A class of degenerate pseudo-parabolic equations: existence, uniqueness of weak solutions, and error estimates for the Euler-implicit discretization

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A class of degenerate pseudo-parabolic equations: existence, uniqueness of weak solutions, and error estimates for the Euler-implicit discretization

by

Y. Fan, I.S. Pop
A class of degenerate pseudo-parabolic equations: existence, uniqueness of weak solutions, and error estimates for the Euler-implicit discretization

Y. Fan, I.S. Pop

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Abstract

In this paper, we investigate a class of degenerate pseudo-parabolic equations. Such equations model two-phase flow in porous media where dynamic effects are included in the capillary pressure. The existence and uniqueness of a weak solution are proved, and error estimates for an Euler implicit time discretization are obtained.

1 Introduction

In this paper, we focus on the following pseudo-parabolic equation:

\begin{equation}
  u_t + \nabla \cdot \vec{F}(u) = \nabla \cdot (H(u)\nabla u) + \tau \Delta u_t,
\end{equation}

where \( H : \mathbb{R} \to [0, +\infty) \) is smooth and bounded. Note that in particular \( H \) may become 0 for some value of \( u \), we call this situation degenerate. A two-phase porous media flow taking into account dynamic effects in the phase pressure difference is proposed in [12],

\begin{equation}
  u_t + \nabla \cdot \vec{F}(u) = \nabla \cdot (H(u)\nabla p) + \tau \partial_t u,
\end{equation}

with \( p = p_c(u) + \tau \partial_t u \). Clearly, (1.1) is a simpler version of (1.2), as the degeneracy is encountered only in the second order term. Here we study the existence and uniqueness of weak solutions to (1.1), complemented with initial and boundary conditions. We do so by applying a discretization in time, for which we also give error estimates.

Pseudo-parabolic equations arise in many real life applications such as radiation with time delay [17], degenerate double-diffusion models [3], heat conduction models [23] and models for lightning propagation [2], etc. Existence and uniqueness of weak solutions to nonlinear pseudo-parabolic equations are proved in [20], while existence of weak solutions for some degenerate cases is studied in [18], [19]. A nonlinear parabolic-Sobolev equation is studied in [25]. In [21], homogenization of a pseudoparabolic system is considered. Travelling wave solutions and their relation to non-standard shock solutions to hyperbolic conservation laws are investigated in [4], [7] for linear higher order terms. This analysis is pursued in [6] for degenerate situations. Numerical schemes for dynamic capillary effects in heterogeneous porous media are given in [13] and a numerical scheme for the pore-scale simulation of crystal dissolution and precipitation in porous media is studied in
The case of discontinuous initial data is analyzed in [5]. Superconvergence of a finite element approximation to similar equation is investigated in [1] and time-stepping Galerkin methods are analyzed in [10] and [11], where two difference-approximation schemes are considered. In [22], Fourier spectral methods for pseudo-parabolic equations are analyzed.

The analysis below is carried out under the following assumptions:

1. (A1) $\Omega$ is an open, bounded and connected domain in $\mathbb{R}^d$, with Lipschitz continuous boundary. With $T > 0$ given we denote $Q = \Omega \times (0,T]$.

2. (A2) $\tau$ is a given strictly positive number.

3. (A3) The vector valued $\vec{F}$ satisfies $\vec{F} = \vec{v} f(u)$, where $\vec{v} \in \mathbb{R}^d$ is a fixed vector. The functions $f$ and $H$ are $C^{1,1}$ satisfying $0 \leq f \leq 1$, $0 \leq H \leq M$ for some $M > 0$. We denote $L$ an upper bound for the Lipschitz constants of $f, H, f', H'$.

**Remark 1.1** We take $\vec{v} \in \mathbb{R}^d$ for the ease of presentation. However, the results below can be extended to more general cases, such as $\vec{v}$ is a divergence free vector field, or $\vec{F}$ is a given $C^{1,1}$ vector valued function.

In this paper, we use standard notations. In particular, $L^2(\Omega)$ stands for the square Lebesgue integrable functions on $\Omega$, $W^{1,2}(\Omega)$ requests the same also for the derivatives of first order. $W^{1,2}_0(\Omega)$ is a subset of $W^{1,2}(\Omega)$ whose elements have zero boundary values. Furthermore, $W^{-1,2}(\Omega)$ is the dual space of $W^{1,2}_0(\Omega)$.

The initial and boundary conditions of (1.1) are given as follows:

\begin{align*}
(1.3) & \quad \ u(\cdot,0) = u^0, \quad \text{and} \quad u|_{\partial\Omega} = 0,
\end{align*}

where $u^0 \in W^{1,2}_0(\Omega)$. We seek a weak solution to the following

**Problem P** Find $u \in W^{1,2}(0,T;W^{1,2}_0(\Omega))$ such that

\begin{align*}
(1.4) & \quad \int_0^T \int_{\Omega} u_t \phi dx dt - \int_0^T \int_{\Omega} \vec{F}(u) \nabla \phi dx dt \\
& \quad + \int_0^T \int_{\Omega} H(u) \nabla u \nabla \phi dx dt + \tau \int_0^T \int_{\Omega} \nabla u_t \nabla \phi dx dt = 0,
\end{align*}

for any $\phi \in L^2(0,T;W^{1,2}_0(\Omega))$.

This paper is organized as follows: Section 2 provides the existence of weak solutions to **Problem P**. The uniqueness of the weak solution is proved in Section 3. In Section 4, some error estimates for an Euler implicit time discretization scheme are obtained, and in Section 5, an iterative approach for solving the time discretization nonlinear problems is discussed and some numerical computations are given to verify the theoretical results. In the last section, some conclusions are given.
2 Existence

We show the existence of a weak solution to Problem P by the method of Rothe (see [14]), based on the Euler implicit time stepping. Before defining the time discretization we mention the following elementary inequality, which will be used several times later:

\[(2.1) \quad ab \leq \frac{1}{2\delta} a^2 + \frac{\delta}{2} b^2, \quad \text{for any} \quad a, b \in \mathbb{R} \quad \text{and} \quad \delta > 0.\]

2.1 Time discretization

With \(N \in \mathbb{N}\), let \(\Delta t = T/N\) and consider the following:

**Problem P\(_{n+1}\)** Given \(u^n \in W^{1,2}_0(\Omega), n \in \{0, 1, 2, ..., N-1\}\), find \(u^{n+1} \in W^{1,2}_0(\Omega)\) such that

\[(2.2) \quad (u^{n+1} - u^n, \phi) + \Delta t(\nabla \cdot \vec{F}(u^{n+1}), \phi) + \Delta t(H(u^{n+1})\nabla u^{n+1}, \nabla \phi) + \tau(\nabla (u^{n+1} - u^n), \nabla \phi) = 0,\]

for any \(\phi \in W^{1,2}_0(\Omega)\), here \((\cdot, \cdot)\) means \(L^2\) inner product. Note that this is the weak formulation of

\[(2.3) \quad \frac{u^{n+1} - u^n}{\Delta t} + \nabla \cdot \vec{F}(u^{n+1}) = \nabla (H(u^{n+1})\nabla u^{n+1}) + \tau \Delta \frac{(u^{n+1} - u^n)}{\Delta t}.\]

We have the following:

**Lemma 2.1** Problem P\(_{n+1}\) has a solution.

**Proof.** Note that \(u^{n+1}\) can be identified formally with the solution of the following equation:

\[(2.4) \quad -\nabla \cdot ((\Delta tH(X) + \tau)\nabla X) + \Delta t\nabla \cdot \vec{F}(X) + X - u^n + \tau \Delta u^n = 0.\]

If \(u^n \in C^{2,1}_0(\Omega)\), Theorem 8.2 from Chapter 4 in [16] provides the existence of \(u^{n+1} = X \in C^{2,1}_0(\Omega)\) solving (2.4).

If \(u^n \in W^{1,2}_0(\Omega)\), there exists a sequence \(\{u^n_k\}_{k \in \mathbb{N}} \subseteq C^{2,1}_0(\Omega)\) converging to \(u^n\) in \(W^{1,2}(\Omega)\). Solving (2.4) gives the sequence \(\{X_k\}_{k \in \mathbb{N}} \subseteq C^{2,1}_0(\Omega)\) with \(u^n_k\) instead of \(u^n\). Consider the weak form of (2.4):

\[(2.5) \quad \Delta t(H(X_k)\nabla X_k, \nabla \phi) + \tau(\nabla X_k, \nabla \phi) - \Delta t(\vec{F}(X_k), \nabla \phi) + (X_k, \phi) = (u^n_k, \phi) + \tau(\nabla u^n_k, \nabla \phi),\]

for any \(\phi \in W^{1,2}_0(\Omega)\).

Taking \(\phi = X_k \in W^{1,2}_0(\Omega)\) with \(\vec{F}(X_k) = \int_0^{X_k} \vec{F}(v)dv\) gives

\[(2.6) \quad (\vec{F}(X_k), \nabla X_k) = \int_\Omega \vec{F}(X_k)\nabla X_k dx = \int_{\partial \Omega} \vec{u} \cdot \vec{F}(0) dx = 0,\]

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where \( \vec{\nu} \) is the outer normal vector to \( \partial \Omega \). By (2.1),

\[
\tau \int_{\Omega} |\nabla X_k|^2 \, dx + \int_{\Omega} |X_k|^2 \, dx \leq \tau \int_{\Omega} |\nabla u^n_k|^2 \, dx + \int_{\Omega} |u^n_k|^2 \, dx \leq C,
\]

where \( C \) is a positive constant.

By the construction of \( \{u^n_k\} \), we have

\[
(u^n_k, \phi) \to (u^n, \phi),
\]
\[
(\nabla u^n_k, \nabla \phi) \to (\nabla u^n, \nabla \phi).
\]

for any \( \phi \in W_0^{1,2}(\Omega) \).

Further, since \( \{X_k\}_{k\in\mathbb{N}} \) and \( \{\nabla X_k\}_{k\in\mathbb{N}} \) are uniformly bounded in \( L^2(\Omega) \), there exists a subsequence (still denoted as \( X_k \)) weakly converging to some \( X \) in \( W_0^{1,2}(\Omega) \). Clearly,

\[
(X_k, \phi) \to (X, \phi),
\]
\[
(\nabla X_k, \nabla \phi) \to (\nabla X, \nabla \phi),
\]
\[
(\vec{F}(X_k), \nabla \phi) \to (\vec{F}(X), \nabla \phi),
\]

for any \( \phi \in W_0^{1,2}(\Omega) \).

Define

\[
\mathcal{H}(y) := \int_0^y H(v) \, dv.
\]

Since \( X_k \to X \) strongly in \( L^2(\Omega) \) and according to (A3), we know that \( \mathcal{H}(X_k) \to \mathcal{H}(X) \) strongly in \( L^2(\Omega) \). Further, \( \mathcal{H}(X_k) \) is uniformly bounded in \( W_0^{1,2}(\Omega) \). Therefore

\[
(\nabla \mathcal{H}(X_k), \nabla \phi) \to (\nabla \mathcal{H}(X), \nabla \phi).
\]

Then from (2.8), (2.9), (2.10), (2.11), (2.12) and (2.14), \( X \) is a solution to **Problem P\( ^{n+1}_n \)**.

**Lemma 2.2** The solution of **Problem P\( ^{n+1}_n \)** is unique, at least if \( \Delta t \) is small enough.

**Proof**. Assume we have two solutions \( X \) and \( Y \). Define

\[
\mathcal{G}(y) = \int_0^y (H(v) + \frac{\tau}{\Delta t}) \, dv,
\]

and subtract the equation for \( Y \) from the equation for \( X \), taking \( \phi = \mathcal{G}(X) - \mathcal{G}(Y) \) in the result gives

\[
\Delta t ||\nabla (\mathcal{G}(X) - \mathcal{G}(Y))||_{L^2(\Omega)}^2 - \Delta t (\vec{F}(X) - \vec{F}(Y), \nabla (\mathcal{G}(X) - \mathcal{G}(Y))) + (X - Y, \mathcal{G}(X) - \mathcal{G}(Y)) = 0.
\]

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Therefore,

\[ (2.17) \Delta t \|
\n\| \nabla(G(X) - G(Y)) \|_{L^2(\Omega)}^2 + \frac{\tau}{\Delta t} \|
\n\| X - Y \|_{L^2(\Omega)}^2 \leq \frac{\Delta t}{2} \| L \| X - Y \|_{L^2(\Omega)}^2 + \| \nabla(G(X) - G(Y)) \|_{L^2(\Omega)}^2. \]

If \( \Delta t^2 < \frac{\tau}{\tau} \),

\[ (2.18) \|
\n\| X - Y \|_{L^2(\Omega)} = 0, \]

implying the uniqueness.

### 2.2 A priori estimates

Having established the existence for the time discretization problems, we proceed with investigating Problem P. To this end, we obtain some a priori estimates.

**Lemma 2.3** For any \( n \in \{0, 1, ..., N - 1\} \), we have:

\[ (2.19) ||u^{n+1}||_{L^2(\Omega)}^2 + \tau ||\nabla u^{n+1}||_{L^2(\Omega)}^2 \leq C, \]

\[ (2.20) ||u^{n+1} - u^n||_{L^2(\Omega)}^2 + \tau ||\nabla(u^{n+1} - u^n)||_{L^2(\Omega)}^2 \leq C(\Delta t)^2, \]

here \( C \) denotes a positive constant.

**Proof.**

1. Taking \( \phi = u^{n+1} \) in (2.2) gives

\[ (2.21) \|
\n\| u^{n+1} \|_{L^2(\Omega)}^2 + \tau \|\nabla u^{n+1} \|_{L^2(\Omega)}^2 + \Delta t \int_\Omega H(u^{n+1})||\nabla u^{n+1}||^2 dx = (u^n, u^{n+1}) + \tau (\nabla u^n, \nabla u^{n+1}). \]

Since \( u^{n+1} \) vanishes on \( \partial \Omega \), with \( F(u^{n+1}) = \int_0^{u^{n+1}} \bar{F}(v) dv \) we have

\( (\nabla \cdot \bar{F}(u^{n+1}), u^{n+1}) = - \int_\Omega \bar{F}(u^{n+1})\nabla u^{n+1} dx = \int_{\partial \Omega} \bar{v} \cdot \bar{F}(0) dx = 0, \)

together with (2.1) yields

\[ (2.22) \||u^{n+1}||_{L^2(\Omega)}^2 + \tau \|\nabla u^{n+1} \|_{L^2(\Omega)}^2 \leq ||u^n||_{L^2(\Omega)}^2 + \tau ||\nabla u^n||_{L^2(\Omega)}^2. \]

Since \( u^0 \in W_0^{1,2}(\Omega) \), this implies

\[ (2.23) \||u^n||_{L^2(\Omega)}^2 + \tau ||\nabla u^n||_{L^2(\Omega)}^2 \leq C. \]

2. Taking \( \phi = u^{n+1} - u^n \) in (2.2) gives

\[ (2.24) \|u^{n+1} - u^n\|^2_{L^2(\Omega)} - \Delta t(\bar{F}(u^{n+1}), \nabla (u^{n+1} - u^n)) + \Delta t(H(u^{n+1})\nabla u^{n+1}, \nabla (u^{n+1} - u^n)) + \tau \|\nabla(u^{n+1} - u^n)||_{L^2(\Omega)}^2 = 0, \]

Using (2.1) and the boundedness of \( \bar{F} \), we have

\[ (2.25) \|u^{n+1} - u^n\|^2_{L^2(\Omega)} + \frac{\tau}{2} \|\nabla(u^{n+1} - u^n)||_{L^2(\Omega)}^2 \leq C(\Delta t)^2. \]
Remark 2.1 From the proof of (2.19), if we define \( \|u^n\|^2 = \|u^n\|^2_{L^2(\Omega)} + \tau \|\nabla u^n\|^2_{L^2(\Omega)} \), then \( \|u^n\| \) decreases as \( n \) increases. Further, from (2.20) one immediately obtains

\[
\sum_{k=1}^{N} ||u^k - u^{k-1}||^2_{L^2(\Omega)} \leq C \Delta t,
\]
(2.26)

\[
\sum_{k=1}^{N} ||\nabla(u^k - u^{k-1})||^2_{L^2(\Omega)} \leq C \Delta t.
\]
(2.27)

2.3 Existence

To show the existence of a solution to Problem P, we start by defining

\[
U_N(t) = u^{k-1} + \frac{t - t^{k-1}}{\Delta t} (u^k - u^{k-1}), \quad \text{and} \quad \overline{U}_N(t) = u^k,
\]
(2.28)

for \( t^{k-1} = (k-1)\Delta t \leq t < t^k = k\Delta t, k = 1, 2...N \). We have the following result:

**Theorem 2.1** Problem P has a solution.

**Proof.** According to the a priori estimates in Lemma 2.3,

\[
\int_0^T ||U_N(t)||^2_{L^2(\Omega)} dt = \sum_{k=1}^{N} \int_{t^{k-1}}^{t^k} ||u^{k-1} + \frac{t - t^{k-1}}{\Delta t} (u^k - u^{k-1})||^2_{L^2(\Omega)} dt
\]

\[
\leq 2 \sum_{k=1}^{N} \int_{t^{k-1}}^{t^k} (||u^{k-1}||^2_{L^2(\Omega)} + ||u^k - u^{k-1}||^2_{L^2(\Omega)}) dt
\]

\[
\leq C.
\]
(2.29)

Similarly,

\[
\int_0^T ||\nabla U_N(t)||^2_{L^2(\Omega)} dt \leq C,
\]
(2.30)

\[
\int_0^T ||\partial_t U_N||^2_{L^2(\Omega)} dt = \frac{1}{\Delta t} \sum_{k=1}^{N} ||u^k - u^{k-1}||^2_{L^2(\Omega)} \leq C,
\]
(2.31)

and

\[
\int_0^T ||\partial_t \nabla U_N||^2_{L^2(\Omega)} dt = \sum_{k=1}^{N} \int_{t^{k-1}}^{t^k} \frac{1}{\Delta t} ||\nabla(u^k - u^{k-1})||^2_{L^2(\Omega)} dt
\]

\[
= \frac{1}{\Delta t} \sum_{k=1}^{N} ||\nabla(u^k - u^{k-1})||^2_{L^2(\Omega)} \leq C.
\]
(2.32)

Therefore \( \{U_N\}_{N \in \mathbb{N}} \) is uniformly bounded in \( W^{1,2}(0,T; W_0^{1,2}(\Omega)) \), so it has a subsequence (still denoted as \( \{U_N\} \)) that converges weakly to some \( U \in W^{1,2}(0,T; W_0^{1,2}(\Omega)) \). Therefore
$U_N$ converges strongly to $U$ in $L^2(Q)$.

We now exploit a general principle that relates the piecewise linear and the piecewise constant interpolation (see e.g. [15] for a proof of the corresponding lemma): if one interpolation converges strongly in $L^2(Q)$, then the other interpolation also converges strongly in $L^2(Q)$. From the convergence of $U_N$, we conclude that $U_N$ also converges strongly in $L^2(Q)$. With $H$ defined in (2.13), the boundedness of $H$ implies that $H(U_N)$ is uniformly bounded in $L^2(0,T;W^{1,2}_0(\Omega))$. As in the proof of Lemma 2.1, one gets

$$\nabla H(U_N) \rightharpoonup \nabla H(U).$$

(2.33)

From (2.2), we know

$$\int_0^T \int_\Omega \partial_t U_N(t)\phi dx dt - \int_0^T \int_\Omega \tilde{F}(U_N(t))\nabla \phi dx dt$$

$$+ \int_0^T \int_\Omega \nabla H(U_N(t))\nabla \phi dx dt + \tau \int_0^T \int_\Omega \partial_t \nabla U_N(t)\nabla \phi dx dt = 0,$$

(2.34)

for any $\phi \in L^2(0,T;W^{1,2}_0(\Omega))$.

Using the weak convergence of $U_N$ and $H(U_N)$, we consider a sequence $\Delta t \to 0$ and pass to the limit in (2.34). This shows that $U$ is a solution to Problem $P$. $\square$

**Remark 2.2** As will be proved in the following section, the solution of Problem $P$ is unique. Therefore the convergence holds along any $\Delta t \searrow 0$.

### 3 Uniqueness

Here we show that the solution to Problem $P$ is unique. To do so, we use the following result (see e.g. Chapter 6 in [9]):

**Proposition 3.1** Let $g \in L^2(\Omega)$. The equation

$$-\Delta G = g \quad \text{in} \quad \Omega,$$

(3.1)

with boundary condition $G|_{\partial \Omega} = 0$ has a unique weak solution $G \in W^{1,2}_0(\Omega)$, satisfying

$$||\nabla G||_{W^{1,2}(\Omega)} = ||g||_{W^{-1,2}(\Omega)} \leq C ||g||_{L^2(\Omega)}.$$

(3.2)

We use this for proving the uniqueness result:

**Theorem 3.1** The solution of Problem $P$ is unique.

**Proof.** Assume $u$ and $v$ are two solutions, we have $(u - v)(\cdot, 0) = 0$ and for any $\tilde{t} > 0$,

$$\int_0^{\tilde{t}} \int_\Omega (u - v)\phi dx dt - \int_0^{\tilde{t}} \int_\Omega (\tilde{F}(u) - \tilde{F}(v))\nabla \phi dx dt$$

$$+ \int_0^{\tilde{t}} \int_\Omega \nabla (H(u) - H(v))\nabla \phi dx dt + \tau \int_0^{\tilde{t}} \int_\Omega \nabla (u - v)\nabla \phi dx dt = 0,$$

(3.3)
for any $\phi \in L^2(0, T; W_0^{1,2}(\Omega))$.

Taking $g = u - v$ in Proposition 3.1, there exists a $G_{u-v} \in W_0^{1,2}(\Omega)$ such that

$$
(\nabla G_{u-v}, \nabla \psi) = (u - v, \psi),
$$

for any $\psi \in W_0^{1,2}(\Omega)$, satisfying

$$
||G_{u-v}||_{W^{1,2}(\Omega)} \leq C ||u - v||_{L^2(\Omega)}.
$$

Note that through $u$ and $v$, $G_{u-v}$ also depends on $t$. First, by (3.4) for any $\tilde{t} > 0$

$$
\int_0^{\tilde{t}} \int_{\Omega} (u - v) G_{u-v} dx dt
\leq \int_{\Omega} (u - v) G_{u-v} \big|_0^{\tilde{t}} dx - \int_0^{\tilde{t}} \int_{\Omega} (u - v) \partial_t G_{u-v} dt dx
\leq \int_{\Omega} |\nabla G_{u-v}|^2 \big|_0^{\tilde{t}} dx - \int_0^{\tilde{t}} \int_{\Omega} \nabla G_{u-v} \nabla \partial_t G_{u-v} dt dx
= \frac{1}{2} \int_{\Omega} |\nabla G_{u-v}(\cdot, \tilde{t})|^2 dx,
$$

as $G_{u-v}(\cdot, 0) = 0$. Further, by (A3)

$$
\int_0^{\tilde{t}} \int_{\Omega} (\vec{F}(u) - \vec{F}(v)) \nabla G_{u-v} dx dt \leq C \int_0^{\tilde{t}} \int_{\Omega} |u - v||\nabla G_{u-v}| dx dt \leq C \int_0^{\tilde{t}} \int_{\Omega} |u - v|^2 dx dt.
$$

Next the monotonicity of $\mathcal{H}$ implies

$$
\int_0^{\tilde{t}} \int_{\Omega} \nabla (\mathcal{H}(u) - \mathcal{H}(v)) \nabla G_{u-v} dx dt = \int_0^{\tilde{t}} \int_{\Omega} (\mathcal{H}(u) - \mathcal{H}(v))(u - v) dx dt \geq 0.
$$

Finally,

$$
\tau \int_0^{\tilde{t}} \int_{\Omega} \partial_t \nabla (u - v) \nabla G_{u-v} dx dt
= \tau \int_0^{\tilde{t}} \int_{\Omega} \partial_t (u - v)(u - v) dx dt
= \frac{\tau}{2} \int_{\Omega} (u - v)(\cdot, \tilde{t})^2 dx.
$$

Therefore taking $\phi = G_{u-v}$ in (3.3) gives

$$
\frac{1}{2} ||\nabla G_{u-v}(\cdot, \tilde{t})||_{L^2(\Omega)}^2 + \frac{\tau}{2} ||(u - v)(\cdot, \tilde{t})||_{L^2(\Omega)}^2 \leq C \int_0^{\tilde{t}} \int_{\Omega} |u - v|^2 dx dt.
$$

By Gronwall’s inequality, $||(u - v)(\cdot, \tilde{t})||_{L^2(\Omega)} = 0$. Since $\tilde{t}$ is arbitrary, this gives uniqueness.
4 Error estimates

From the above we see that the approximating sequence $U_N$ converges strongly to $U$ in $L^2(Q)$. In this section we will estimate the error $U_N - U$. Recalling (3.3), we have

\[
\int_0^T \int_\Omega \partial_t U_N(t) \phi dx dt - \int_0^T \int_\Omega \bar{F}(U_N(t)) \nabla \phi dx dt \\
+ \int_0^T \int_\Omega \nabla H(U_N(t)) \nabla \phi dx dt + \tau \int_0^T \int_\Omega \partial_t \nabla U_N(t) \nabla \phi dx dt = 0.
\]

Denote

\[
e_u(t) = u(t) - U_N(t), \quad \text{and} \quad e_H(t) = H(u(t)) - H(U_N(t)).
\]

Obviously, $e_u, e_H \in W^{1,2}_0(\Omega)$ and $e_u(\cdot, 0) = e_H(\cdot, 0) = 0$.

**Theorem 4.1** The following estimate holds:

\[
||e_u||_{L^{\infty}(0,T; L^2(\Omega))} \leq C \Delta t.
\]

**Proof**. Subtracting (4.1) from (1.4) gives

\[
\int_0^t \int_\Omega \partial_t e_u \phi dx dt - \int_0^t \int_\Omega (\bar{F}(u(t)) - \bar{F}(U_N(t))) \nabla \phi dx dt \\
+ \int_0^t \int_\Omega \nabla H(u(t)) - \nabla H(U_N(t)) \nabla \phi dx dt + \tau \int_0^t \int_\Omega \partial_t \nabla e_u \nabla \phi dx dt = 0.
\]

Taking $g = e_u$ in Proposition 3.1 provides a $G_{e_u} \in W^{1,2}_0(\Omega)$ satisfying

\[
(\nabla G_{e_u}, \nabla \psi) = (e_u, \psi),
\]

for any $\psi \in W^{1,2}_0(\Omega)$, and

\[
||G_{e_u}||_{W^{1,2}(\Omega)} \leq C ||e_u||_{L^2(\Omega)}.
\]

We will use $G_{e_u}$ as test function in (4.4). As in Section 3 we have for any $\tilde{t} > 0$

\[
\int_0^\tilde{t} \int_\Omega \partial_t e_u G_{e_u} dx dt = \frac{1}{2} \int_\Omega (\nabla G_{e_u}(\cdot, \tilde{t}))^2 dx = \frac{1}{2} ||e_u(\tilde{t})||_{W^{-1,2}}^2.
\]

Further,

\[
\int_0^\tilde{t} \int_\Omega (\bar{F}(u(t)) - \bar{F}(U_N(t))) \nabla G_{e_u} dx dt \\
\leq C_1 \int_0^\tilde{t} ||e_u||_{L^2(\Omega)}^2 dx dt + \int_0^\tilde{t} \int_\Omega (\bar{F}(U_N(t)) - \bar{F}(U_N(t))) \nabla G_{e_u} dx dt \\
\leq C_1 \int_0^\tilde{t} ||e_u||_{L^2(\Omega)}^2 dx dt + C_2 \int_0^\tilde{t} ||U_N - U_N||_{L^2(\Omega)} ||\nabla G_{e_u}||_{L^2(\Omega)} dt
\]
Since $U_N - \overline{U}_N = \frac{u_{k-1} - u_k}{\Delta t}$, for $t \in (t_{k-1}, t_k)$. By (2.26), we get $||U_N - \overline{U}_N||_{L^2(\Omega)} \leq C\Delta t$, therefore

\begin{equation}
\int_0^\ell \int_\Omega (\overline{F}(u(t)) - \overline{F}(\overline{U}_N(t)))\nabla G_{e_u} dx dt \\
\leq (C_1 + \frac{1}{2}) \int_0^\ell ||e_u||^2_{L^2(\Omega)} dx dt + C_3(\Delta t)^2
\end{equation}

Similarly,

\begin{equation}
\int_0^\ell \int_\Omega \nabla (H(u(t)) - H(\overline{U}_N(t)))\nabla G_{e_u} dx dt \\
= \int_0^\ell \int_\Omega \nabla e_H \nabla G_{e_u} dx dt + \int_0^\ell \int_\Omega \nabla (H(U_N(t)) - H(\overline{U}_N(t)))\nabla G_{e_u} dx dt \\
\geq \int_0^\ell \int_\Omega \nabla (H(U_N(t)) - H(\overline{U}_N(t)))\nabla G_{e_u} dx dt \\
= \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \int_\Omega \left(H(U_N(t)) - H(\overline{U}_N(t))\right) e_u dt dx \\
\geq -C \int_{t_{k-1}}^{t_k} \int_\Omega \left|u_k - u_{k-1}\right|^2 + \left|e_u\right|^2 dt dx \\
\geq -C(\Delta t)^2 \frac{1}{2} \int_\Omega \int_0^\ell \left|e_u\right|^2 dx dt,
\end{equation}

and

\begin{equation}
\tau \int_0^\ell \int_\Omega \partial_t \nabla e_u \nabla G_{e_u} dx dt = \tau \int_\Omega \partial_t e_u e_u dx dt = \frac{\tau}{2} \int_\Omega e_u(\cdot, \tilde{t})^2 dx.
\end{equation}

Using above, taking $\psi = G_{e_u}$ in (4.4) gives

\begin{equation}
\frac{1}{2} \int_\Omega (\nabla G_{e_u}(\cdot, \tilde{t}))^2 dx + \frac{\tau}{2} \int_\Omega e_u(\cdot, \tilde{t})^2 dx \leq C_1(\Delta t)^2 + C_2 \int_0^\ell \int_\Omega \left|e_u\right|^2 dx dt.
\end{equation}

Using Gronwall’s inequality, we obtain the estimate

\begin{equation}
||e_u||_{L^\infty(0,T;L^2(\Omega))} \leq C\Delta t.
\end{equation}

$\square$

\textbf{Remark 4.1} From (4.13), since $H$ is Lipschitz continuous, we immediately obtain

\begin{equation}
||e_H(\cdot, t)||_{L^\infty(0,T;L^2(\Omega))} \leq C\Delta t.
\end{equation}
5 Numerical example

In this section, we give a numerical example to verify the theoretical findings. We solve the following equation in $\mathbb{Q} = (0, 1) \times (0, 1)$

\begin{equation}
\frac{\partial u}{\partial t} = \frac{1}{6} \frac{\partial}{\partial x} ([u]_+ + \partial_x u) + \frac{1}{6} \frac{\partial^3 u}{\partial x^3 \partial t} - \frac{1}{2(1 + t)^2},
\end{equation}

with initial and boundary conditions

\begin{equation}
u(x, 0) = x(1 - x), \quad u(0, t) = u(1, t) = 0.
\end{equation}

Here

\begin{equation}[u]_+ = \begin{cases} u & \text{if } u > 0, \\ 0 & \text{if } u \leq 0. \end{cases}
\end{equation}

therefore the equation becomes degenerate whenever $u \leq 0$. For the equation (5.1), the exact solution is

\begin{equation}u(x, t) = \frac{x(1 - x)}{1 + t}.
\end{equation}

In the following, we use this solution to test the numerical scheme.

5.1 Numerical scheme

Before giving the numerical results, we present an iterative scheme to solve the time discretization problems. To do so, taking $\Delta t = 1/N (N \in \mathbb{N})$ and denoting $f(t) = \frac{1}{2(1 + t)^2}$, formally we get

\begin{equation}\frac{u^n - u^{n-1}}{\Delta t} = \frac{1}{6} \partial_x ([u^n]_+ + \partial_x u^n) + \frac{1}{6} \partial_{xx}(u^n - u^{n-1}) - f(t^n).
\end{equation}

Define the Kirchhoff transform

\begin{equation}v = \beta(u) := \frac{1}{6} \int_0^u (\Delta t[s]_+ + 1)ds = \begin{cases} \frac{\Delta t}{12} u^2 + \frac{1}{6} u, & \text{if } u > 0 \\ \frac{1}{6} u, & \text{if } u \leq 0, \end{cases}
\end{equation}

instead of solving (5.5), we seek $v^n = \beta(u^n)$ such that

\begin{equation}\beta^{-1}(v^n) - \partial_{xx} v^n = u^{n-1} - \frac{1}{6} \partial_{xx} u^{n-1} - \Delta t f(t^n).
\end{equation}

with $v^n = 0$ at $x = 0$ and $x = 1$. To solve (5.7), we use the following iteration method inspired from [26], pp. 90-100 (also see e.g. [8], [24]):

\begin{equation}6v^{n,i} - \partial_{xx} v^{n,i} = 6v^{n,i-1} - \beta^{-1}(v^{n,i-1}) + \alpha(u^{n-1}, v^n),
\end{equation}
where \( i = 1, 2 \ldots \) and

\[
\alpha(u^{n-1}, t^n) = u^{n-1} - \frac{1}{6} \partial_{xx} u^{n-1} - \Delta t f(t^n).
\]

This iteration requires a starting point \( v^{n,0} \). As will be proved below, the iteration is convergent for any \( v^{n,0} \). However, for the practical reasons, we choose \( v^{n,0} = v^{n-1} = \beta(u^{n-1}) \).

**Lemma 5.1** The iteration method (5.8) is convergent in the \( W^{1,2}(0,1) \) norm.

**Proof**. We write (5.9) in weak form, find \( v^{n,i} \in W^{1,2}_0(0,1) \) such that

\[
(6u^{n,i}, \phi) + (\partial_x v^{n,i}, \partial_x \phi) = (6v^{n,i-1} - \beta^{-1}(v^{n,i-1}), \phi) + (\alpha(u^{n-1}, t^n), \phi).
\]

for any \( \phi \in W^{1,2}_0(0,1) \). Similarly,

\[
(6v^{n,i-1}, \phi) + (\partial_x v^{n,i-1}, \partial_x \phi) = (6v^{n,i-2} - \beta^{-1}(v^{n,i-2}), \phi) + (\alpha(u^{n-1}, t^n), \phi).
\]

Subtracting (5.10) from (5.11),

\[
6(v^{n,i} - v^{n,i-1}, \phi) + (\partial_x(v^{n,i} - v^{n,i-1}), \partial_x \phi)
\]

\[
= 6(v^{n,i-1} - v^{n,i-2}, \phi) - (\beta^{-1}(v^{n,i-1}) - \beta^{-1}(v^{n,i-2}), \phi).
\]

Taking \( \phi = v^{n,i} - v^{n,i-1} \) gives,

\[
6\|v^{n,i} - v^{n,i-1}\|^2_{L^2(\Omega)} + \|\partial_x(v^{n,i} - v^{n,i-1})\|^2_{L^2(\Omega)} \leq \|v^{n,i} - v^{n,i-1}\|^2_{L^2(\Omega)} \cdot 6\|v^{n,i-1} - v^{n,i-2}\|^2_{L^2(\Omega)} - (\beta^{-1}(v^{n,i-1}) - \beta^{-1}(v^{n,i-2}))\|L^2(\Omega).
\]

From the definition of \( \beta \), we have

\[
\beta'(u) = \begin{cases} 
\frac{1}{6}(u\Delta t + 1), & \text{if } u \geq 0 \\
\frac{1}{6}, & \text{otherwise}.
\end{cases}
\]

Therefore

\[
(\beta^{-1})'(v) = \frac{1}{\beta'(u)} \in (0, 6].
\]

From (5.13), we obtain

\[
6\|v^{n,i} - v^{n,i-1}\|^2_{L^2(\Omega)} + \|\partial_x(v^{n,i} - v^{n,i-1})\|^2_{L^2(\Omega)} \leq 6\|v^{n,i} - v^{n,i-1}\|^2_{L^2(\Omega)} \cdot 6\|v^{n,i-1} - v^{n,i-2}\|^2_{L^2(\Omega)}.
\]

Using Poincaré inequality, \( ||u||_{L^2(0,1)} \leq ||\partial_x u||_{L^2(0,1)} \) for any \( u \in W^{1,2}_0(0,1) \). Therefore

\[
(5.16)\|v^{n,i} - v^{n,i-1}\|^2_{L^2(\Omega)} + \frac{1}{6}\|\partial_x(v^{n,i} - v^{n,i-1})\|^2_{L^2(\Omega)}
\]

\[
\leq \frac{1}{2}\|v^{n,i} - v^{n,i-1}\|^2_{L^2(\Omega)} + ||v^{n,i-1} - v^{n,i-2}||^2_{L^2(\Omega)}
\]

\[
\leq \frac{1}{2}\|v^{n,i} - v^{n,i-1}\|^2_{L^2(\Omega)} + \frac{3}{8}\|v^{n,i-1} - v^{n,i-2}\|^2_{L^2(\Omega)} + \frac{1}{8}\|\partial_x(v^{n,i-1} - v^{n,i-2})\|^2_{L^2(\Omega)}
\]

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Define $\|v^{n,i}\|^2 = \|v^{n,i} - v^{n,i-1}\|^2_{L^2(\Omega)} + \frac{1}{3} \|\partial_x (v^{n,i} - v^{n,i-1})\|^2_{L^2(\Omega)}$ (equivalent to the $W^{1,2}$ norm), we obtain

$$\|v^{n,i}\|^2 \leq \frac{3}{4} \|v^{n,i-1}\|^2,$$

using Banach fixed point theorem, we obtain the convergence of the iteration method (5.9).

5.2 Numerical results

We compute the numerical solution $u^N$ of (5.1) and estimate the error $e_u = u - u_N$, with $u$ the exact solution of (5.1). For simplicity, we only compute $e_u$ at $t = 1$. To this aim, finite difference scheme on uniform mesh with $dx = 10^{-5}$ is coupled with different time stepping $dt = 10^{-1}, 10^{-2}, 10^{-3}$ and $10^{-4}$. To solve the nonlinear problem at any two steps, we perform 3 to 4 iterations. This is sufficient to achieve $\|v^{n,i} - v^{n,i-1}\|_{L^2(\Omega)} \leq 10^{-5}$. The numerical results are presented in Table 1. As follows from Theorem 4.1, the error satisfies

$$\|e_u(\cdot, 1)\|_{L^2(\Omega)} \leq C \Delta t.$$

This is confirmed by the Table 1. In particular, we estimate $C$ to 0.066.

<table>
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<tr>
<th>$dt$</th>
<th>$|e_u(\cdot, 1)|_{L^2(\Omega)}$</th>
<th>ratio($|e_u|/dt$)</th>
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<tr>
<td>$10^{-1}$</td>
<td>$6.1997 \times 10^{-3}$</td>
<td>$6.1997 \times 10^{-2}$</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>$6.447 \times 10^{-4}$</td>
<td>$6.447 \times 10^{-2}$</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>$6.4632 \times 10^{-5}$</td>
<td>$6.4632 \times 10^{-2}$</td>
</tr>
<tr>
<td>$10^{-4}$</td>
<td>$6.5842 \times 10^{-6}$</td>
<td>$6.5842 \times 10^{-2}$</td>
</tr>
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</table>

Table 1: Errors $e_u(\cdot, 1)$ for different $dt$

Figure 1 also displays numerical solutions for $u(\cdot, 1)$ at $t = 1$, compared to the exact solution for $dt = 10^{-1}$ and $dt = 10^{-2}$. For $dt = 10^{-3}$ and $dt = 10^{-4}$, one could not distinguish between the numerical solution and the analytical one.

6 Conclusion

In this paper, a class of degenerate pseudo-parabolic equations is investigated. This involves a vanishing nonlinear factor in the second order differential operator. We employ the Rothe method for proving the existence of a solution, and use a Green function approach for the uniqueness. Further, we estimate the error between the exact and the time discrete solution. Finally, these theoretical estimates are confirmed by a numerical example.

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Figure 1: Numerical solution and exact solution for $dx = 10^{-5}, dt = 10^{-1}$ (left) and $dt = 10^{-2}$ (right)

References


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