and the magnetic field induces a Lorentz force that sets the fluid in motion. Our experimental setup is similar to that used in several other studies by, e.g., Dolzhanskii *et al.*,⁵ Danilov *et al.*,⁶ and Tabeling and co-workers^{16,17} (although the latter authors used a stably stratified two-layer system). A schematic of the setup is depicted in Fig. 1. The left-hand side of this figure shows a top view of the 52×52 cm² square tank with one disk-shaped permanent magnet below the bottom. The bottom plate has a thickness of 1 mm. Two rectangular-shaped electrodes are placed on the opposite sides of the tank, leading to a uniform current density in the x -direction. A single layer of sodium chloride solution (NaCl, 15% Brix) serves as the conducting fluid enabling the electromagnetic forcing. The magnet is placed approximately at the middle of the tank to minimize the influence of the lateral walls and nonuniformities in the current density.

We adopt a Cartesian coordinate frame, with the x - and y -axes spanning a plane parallel to the bottom of the tank, and the z -axis is taken vertically upward. The origin of the coordinate system lies above the center of the magnet at the bottom of the tank. The three velocity components in the x -, y -, and z -directions are denoted by u , v , and w , respectively. The cylindrical magnet (indicated in Fig. 1), with a diameter of 25 mm and thickness of 5 mm, is assumed to be uniformly magnetized in its axial direction and produces a magnet field with a magnitude of the order of 1 T.

The right-hand side of Fig. 1 shows a cross section of the experimental setup. Two cameras, placed at an angle, enable the use of SPIV (Ref. 23) to measure the full three-component velocity field in a horizontal plane inside the fluid layer. This method consists of a calibration procedure using

polynomial mapping functions and a light sheet misalignment correction procedure. The latter utilizes the disparity field to compute the true light sheet position compared to the calibration plane. The measurement images from both cameras are evaluated with a cross-correlation algorithm, and subsequently recombined with the aid of the calibration information (utilizing triangulation) to obtain the three-component velocity field.

In the SPIV experiments we use a dual pulse Nd:YAG (yttrium aluminum garnet) laser (Spectron Laser SL454, 200 mJ/pulse) to produce a horizontal light sheet of 1 mm thickness. In order to limit the in-plane particle loss²³ and for correct temporal sampling of the signal, a delay time between laser pulses of 10 ms is chosen.

The fluid is seeded with polystyrene particles having a mean diameter d_p of 20 μm and a specific density ρ_p of $1.03 \times 10^3 \text{ kg/m}^3$. The volume fraction of the particles is of the order of 10^{-5} , so that the seeding particles have a negligible influence on the flow properties. How well these particles follow the flow is characterized by the Stokes number $\text{St} = \tau_p / \tau_f$, where $\tau_p = d_p^2 \rho_p / 18 \mu_f$ is the particle response time to acceleration. The flow time scale τ_f is estimated as $1 / \omega_{z,\text{max}} \approx 0.08 \text{ s}$, yielding $\text{St} = 2 \times 10^{-4}$, indicating that the particles follow the flow passively.

Two cameras (Kodak Megaplug ES1.0 with sensor resolution of 1008×1019 pixels, $f_\# = 2.8$) are mounted on Scheimpflug adaptors to enable in-focus imaging of the entire field of view, as the stereoscopic angle is approximately 80° . The cameras and the light source are synchronized with a delay generator. With this setup, image pairs are acquired at 15 Hz. The typical field of view is approximately $8.5 \times 7 \text{ cm}^2$ in the x - and y -directions. The area covered by the cameras is indicated schematically in Fig. 1 by the two dashed rectangles. The rectangle around the position of the magnet is our viewing area during the forcing phase. As the dipole will be propagating in the positive y -direction, the upper rectangle represents the field of view used to study its evolution. During postprocessing, all images are sampled at a resolution of 896×896 pixels; the PIV analysis involves square interrogation windows of 32×32 pixels and 50% overlap between neighboring windows. After postprocessing, these settings result in SPIV velocity fields that are resolved on a 55×55 spatial grid, corresponding to physical grid spacing of (1.55; 1.25) mm in the x - and y -directions, respectively. In a correlation window of 32×32 pixels there are, on average, 16 seeding particles present.

Table I provides an overview of the different experiments. Although experiments were carried out with three different fluid layer depths H and various electrical current strengths I (see Table I), in this paper we will present results mainly for a fluid depth of $H = 9.3 \text{ mm}$ and a forcing protocol consisting of a single 1 s constant-current pulse of $I = 4.4 \text{ A}$. It is stressed that the results presented for this case are indicative for all the other fluid layer depths and forcing protocols. Measurements were performed at several levels h_{ls} inside the different fluid depths. Time t was set to zero at the onset of forcing for all the results presented in Sec. IV. By changing the fluid depth H and forcing current I , we effectively change the Re_α -value and initial Re -value, respec-

TABLE I. Experimental parameter values for the SPIV measurements: total fluid depth H , measurement level h_{ls} , electrical current I , current density j_x , and ratio vertical/horizontal Reynolds numbers.

H (mm)	h_{ls} (mm)	I (A)	j_x (A/cm ²)	$\text{Re}_\alpha / \text{Re}$ (-)
6.0	2.0; 3.5	2.4; 3.7	0.8; 1.2	0.02
9.3 ^a	2.0; 5.0; 9.0	2.4; 4.4 ^a ; 5.4	0.5; 0.9; 1.1	0.06
11.4	2.0; 5.0; 9.0; 10.0	2.4; 4.4; 6.4	0.4; 0.7; 1.1	0.08

^aResults of this experimental (and numerical) case, i.e., $H = 9.3 \text{ mm}$ and $I = 4.4 \text{ A}$, presented in Secs. IV A and IV B are indicative for all the other cases.

tively. Note that the current density j_x is a measure of the forcing strength. This current density is computed as the electrical current I divided by the cross-sectional area, which equals HL_{tank} (the latter being 52 cm). The last column in Table I presents values of the quantity $\text{Re}_\alpha / \text{Re}$, or equivalently $(2\mathcal{H} / \pi \mathcal{L})^2$, where the magnet diameter and fluid depth were taken as measures of the characteristic horizontal and vertical length scales, respectively. The small values of the ratio $\text{Re}_\alpha / \text{Re}$ indicate that all experiments are dominated by bottom friction effects. Note that the Reynolds number Re based on the maximum horizontal velocity $\mathcal{U}$ at the end of the forcing period (forcing current strength of $I = 4.4 \text{ A}$) measured approximately 1400. The Reynolds number based on the bottom friction coefficient for this case equals approximately 75. In the regime diagram of Satijn *et al.* (see Fig. 6 in Ref. 21), this $(\text{Re}_\alpha / \text{Re})$ combination lies outside the Q2D regime (note that their diagram is for a monopole). However, as noted before, the results presented for this case, i.e., $H = 9.3 \text{ mm}$ and $I = 4.4 \text{ A}$ are representative of the other fluid layer depths and forcing strengths.

As is to be expected, the influence of the flow on the magnetic field is negligibly small since the magnetic Reynolds number Re_m , defined as $\mu \sigma \mathcal{U} \mathcal{L}$, is small [$\text{Re}_m \sim \mathcal{O}(10^{-7})$]. Here μ and σ denote the magnetic permeability and electric conductivity, respectively.

III. NUMERICAL METHOD

The numerical simulations are based on the Navier-Stokes equation, i.e.,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v} + \frac{1}{\rho} \mathbf{f}_L, \quad (1)$$

where $\mathbf{v}$ is the 3D velocity, p is the pressure, ρ is the mass density, ν is the kinematic viscosity, and $\mathbf{f}_L$ is the external body force.

This Navier-Stokes equation is solved using a finite element method²⁴ in the three spatial dimensions in conjunction with adaptive meshing and error control. The external body force $\mathbf{f}_L$ in Eq. (1) represents the electromagnetically generated Lorentz force, resulting from the interaction of the magnetic field $\mathbf{B}$ with the current density $\mathbf{j}$, i.e.,

$$\mathbf{f}_L = \mathbf{j} \times \mathbf{B}. \quad (2)$$

Here the current density is a uniform and constant pulse of 1 s duration in the x -direction ($\mathbf{j} = j_0 \mathbf{e}_x$ for $0 < t \leq 1 \text{ s}$ and $\mathbf{j}$

$\equiv 0$ for $t > 1$ s), similar to the forcing applied in the experiments (see Table I). Since the electromagnetic forcing inside the fluid layer does not take place in the so-called far field of the permanent magnet, a point-dipole approximation of the magnetic field is not justified here. Therefore we use exact expressions known in analytical form for the magnet's magnetic field, derived under the assumption that the magnet's magnetization current can be modeled as a stack of current-carrying circular wires, the magnetic field of which can be written in terms of elliptic integrals (see, e.g., Ref. 25).

As the magnet's strength is not exactly known, we adjust this strength in such a way that the numerical simulation matches the corresponding laboratory experiment at a certain moment in time. For this matching one can use different criteria, such as the maximum of the vertical vorticity component or the horizontal kinetic energy, both at the end of the forcing period. The former matching criterion is used for the numerical results presented in the remainder of this paper.

The computational domain is identical to the experimental one (see Fig. 1), with the exception of the lateral (outer) domain boundary. This is taken circular (for computational efficiency) with a diameter of six times the magnet diameter, whereas the experimental setup has a square outer boundary. It has been checked by simulations with a larger circular domain that its size did not affect the result. To acquire the desired accuracy in a typical run, the computational domain is discretized with approximately 50 000 mesh elements, with finer elements being used near the bottom, near the free surface, and close to the magnet (where the forcing is strongest) in order to resolve the gradients in the local flow field. The resulting number of degrees of freedom solved for is approximately 250 000. Convergence of the solution was checked by additional mesh refinement, after which the result appears indistinguishable from the simulation carried out with 50 000 mesh elements.

Two types of numerical simulations have been performed. In all runs the upper free surface was taken to be stress-free and flat. The latter implies that the generation of surface gravity waves is excluded in the numerical simulations. The first set of runs was aimed to simulate the experimental results. For that purpose we impose a no-slip boundary condition at the bottom of our computational domain. In order to examine how the bottom boundary layer at a no-slip bottom is related to the 3D flow structures found numerically, a second set of simulations has been performed. In these runs the bottom, just like the upper surface of the fluid layer, was taken to be stress-free rather than no-slip, leaving all other parameters and settings unchanged.

IV. EXPERIMENTAL AND NUMERICAL RESULTS

A. The generation and evolution of the dipolar vortex

In this subsection we will present results for a fluid depth of $H=9.3$ mm and electrical current of $I=4.4$ A (see Table I), as the flow evolution of this case is indicative for all the other fluid depths and forcing strengths.

Figure 2 shows plots of the instantaneous vertical (shades/colors) and horizontal (vectors) velocities in a horizontal plane at 5 mm above the bottom. For clarity of representation the vectors are undersampled: only every second

vector is shown in the x - and in the y -directions, so that only 25% of the total set is shown. The disk-shaped magnet (indicated by the circle) is centered at $(x,y)=(0,0)$. The upper row of figures show experimental results at three instances of time and the lower row shows numerically obtained results at the same times. Since the total forcing time $\Delta t=1$ s and the forcing is started at $t=0$, Figs. 2(a), 2(d), 2(b), and 2(e) correspond to the midstage ($t=0.5$ s) and to the final stage ($t=0.96$ s) of the forcing, while Figs. 2(c) and 2(f) show snapshots of the flow field 0.5 s after the forcing has stopped.

As can be seen in Fig. 2(a), two regions of downward flow have developed in the two vortex cores. At the late stage of the forcing [see Fig. 2(b)], these regions of downward flow are still present inside the vortices, while upward motion is observed at the tail of the dipole. During the entire forcing phase, a buildup of downward motion is seen inside the vortices. Comparison of the numerical simulation results shown in Figs. 2(d) and 2(e) with Figs. 2(a) and 2(b) reveals a striking resemblance, although the upward motion at the tail side found in the simulations is not so clearly visible in the experimental result shown in Fig. 2(b). It should be noted that the downwelling in both vortex cores is not driven directly by the vertical component of the Lorentz force but can be understood as follows. From the numerical simulations we observe that the initial (linear) flow evolution is governed by a magnetohydrostatic balance between the vertical component of the Lorentz force and the vertical pressure gradient (see Appendix A). This implies that the vertical component of the Lorentz force does not drive a flow at the early stages of the electromagnetic forcing. While the magnitude of the horizontal flow velocities gradually increases during the forcing, a cyclostrophic balance between the horizontal pressure gradient and the centrifugal force in both swirls is established. Due to the stronger forcing close to the magnet the swirl is strongest close to the bottom, implying a vertical pressure gradient along the vortex axes in the positive z -direction. As a result, the initial magnetohydrostatic balance [see Eq. (A5)] in each of the two vortex cores is gradually distorted by additional vertical pressure gradients that start to drive a downward motion in both swirls during the whole forcing phase, as is observed in Figs. 2(a) and 2(b) as well as in Figs. 2(d) and 2(e).

Figure 2(c) [or Fig. 2(f)] shows a snapshot of the experimentally (numerically) observed flow field 0.5 s after the forcing has stopped. Comparison with Fig. 2(b) [or Fig. 2(e)] reveals an essentially different flow distribution. One now observes well-defined upward motion in the vortex cores surrounded by downwelling similar to the secondary circulation observed in a single monopolar vortex.²¹ This upward motion appears shortly after the forcing has stopped. In the monopolar vortex case the emergence of secondary circulation is driven by the interaction with the viscous boundary layer at the bottom. Although this mechanism is also working in the dipolar vortex case, it cannot explain the velocity magnitude observed. Estimation of the vertical velocity based on the Bödewadt flow model (see, e.g., Refs. 26 and 27) leads to a typical vertical velocity of 4 mm/s, whereas from Fig. 2(c) we read a substantially larger velocity of approximately 10

FIG. 2. (Color online) Instantaneous velocity fields of a dipolar vortex in a horizontal plane at $z=5$ mm (see Table I for the experimental and numerical parameters). Vectors represent horizontal velocity components and shades/colors indicate the magnitude of the vertical velocity. The circle denotes the position of the disk-shaped magnet. Experimental results obtained with SPIV at (a) $t=0.50$ s, (b) $t=0.96$ s, and (c) $t=1.50$ s. Numerical results at (d) $t=0.50$ s, (e) $t=0.96$ s, and (f) $t=1.50$ s.

mm/s. This difference suggests a relaxation of the flow after the forcing phase that has a different origin than linear bottom friction. This questions the use of linear friction to model the influence of the bottom in 2D turbulence simulations, as this relaxation is not a bottom friction effect. The mechanism of this relaxation will be discussed in more detail later in this section.

Figures 3(a)–3(c) display three numerically obtained snapshots of the vertical vorticity evolution at the same time instants as Figs. 2(d)–2(f) and evaluated at 5 mm above the bottom.

In these figures oppositely signed vorticity patches are observed, a characteristic for a 2D dipolar vortex. Initially, these vorticity patches have a kidney shape, typical for the

FIG. 3. (Color online) Snapshots of the numerically simulated flow evolution showing the horizontal velocity components (vectors) and the vertical vorticity component ω_z (magnitude indicated by shades/colors) in the horizontal plane at $z=5$ mm. The circle indicates the position of the disk-shaped magnet. Snapshots taken at (a) $t=0.50$ s, (b) $t=0.96$ s, and (c) $t=1.50$ s.

FIG. 4. Comparison of experimental (open circles) and numerical (solid line) vorticity profiles at (a) $t = 0.50$ s and $y = 0$ mm, (b) $t = 0.96$ s and $y = 3$ mm, and (c) $t = 1.50$ s and $y = 11$ mm. The profiles are evaluated at height $z = 5$ mm above the bottom.

magnetic field produced by a cylindrical magnet. This kidney-like shape becomes less pronounced for later time and for larger vertical distance from the magnet.

To make a more quantitative comparison between experiment and numerical simulations, vorticity profiles of ω_z are displayed in Fig. 4 for the same time instances as in Fig. 3. In general, a good quantitative agreement is seen. Figure 4(c) shows the development of small-scale structures in the numerically obtained vorticity. This fragmentation of vorticity becomes more pronounced at later stages of the evolution. These regions of high gradients in the vorticity are less well reproduced by the experiments due to the spatial averaging effect of the cross correlation by the SPIV technique.

In order to observe the evolution of the dipole after the forcing has been switched off, the field of view is shifted in the positive y -direction (i.e., the upper dashed rectangle in Fig. 1). Figure 5 shows plots of instantaneous velocities in a horizontal cross section 5 mm above the bottom at $t = 1.50$ s, $t = 2.00$ s, and $t = 2.60$ s. The first row displays the experimentally obtained velocity fields and the second row shows the velocity field obtained numerically. The meaning of the shades/colors and vectors is the same as in Fig. 2.

After the electromagnetic forcing has stopped, one observes the appearance of a well-defined upward motion in the two vortex cores, surrounded by bands of negative vertical velocity, as seen in Fig. 5(a). Although these 3D motions are quite similar to the secondary circulation seen in the monopolar vortex,²¹ as stated before, the driving mechanism is quite different, i.e., the upwelling is not driven by a viscous boundary layer at the bottom. This can be understood as follows.

In the absence of forcing, no swirl is maintained in the lower part of the fluid layer. The continuing downflow and related outflow at the base of the two vortex cores will then, in contrast to the forced case, rapidly reduce the vertical vorticity ω_z close to the bottom. This is basically due to conservation of angular momentum $r v_\theta$: the swirl velocity v_θ of the radially spreading fluid decreases. Our experiments and simulations indicate that this spin-down (as seen in the kinetic energy of the horizontal motion) occurs on a much shorter time scale (approximately 1.5 s) than the typical bot-

tom friction time scale $\tau_E (=2\mathcal{H}^2/\pi^2\nu)$, here equal to approximately 18 s). The cyclostrophic balance then dictates an increase of the pressure in the vortex core near the bottom, while no significant pressure change is expected near the free surface. Eventually, the axial pressure gradient in the vortices is even reversed, resulting in a deceleration of the downflow and actually leading to an upward flow, as is seen in Fig. 5(a). Note that this phenomenon is independent of viscosity but is a consequence of the vertical confinement of the flow. The minor importance of bottom friction was already hinted at before, where it appeared that the Bödewadt model substantially underestimated the magnitude of this upward motion.

Besides this complex 3D motion, upward motion at the plane of symmetry of the dipole and significant negative vertical motion at the front of the dipole are observed. For example, in the region of strong downward motion the vertical velocity turns out to be of the order 25% of the horizontal velocity. At the front of the dipole one observes a feature referred to by Sous *et al.*^{28,29} as “frontal circulation,” i.e., a roll-like flow structure with an upward motion in front of the dipole and a weaker downward motion in front of that [see Figs. 5(b) and 5(c)]. Note that Lin *et al.*²² (who were, to our knowledge, the first to report on this feature) and Sous *et al.*^{28,29} did not use the electromagnetic forcing method, but their dipoles were created by injecting a small amount of fluid horizontally in the tank.

As time progresses one observes an oscillating motion of the vertical velocity component inside the two vortex cores. This is illustrated in Fig. 5(c), where the emergence of oppositely signed vertical motion is seen inside the individual vortex cores as compared to Fig. 5(a). We interpret these oscillations to be of inertial origin³⁰ and resulting from an overshoot of the decelerated upwelling described above. Typically, two to three oscillations are seen in both the experiments and the simulations, depending on the forcing strength. This inertial oscillation will be the subject of future study.

Comparison of the numerically obtained Figs. 5(d)–5(f) with Figs. 5(a)–5(c) reveals a striking resemblance, although the upward motion at the tail side found in the simulations is

FIG. 5. (Color online) Instantaneous velocity fields of a dipolar vortex evaluated in the horizontal plane at $z=5$ mm above the bottom during postforcing phase (see Table I for the experimental and numerical parameters). Meaning of vectors and shades/colors: see caption of Fig. 2. Experimental results at (a) $t=1.50$ s, (b) $t=2.00$ s, and (c) $t=2.60$ s. Snapshots of numerical simulation at (d) $t=1.50$ s, (e) $t=2.00$ s, and (f) $t=2.60$ s.

not so clearly visible in the experimental result. From this resemblance one can conclude that free-surface effects (through surface gravity waves) are of minor importance in generating vertical motions in the laboratory experiments. In the laboratory experiments surface deformations are present, such as the dimple on the free surface in the vortex core, whereas the generation of surface gravity waves is excluded in the numerical simulations (the surface is not allowed to deform). One can also estimate the phase speed c of a surface gravity wave in a shallow fluid layer, see, e.g., Ref. 31. The phase velocity of a surface gravity wave equals $c = \sqrt{g\lambda \tanh(2\pi H/\lambda)/2\pi}$, where g is the gravitational acceleration, λ is the wavelength, and H is the fluid depth. Taking for the wavelength λ the magnet diameter D as a characteristic horizontal length scale, one obtains a phase speed c of 23 cm/s, which is much larger than the fluid velocities.

To summarize, the necessary condition for the occurrence of the dynamical flow behavior inside the dipolar vortex as explained above is a z -dependence of the vertical vorticity component, i.e., $\omega_z = f(z)$. This condition can be understood in the following way. When ω_z is a function of height, the cyclostrophic balance dictates that the pressure p must also vary with z . This vertical pressure gradient will drive a vertical motion. In general, any height variation in

the vertical vorticity, as a result of the forcing and/or dictated by boundary conditions, will lead to vertical motions.

B. The structure of the dipole: Stagnation points and horizontal vorticity rolls

Figures 6(a)–6(c) present the numerically obtained distributions of the z -component of the vorticity for the same three instances of time as in Figs. 5(d)–5(f). For $t=1.50$ s, i.e., 0.5 s after the forcing has been stopped, Fig. 6(a) shows two coherent, symmetrical patches of oppositely signed vorticity. Moreover, one observes weak bands of vorticity at the locations of the frontal circulation band. As time progresses we see in Figs. 6(b) and 6(c) an increasing fragmentation of the vorticity, as was also observed by Lin *et al.*²²

To investigate the vertical flow structures at the front and tail sides of the dipolar vortex, we show in Fig. 7(a) the numerically obtained instantaneous streamline pattern for the horizontal flow field $\mathbf{v}_H = (u, v)$ at mid-depth and in a reference frame that is comoving. In this figure spiraling streamlines are seen inside the individual vortex cores, inward or outward depending on the net in- or outflux (i.e., $\partial w / \partial z$) at the specific evaluation height. Furthermore, in front and tail of the dipole two hyperbolic points can be identified. It is found that these points coincide with regions of significant

FIG. 6. (Color online) Numerically observed vertical vorticity evolution at $z=5$ mm (see Table I for the numerical parameters). Meaning of vectors and shades/colors: see caption of Fig. 3. Numerical snapshots at (a) $t=1.50$ s, (b) $t=2.00$ s, and (c) $t=2.60$ s.

vertical motion. Remarkable is that the band of the upward motion in the frontal circulation structure delineates the instantaneous “separatrix” that is associated with the frontal stagnation point. The separatrix is not closed at the rear of the dipole, implying advection of fluid out of the dipole into the tail. This is directly associated with the fluid entering this midplane near the vortex cores, as is evident from the locally spiral-shaped streamlines. Regions of upwelling flow are observed near the frontal and rear (hyperbolic) stagnation points. The elongated shape of these regions is directly linked with the orientation of the streamlines near these hyperbolic points.

Figure 7(b) shows a numerically obtained distribution of the y -component of the vorticity ω_y in a vertical slice approximately through the instantaneous stagnation point at the tail of the dipole, i.e., $y=6.0$ mm for $t=2.60$ s. Clearly visible is a dipolar vortical structure that is associated with the

up- and downwelling close to this hyperbolic point [see Fig. 7(a)]. The magnitude of the vorticity component ω_y in this comoving vertical slice turns out to evolve to significantly larger values than that of the “primary” vorticity component ω_z . In Fig. 7(c) the frontal circulation is seen in front of the dipole indicated by the negative ω_x , which originates from the bottom boundary layer. Furthermore, a spanwise vortex with high (positive) spanwise vorticity is seen.

To further illustrate the 3D structure inside the vortices during and after the forcing phase, the velocity distribution in a vertical slice defined by the plane $x=-12.5$ mm is displayed in Fig. 8 for $t=0.50$ s, $t=0.96$ s, $t=1.50$ s, $t=2.00$ s, and $t=2.60$ s. Here vectors represent the v - and w -velocity components. The magnitude of the vertical velocity component w has also been indicated by shades/colors. The vertical plane defined by $x=-12.5$ mm is chosen such that it cuts approximately through one of the vortex cores.

FIG. 7. (Color online) (a) Numerically obtained streamline plot for $t=2.60$ s at mid-depth of the 9.3 mm fluid layer (see Table I for the numerical parameters). Shades/colors indicate the magnitude of vertical velocity component. Arrows at the two hyperbolic points indicate the directions of advection. (b) Vertical slice ($y=6.0$ mm for $t=2.60$ s) of the numerically obtained distribution of the ω_y vorticity component. The shades/colors indicates the magnitude of ω_y , while the vectors represent the flow in the xz -plane. (c) Vertical slice through the symmetry plane of the dipole ($x=0$) at time $t=2.60$ s showing the ω_x distribution (shades/color) with vectors representing the flow in the yz -plane.

FIG. 8. (Color online) Numerically obtained snapshots of vertical slice of the plane $x = -12.5$ mm at (a) $t = 0.50$ s, (b) $t = 0.96$ s, (c) $t = 1.50$ s, (d) $t = 2.00$ s, and (e) $t = 2.60$ s (see Table I for the numerical parameters). Vectors represent the v - and w -velocity components, while shades/colors indicate the magnitude of the vertical velocity w .

Figures 8(a) and 8(b) clearly illustrate the buildup of downward motion inside the vortex core, located around $y = 0$, during the forcing phase. For $t = 0.96$ s it is seen that a large recirculation cell is present inside the vortex, with the largest downward motion located near the axis of the vortex at $y = 0$. Shortly after the forcing has stopped [see Fig. 8(c)], a reversal of the vertical velocity just inside the vortex core is observed, i.e., upward motion at approximately $y = 10$ mm [cf. Fig. 5(d)]. Figures 8(c)–8(e) illustrate the complex flow motion in the vertical slice, which persists after the forcing has been switched off. The second reversal of the vertical

velocity is only weakly visible (approximately at $y = 26$ mm) in Fig. 8(e) as the slice $x = -12.5$ mm barely touches the region where this second reversal takes place [see Fig. 5(d)]. Clearly, in view of the vertical structure of the flow field both during the forcing phase and during the postforcing phase, the flow does not behave as a Q2D flow, i.e., planar flow with a Poiseuille-like vertical structure. The emergence of these significant 3D structures during the forcing phase raises serious questions regarding the use of electromagnetic forcing in a shallow fluid flow to study forced Q2D turbulence.

FIG. 9. (Color online) Numerically obtained snapshots of the velocity fields (5 mm measurement height) with stress-free condition at both the bottom and the free surface (see Table I for the numerical parameters) at (a) $t=0.96$ s, (b) $t=1.50$ s, and (c) $t=2.00$ s. Meaning of vectors and shades/colors: see caption of Fig. 5.

C. Alternative initial and boundary conditions

As argued above, the no-slip boundary is not a necessary condition for the 3D structure of the dipolar vortex and its subsequent evolution. To substantiate this, we show in Figs. 9(a)–9(c) and 10 the numerically calculated velocity fields where the bottom and the free surface are both taken to be stress-free, leaving all other parameters and settings unchanged. Comparing Figs. 9(a)–9(c) and 10 with the corresponding results in Secs. IV A and IV B (with a no-slip bottom), one observes that the spatial and temporal evolutions

of the flow is qualitatively the same except the frontal circulation is not present. This frontal circulation does seem to be dependent on the no-slip condition. However, the spanwise vortex with strong positive ω_x [see Fig. 7(c)] is still present. Furthermore, we now observe a higher propagation velocity of the dipole as a consequence of the absence of damping due to the bottom friction. Obviously the no-slip bottom boundary layer is *not* the only actor in the generation of three-dimensionality in the flow. Also the vertical component of the (3D) Lorentz force plays a negligible role in generat-

FIG. 10. (Color online) Numerically obtained snapshots of the velocity in the vertical slice at $x=-12.5$ mm with stress-free condition at both the bottom and the free surface (see Table I for the numerical parameters) for (a) $t=0.96$ s, (b) $t=1.50$ s, and (c) $t=2.00$ s. Meaning of vectors and shades/colors: see caption of Fig. 8.

FIG. 11. Evolution of the horizontal kinetic energy E_H (left axis) and vertical kinetic energy E_V (right axis) at different levels $z=h$ inside the fluid. Numerical results evaluated at (a) $h=2.00$ mm, (b) $h=5.00$ mm, and (c) $h=9.00$ mm above the bottom. Experimental results measured at (d) $h_{ls}=2.00$ mm, (e) $h_{ls}=5.00$ mm, and (f) $h_{ls}=9.00$ mm for fluid depth of $H=9.3$ mm and current strength of $I=4.4$ A.

ing three-dimensionality and associated vertical motion, which was confirmed by numerical simulations with the vertical component of the Lorentz force (artificially) set to zero. It is rather the nonuniformity in the vertical direction of the *horizontal* components of the Lorentz force that results in the observed 3D motion, as this introduces a z -dependence of the vertical vorticity ω_z , or, equivalently, in the swirl velocity v_θ . Of course, during the forcing period, this nonuniformity is the main driving mechanism of vertical motion, and explains why the numerical simulations with no-slip and stress-free bottom condition show such close resemblance [cf. Figs. 8(b) and 10(a)]. However, one should not conclude from this that the observed 3D structures in the postforcing phase are solely due to three-dimensionality introduced by the electromagnetic forcing. Numerical simulations with the flow initialized in an alternative way, e.g., by imposing a Lamb-like planar dipole structure with a purely horizontal flow field that is divergence-free and that satisfies a no-slip bottom boundary with a Poiseuille-like structure in the vertical, show that such an initial dipolar flow evolves similar as observed in the laboratory experiments, i.e., with qualitatively the same 3D flow structures as shown in Figs. 2 and 5.

From the above we conclude that the no-slip boundary condition is not the main cause of three-dimensionality but it is rather the vertical confinement of the shallow layer in

combination with the vertical gradient in the horizontal forcing that leads to the complicated 3D structures.

D. Quantifying two-dimensionality of the shallow flow

The 3D effects in shallow fluid flows may be quantified in different ways. One possibility is to use the kinetic energy ratio $q=E_V/E_H$, where the instantaneous kinetic energies E_H and E_V associated with the horizontal and vertical flow components, respectively, are computed as

$$E_H = \frac{1}{2} \mathcal{H} \iint_S \rho (u^2 + v^2) dx dy, \quad (3a)$$

$$E_V = \frac{1}{2} \mathcal{H} \iint_S \rho w^2 dx dy. \quad (3b)$$

Note that E_H and E_V defined in this way represent the kinetic energy evaluated in a horizontal plane S at $z=h$. The fluid depth H will be used as a measure of the vertical length scale $\mathcal{H}$.

Figures 11(a)–11(c) present the numerically obtained horizontal kinetic energy E_H (solid line) and vertical kinetic energy E_V (dashed line) for a fluid depth of 9.3 mm, evaluated at three different heights $z=h$ above the (no-slip) bot-

FIG. 12. Numerically obtained evolution of the kinetic energy ratio q . (a) Varying evaluation level h , whereas fluid depth ($H=9.3$ mm) and forcing strength ($j_x=0.9$ A/cm²) are kept constant. (b) Varying forcing strength j_x , whereas fluid depth ($H=9.3$ mm) and evaluation level ($h=5.0$ mm) are kept constant. (c) Varying fluid depth H , whereas forcing strength ($j_x \approx 0.9$ A/cm²) is kept constant and evaluated at mid-depth of the fluid (i.e., $h/H \approx 0.5$).

tom, i.e., at $h=2.00$, 5.00 , and 9.00 mm. The second row of Figs. 11(d)–11(f) shows the energies E_H and E_V as measured in an experiment carried out under the same conditions, i.e., with a no-slip bottom. Since the total fluid depth $H=9.3$ mm, Figs. 11(a) and 11(d) show the evolution as evaluated just above the bottom, Figs. 11(b) and 11(e) at approximately mid-depth of the fluid layer, and Figs. 11(c) and 11(f) just below the free surface.

In Figs. 11(a)–11(c) one observes an increase in E_H during the forcing phase for all three evaluation heights. A global maximum is attained at $t=1$ s [for Fig. 11(c) a little bit later] when the forcing stops, after which decay sets in. For the vertical component of the kinetic energy E_V (note the different scaling of the left and right axes in the figures), it is seen that the maximum is reached some time after the forcing has stopped, roughly at $t=1.7$ s. As to be expected, E_V is largest at mid-depth and smaller near the bottom and the free surface. For early times, it is seen that E_V remains small, consistent with the fact that initially the vertical component of the Lorentz force does not drive the flow, as discussed in Sec. IV A.

Comparison of the numerical simulation results [Figs. 11(a)–11(c)] with the corresponding experimentally obtained results [Figs. 11(d)–11(f)] reveals a good qualitative agreement. Quantitatively, differences are seen which we attribute to the matching of the numerical simulations to the experiments, as discussed in Sec. III. Besides these differences, the initial time behavior of E_V and the time at which E_V attains its maximum are nicely captured [except for Fig. 11(f) where the measurement was performed close to the free surface].

Figure 12 shows the evolution of the kinetic energy ratio $q=E_V/E_H$ for three cases to illustrate the influence of (i) the evaluation level [Fig. 12(a)], (ii) the forcing strength [Fig. 12(b)], and (iii) the fluid depth [Fig. 12(c)].

- (i) Figure 12(a) reveals that the energy ratio q is highly dependent on the evaluation level. Near the free surface and no-slip bottom the q -value is minimal due to the impermeability condition, leading to small values of w , whereas the kinetic energy ratio attains a maximum at mid-depth, where w approximately attains its maximum magnitude. Clearly, based on the kinetic

energy ratio q , the free surface is not representative of the two-dimensionality inside the bulk of the fluid. The ratio q attains a maximum after the forcing has been switched off (approximately at $t=1.7$ s) as E_V attains its maximum there [see, e.g., Fig. 11(e)].

- (ii) Figure 12(b) demonstrates the effect of changes in strength of the forcing: An increase in the forcing strength corresponds with an increase in the kinetic energy ratio q .
- (iii) Figure 12(c) shows the numerically obtained evolution of the q -ratio for varying fluid depth H . Apparently, the flow behaves more 2D-like with decreasing fluid depth. However, it should be noted that experimentally there exists a practical lower bound for the fluid depth, as the damping of the flow (due to the bottom friction) becomes stronger for decreasing fluid depth.

Figure 13 displays the numerically obtained ratio Q of the vertical and horizontal kinetic energies contained in the en-

FIG. 13. Numerically simulated evolution of the kinetic energy ratio $Q \cdot D^2/H^2$ for five different fluid depths H . Note that the vertical and horizontal kinetic energy components are computed through integration over the entire 3D domain.

FIG. 14. Numerically calculated evolution of the normalized horizontal divergence Λ at three different evaluation levels (2.0, 5.0, and 9.0 mm). For all cases $H=9.3$ mm and $I=4.4$ A [same settings as Figs. 11(a)–11(c)].

tire 3D fluid volume for decreasing values of the fluid depth H . Since one expects a scaling of the kinetic energy ratio Q with H^2/D^2 , the scaled version $Q \cdot D^2/H^2$ is depicted in Fig. 13. Qualitatively, similar behavior is seen as the q -ratio displayed in Fig. 12(c). Clearly, for the $H=1$ mm fluid depth $Q \approx 0$, the flow is very close to two-dimensionality. However, a strong increase in the Q -value is seen when increasing the fluid depth. This increase is stronger than expected from the scaling, which might be due to the inertial oscillations.

An alternatively quantity that can be used to characterize the deviation from two-dimensionality in the flow is the normalized horizontal divergence Λ , which is computed in a horizontal plane as

$$\Lambda = \frac{\mathcal{H} \iint_S |\nabla_H \cdot \mathbf{u}| dx dy}{\mathcal{L} \iint_S |\omega_z| dx dy}, \quad (4)$$

where ∇_H denotes the divergence with respect to only the horizontal components. In the expression for Λ , ω_z is the vertical component of the vorticity, and the magnet diameter is taken as a measure of the horizontal length scale $\mathcal{L}$. The normalization factor is a measure for the characteristic horizontal velocity.

Figure 14 displays the numerically calculated evolution of Λ at three different levels, i.e., at $h=2.0$, 5.0, and 9.0 mm for the same fluid depth ($H=9.3$ mm) and current strength ($I=4.4$ A) as in Fig. 11. Obviously, the normalized horizontal divergence Λ is nonzero at all three measurement planes, whereas in pure 2D flow it should be zero. The highest Λ -values (maximum over 40%) are observed at the level closest to the surface ($h=9.0$ mm).

It is remarkable that the behavior of the normalized horizontal divergence Λ suggests deviations from Q2D behavior in a different way than that revealed by the kinetic energy ratio q : while q approaches zero at the free surface (where w becomes zero) and attains a maximum at mid-depth of the fluid layer, Λ reaches a minimum at mid-depth and maximum at the free surface.

To investigate the tendency of relaxation to a Poiseuille-like profile of the (3D) velocity field with decreasing fluid

FIG. 15. Numerically obtained temporal evolution of the deviation from Poiseuille-like profile P [see Eq. (B6)] for different values of the fluid depth H . The current density j_x is kept constant at 0.9 A/cm².

depth H , we now consider quantity P derived in Appendix B [see Eq. (B6)] being a measure of the difference between the actual 3D velocity field and a velocity field with a Poiseuille-like structure in the vertical direction. If $P \approx 0$, then the flow is very close to a flow with a Poiseuille-like vertical structure. Figure 15 shows the numerically obtained temporal evolution of P for several fluid depths. The graph displays a sharp decrease in P during the forcing phase. As soon as the forcing stops, a relaxation process sets in (except for $H=1$ mm, which is due to the strong bottom friction). The oscillatory behavior visible in Fig. 15 during this relaxation phase is related to the previously mentioned inertial oscillations. For large times an exponential decay of P is observed whose time scale is comparable to the bottom friction time scale τ_E . The deviation from the Poiseuille-like vertical structure decreases with decreasing fluid depth, although one has to bear in mind that the damping due to the bottom friction becomes stronger with decreasing fluid depth (and may therefore become experimentally unpractical).

V. CONCLUSIONS AND DISCUSSION

Many experiments on 2D or Q2D turbulence have been performed in shallow layers of fluid under the assumption that the shallowness assures sufficient suppression of vertical motions. Moreover, vertical dependence due to the no-slip bottom is commonly modeled through the introduction of Rayleigh damping, which is presumed to be justified by a Poiseuille-like flow profile in the vertical.

In this paper we have examined the validity of these assumptions by studying experimentally as well as numerically the 3D structure of a generic vortex structure in Q2D turbulence, i.e., a dipolar flow in a single shallow layer of electrolyte. Our results show that significant and remarkably complex 3D structures and vertical motions occur throughout the flow evolution, i.e., during and after the forcing.

The development of vertical motion in the dipole is directly linked to the horizontal flow field varying with depth.

The cyclostrophic balance then dictates a pressure that also varies with height, which subsequently drives a vertical motion. Moreover, since free-surface deformation is absent in the numerical model, the close resemblance of the simulations and laboratory experiments implies that surface gravity waves are of minor importance in generating this vertical motion in the laboratory experiments.

Due to the type of boundary conditions, vertical confinement or z -dependent forcing, vertical variations in the horizontal flow field are inevitable in every experimental realization of one- and two-layer shallow flows. The *only* situation in which these 3D flow features with vertical velocities will not develop is a flow with (i) stress-free bottom and surface boundary conditions and (ii) a forcing mechanism that has no vertical component, with its horizontal components uniform over the fluid depth and which generates a horizontal divergence-free velocity field. This is a situation that is certainly hard to achieve in practice with the current experimental setup (although these conditions might be more closely met in soap-film experiments).

Moreover, based on the criteria introduced in this paper it is shown that the vertical structure of such shallow flows relaxes toward a Poiseuille-like profile on a timescale that is comparable to the bottom friction time scale. The latter result questions the use of a linear (Rayleigh) friction model to model a no-slip bottom. Since the vortex dipole can be considered as a prototype flow structure in 2D turbulence, the conclusions of the present study apply more generally to experimental realizations of 2D turbulence, both for the decaying and the forced case.

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APPENDIX A: INITIAL FLOW EVOLUTION

The initial (linear) flow evolution of the electromagnetically forced flow can be analyzed using the observed magnetohydrostatic balance between the vertical component of the Lorentz force and the vertical pressure gradient. This balance is described by

$$\frac{\partial p}{\partial z} = \mathbf{f}_L \cdot \mathbf{e}_z = j_0 \mathbf{B} \cdot \mathbf{e}_y = j_0 B_y. \quad (\text{A1})$$

Since the magnet is circular and assumed to be axially symmetric, the right-hand-side of this balance can also be written as

$$j_0 B_y = j_0 \sin \varphi B_r = -j_0 \sin \varphi \frac{\partial A_\varphi}{\partial z}, \quad (\text{A2})$$

where we have introduced cylindrical coordinates (r, φ, z) . A_φ is the azimuthal component of the vector potential $\mathbf{A}$ for the magnet's magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}$.

Integrating Eq. (A1) with respect to z using Eq. (A2) yields

$$p = -j_0 \sin \varphi A_\varphi(r, z) + C(r, \varphi), \quad (\text{A3})$$

where $r = \sqrt{x^2 + y^2}$.

It is easily verified from Eq. (1), keeping in mind that inside the fluid (i.e., outside of the permanent magnet) the magnetic field is curl-free, that the pressure needs to satisfy Laplace's equation in the initial (linear) regime that we are considering here. Moreover, the integration “constant” $C(r, \varphi)$ must satisfy

$$\nabla^2 C = \nabla^2 (j_0 \sin \varphi A_\varphi) = 0. \quad (\text{A4})$$

Since $C(r, \varphi) \rightarrow 0$ as r approaches infinity, the solution is $C(r, \varphi) \equiv 0$.

From the above we conclude that in the very beginning of the forcing phase, the spatial pressure distribution inside the fluid is not yet governed by dynamical effects but completely determined by the spatial structure of the magnetic field that permeates the fluid, i.e.,

$$p(x, y, z, t) \propto p_0 = -j_0 \frac{y}{r} A_\varphi(r, z) \quad \text{for } t \downarrow 0. \quad (\text{A5})$$

The latter equation implies that the vertical component of the Lorentz force does not drive the flow in the initial period when the electromagnetic forcing is active.

APPENDIX B: DEVIATION FROM POISEUILLE-LIKE FLOW

The numerically obtained 3D velocity field $\mathbf{v}$ can be compared with a Poiseuille-like profile in the following way. First considering the 3D velocity field, i.e.,

$$\mathbf{v} = \mathbf{v}(x, y, z, t) = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z, \quad (\text{B1})$$

one can then define the Poiseuille-like profile $\mathbf{v}_P$ in the following way:

$$\mathbf{v}_P = \mathbf{v}(x, y, z = H, t) \sin\left(\frac{\pi z}{2H}\right). \quad (\text{B2})$$

The Poiseuille-like flow $\mathbf{v}_P$ is thus the 3D velocity field, taken at the surface with a sine-like dependence in the vertical direction. Equations (B1) and (B2) satisfy $\mathbf{v}(x, y, z = H, t) = \mathbf{v}_P(x, y, z = H, t)$ and $\mathbf{v}(x, y, z = 0, t) = \mathbf{v}_P(x, y, z = 0, t) = 0$.

Now consider the rms value as a measure of the difference between the 3D velocity field and the Poiseuille-like profile, i.e.,

$$F(t, H) = \sqrt{\langle (u - u_P)^2 + (v - v_P)^2 \rangle}, \quad (\text{B3})$$

which is a function of time t and the fluid depth H under consideration. A spatial averaging is implied by the angular brackets. Substitution of the planar components of Eqs. (B1) and (B2) in Eq. (B3) leads to

$$F^2 = \frac{1}{V} \int \int \int_V \left\{ \left[u(x, y, z, t) - u(x, y, z = H, t) \sin\left(\frac{\pi z}{2H}\right) \right]^2 + \left[v(x, y, z, t) - v(x, y, z = H, t) \sin\left(\frac{\pi z}{2H}\right) \right]^2 \right\} dx dy dz, \quad (\text{B4})$$

where V is the volume of the computational domain. The latter equation is normalized with the rms value of the horizontal components of the 3D velocity field

$$F_{\text{norm}}^2 = \frac{1}{V} \int \int \int_V [u^2(x, y, z, t) + v^2(x, y, z, t)] dx dy dz. \quad (\text{B5})$$

A measure P of the relative difference between the 3D velocity field and the Poiseuille-like flow is defined according to

$$P = \sqrt{\frac{F^2}{F_{\text{norm}}^2}}. \quad (\text{B6})$$

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