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A Note on Unser–Zerubia Generalized Sampling Theory Applied to the Linear Interpolator

A. J. E. M. Janssen and T. Kalker

Abstract—In this correspondence, we calculate the condition number of the linear operator that maps sequences of samples $f(2k)$, $f(2k+a)$, $k \in \mathbb{Z}$ of an unknown continuous $f \in L^2(\mathbb{R})$ consistently (in the sense of the Unser–Zerubia generalized sampling theory) onto the set of continuous, piecewise linear functions in $L^2(\mathbb{R})$ with nodes at the integers as a function of $a \in (0, 2)$. It turns out that the minimum condition numbers occur at $a = \sqrt{2/3}$ and $a = 2 - \sqrt{2/3}$ and not at $a = 1$ as we might have expected.

Index Terms—Deinterlacing, error analysis, interpolation, motion estimation, sampling, video.

I. INTRODUCTION AND ANNOUNCEMENT OF RESULTS

Generalized sampling theory can successfully be applied to deinterlacing of television images. The interlaced video format entails that odd and even television lines, referred to as the odd and even *fields*, are sampled at different times. To create a progressive television image, even and odd lines, collectively referred to as a *frame*, are sampled at the same moment. In order to deinterlace a field, it is therefore necessary to reconstruct the missing even or odd lines of a progressive frame from preceding or following fields. If there is no motion in the scene, this reconstruction process is trivial. In general, however, the scene is not static. A common approach to deinterlacing in this nonstatic case is the assumption that successive frames are locally related by uniform motion. The image samples in successive fields can then be viewed as a nonuniform sampling of the underlying progressive frames. Viewed in this way, deinterlacing can be mathematically modeled as the reconstruction of uniformly spaced sampling values $f(k)$, $k \in \mathbb{Z}$ of an unknown function $f(x)$ sampled at positions $2k$ and $2k+a$, $k \in \mathbb{Z}$. The velocity v of the scene and the parameter a are related by $a = 1 - v$. For reasons of symmetry, we may assume without loss of generality that $0 \leq a < 1$. For an example of a successful application of this approach, see [1].

The parameter a is directly related to the assumed motion in the scene. However, this motion model is usually only an approximation, and even if the model were exact, the parameter a could only be estimated. Moreover, the sampling values of the interlaced fields usually have a large noise component. It is therefore relevant to

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Fig. 1. Condition number for discrete reconstruction as a function of the interlace parameter a .

study, as in [2], the stability of the sequence of reconstructed sampling values. Intuitively, we would expect that the most stable reconstruction is obtained if there is no motion in the scene, i.e., if $a = 1$. In this case, reconstruction amounts to copying of sampling values. For $a \neq 1$, a simple and practical solution for consumer devices is the use of a linear interpolator function. However, directly applying the stability theory of [2] to this situation leads to the somewhat surprising result that the most stable situation occurs at $a = \sqrt{2/3}$. The derivation of this result is presented in Section III.

So what is wrong? As it turns out, nothing is wrong. Both our intuition and the theory of [2] are correct. The explanation of these counterintuitive results is based on the fact that the theory of [2] does not directly apply to the case of deinterlacing. The deinterlacing problem can be described as the study of the discrete operator $Q_a: l_2^2 \rightarrow l_2$, mapping nonuniform sampling values to uniform sampling values $c(k)$, $k \in \mathbb{Z}$. The theory of [2], on the other hand, studies the construction of the piecewise linear function

$$\tilde{f}(x) = E(\{c(k)\}) = \sum_{k \in \mathbb{Z}} c(k) \varphi(x - k) \quad (1)$$

i.e., an operator $\tilde{Q}_a = E \circ Q_a: l_2^2 \rightarrow L_2$. The function $\varphi(x)$ in (1) above is the interpolator for piecewise linear functions and is equal to the centered linear spline $\beta^{(1)}(x)$.

In the case of linear interpolation, the operator Q_a is represented by the matrix [see (10)]

$$Q_a(z) = \frac{1}{a} \begin{bmatrix} a & 0 \\ a-1 & 1 \end{bmatrix} \quad (2)$$

where we have used the isomorphism $\iota: l_2 \rightarrow l_2^2$, $\{c(k)\} \rightarrow (\{c(2k)\}, \{c(2k+1)\})$ to view Q_a as an operator $l_2^2 \rightarrow l_2^2$. Following the analysis of [3], the stability of the reconstruction $\alpha_a^{(l_2^2)}$ is measured by the condition number of the matrix $A_a^{(l_2^2)}(z) = Q^T(z^{-1})Q(z)$. It is easy to verify that maximal stability (i.e., a minimal condition number) is obtained for $a = 1$. Fig. 1 gives a graphical representation of the dependency of the condition number as a function of the shift parameter a .

In [2], it is shown that the condition of the operator $\tilde{Q}_a: l_2^2 \rightarrow L_2$ not only involves the matrix Q_a but the autocorrelation properties of the linear interpolator function $\varphi(x)$ as well. In particular, all

Fig. 2. Condition number for continuous reconstruction as a function of the interlace parameter a .

computations are based on the matrix (see [2, (18)])

$$\mathbf{A}_a(\mathbf{z}) = \mathbf{Q}_a^T(\mathbf{z}^{-1})\mathbf{A}_\varphi\mathbf{Q}_a(\mathbf{z}) \quad (3)$$

where the entries of $\mathbf{A}_\varphi$ record the autocorrelation properties of the interpolator function $\varphi(x)$. Stability is measured in this case by the square root of the condition number α_a of the matrix $\mathbf{A}_a(\mathbf{z})$. As opposed to the discrete case, the computation of the minimal α_a is quite involved. However, the computations of Section III show that an analytical solution is possible. In particular, it follows from those computations that maximal stability is reached for $a = \sqrt{2/3}$ (see also Fig. 2). This is conclusive evidence that linear interpolation in the discrete domain and continuous domain are really different from the point of view of stability.

A similar counterintuitive result as above does not hold for some higher order interpolators. For example, as shown in [2], the cardinal cubic spline and sinc interpolator achieve maximal stability in the continuous domain at $a = 1$. The corresponding discrete domain operators Q_a have, of course, the same property. We are not aware of any general condition on interpolator functions that imply stability at $a = 1$.

The organization of this correspondence is as follows. In Section II, we give a short overview of the most important ingredients of the Unser-Zerubia theory. In Section III, we compute analytically the value a for which stability is maximal. This correspondence ends with a short summary of the achieved results.

II. DEINTERLACING IN THE UNSER-ZERUBIA THEORY

For the sake of simplicity, we will assume that we are dealing with 1-D digital video. We will also assume that the video captures a uniformly moving scene $f(x, t) = f(x - vt)$, where v is the motion of the scene. This scene is sampled at time instances $t \in \mathbb{Z}$, but it is sampled differently for t odd and even. If t is even, the scene is sampled at $x \in 2\mathbb{Z}$ (the even field), and if t is odd, the scene is sampled at $2\mathbb{Z} + 1$ (the odd field). If we only consider the sampling moments $t = 0$ and $t = 1$, we obtain two different samplings of the function $f(x)$, viz. the sequences $\{f(2k)\}$ and $\{f(2k + 1 - v)\}$, where $k \in \mathbb{Z}$. We may assume without loss of generality that $0 \leq v \leq 1$; if necessary, the function $f(x)$ is reversed to $f(-x)$ and/or an integer shift is applied to the odd field.

The problem of deinterlacing can now be formulated concisely either as the reconstruction of the function $f(x)$ or, in a more restrictive formulation, as the construction of the *progressive frame* $f(k)$, $k \in \mathbb{Z}$. As deinterlacing is particularly relevant for consumer devices, only simple interpolation functions can be used, in order to limit costs. A common choice is to use bilinear interpolation, using the first order spline $\beta^{(1)}$ as interpolator. Setting $a = 1 - v$, we are

interested in the stability of the reconstruction process. As shown in the previous section, the reconstruction of the discrete progressive frame is most stable if $a = 1$, that is, if there is no motion. In order to analyze the stability for the continuous case, we formulate the deinterlacing problem in terms of the Unser-Zerubia theory.

The sampling of the odd and even fields is modeled by sampling functions $\phi_1(x) = \delta(x)$ and $\phi_2(x) = \delta(x - a)$ (see [2, Sec. III-C]). The interpolator function is the centered linear spline $\beta^{(1)}(x)$. The first step in the stability analysis is the construction of the matrix $\mathbf{Q}(\mathbf{z})$ as the inverse of the matrix $\mathbf{A}_{\phi, \varphi}(\mathbf{z})$

$$[\mathbf{A}_{\phi, \varphi}]_{i,j}(\mathbf{k}) = \langle \varphi(\mathbf{x}), \phi_i(\mathbf{x} - 2\mathbf{k} + (\mathbf{j} - 1)) \rangle \quad i, j \in \{0, 1\}. \quad (4)$$

Second, the autocorrelation matrix $\mathbf{A}_\varphi(\mathbf{z})$ needs to be computed, where

$$[\mathbf{A}_\varphi]_{i,j}(\mathbf{k}) = \langle \varphi(\mathbf{x} - \mathbf{i} - 2\mathbf{k}), \varphi(\mathbf{x} - \mathbf{j}) \rangle, \quad i, j \in \{0, 1\}. \quad (5)$$

Third, the dual autocorrelation matrix $\mathbf{A}_{\tilde{\phi}}(\mathbf{z})$ is computed as

$$\mathbf{A}_{\tilde{\phi}}(\mathbf{z}) = \mathbf{Q}^T(\mathbf{z}^{-1})\mathbf{A}_\varphi(\mathbf{z})\mathbf{Q}(\mathbf{z}). \quad (6)$$

Fourth, the largest and smallest eigenvalues, λ_+ and λ_- , respectively, of $\mathbf{A}_{\tilde{\phi}}(\mathbf{z})$ over the unit circle have to be computed. Note that $\mathbf{A}_{\tilde{\phi}}(e^{j\omega})$ is Hermitian and positive definite by construction and that therefore, $\lambda_{\pm}$ is non-negative. The condition number α is then computed as

$$\alpha = \sqrt{\frac{\lambda_+}{\lambda_-}}. \quad (7)$$

In Section III-B and C this condition number α is analyzed as a function $\alpha(a)$ of the parameter a . In particular, it is proven that the minimum condition number is obtained for $a = \sqrt{2/3}$.

III. DERIVATIONS

A. Computation of Eigenvalues

We have

$$(\mathbf{A}_{\phi, \varphi}(\mathbf{k}))_{i,j} = \varphi(2\mathbf{k} + (\mathbf{i} - 1)\mathbf{a} - (\mathbf{j} - 1)) \quad i, j = 1, 2 \quad (8)$$

so that

$$\mathbf{A}_{\phi, \varphi}(\mathbf{z}) = \begin{bmatrix} 1 & 0 \\ 1 - a & a \end{bmatrix} \quad (9)$$

and

$$\mathbf{Q}(\mathbf{z}) = \mathbf{A}_{\phi, \varphi}^{-1}(\mathbf{z}) = \frac{1}{a} \begin{bmatrix} a & 0 \\ a - 1 & 1 \end{bmatrix}. \quad (10)$$

Now, using Kronecker's delta $\delta_{p,q}$, we have

$$\mathbf{A}_\varphi(\mathbf{k}) = \begin{bmatrix} \frac{2}{3}\delta_{k,0} & \frac{1}{6}\delta_{k,0} + \frac{1}{6}\delta_{k,1} \\ \frac{1}{6}\delta_{k,0} + \frac{1}{6}\delta_{k,-1} & \frac{2}{3}\delta_{k,0} \end{bmatrix}, \quad \mathbf{k} \in \mathbb{Z} \quad (11)$$

so that

$$\mathbf{A}_\varphi(\mathbf{z}) = \frac{1}{6} \begin{bmatrix} 4 & 1 + z^{-1} \\ 1 + z & 4 \end{bmatrix}. \quad (12)$$

Then, (6), (10), and (12) yield

$$\mathbf{A}_{\tilde{\phi}}(\mathbf{z}) = \frac{1}{6a^2} \begin{bmatrix} 10a^2 - 10a + 4 + a(a-1)(z + z^{-1}) & 5a - 4 + az^{-1} \\ 5a - 4 + az & 4 \end{bmatrix}. \quad (13)$$

With

$$x = \frac{1}{2}(z + z^{-1}) = \cos(\omega) \in [-1, 1] \quad (14)$$

we compute the two eigenvalues $\lambda_{\pm}(x)$ of $\mathbf{A}_{\tilde{\phi}}(\mathbf{z})$ as

$$6a^2\lambda_{\pm}(x) = B_1 \pm \sqrt{B_2} \quad (15)$$

where

$$B_1 = 5a^2 - 5a + 4 + a(a-1)x \quad (16)$$

and

$$B_2 = 36(a-2/3)^2 + a^2(1-a)^2(5+x)^2 - 2a(5a-4)(1-x). \quad (17)$$

It is easily seen that B_2 is a smooth and strictly positive function of $(a, x) \in [0, 1] \times [-1, 1] \setminus (1, -1)$ and that it vanishes at $(a, x) = (1, -1)$.

B. Analysis of Eigenvalues

With $y = 5+x \in [4, 6]$, we can write the right-hand side of (15) as

$$\psi_{\pm}(y) := 4 - a(1-a)y \pm \sqrt{E} \quad (18)$$

where

$$E = 16 - 24a^2 + 2a(5a-4)y + a^2(1-a)^2y^2. \quad (19)$$

Differentiating with respect to y , we find

$$\psi'_{\pm}(y) = \frac{1}{\sqrt{E}} \left(-a(1-a)\sqrt{E} \pm (a(5a-4) + a^2(1-a)^2y) \right). \quad (20)$$

Now, observe that

$$\begin{aligned} (a(1-a)\sqrt{E})^2 - (a(5a-4) + a^2(1-a)^2y)^2 \\ = a^3(8 - 33a + 48a^2 - 24a^3) \end{aligned} \quad (21)$$

is independent of y . Thus, when we take an $a \in (0, 1)$ such that

$$P(a) := 8 - 33a + 48a^2 - 24a^3 \neq 0 \quad (22)$$

we see that $\psi'_{\pm}(y)$, being the sum/difference of two smooth, real expressions of unequal modulus, is of one sign. The unique root $a_0 \in (0, +1]$ of $P(a) = 0$ can be computed as

$$\begin{aligned} a_0 &= \frac{2}{3} + \frac{1}{6} \left(\left(\frac{3}{2\sqrt{2}} + 1 \right)^{1/3} - \left(\frac{3}{2\sqrt{2}} - 1 \right)^{1/3} \right) \\ &= 0.813269147. \end{aligned} \quad (23)$$

Hence, when $a \neq a_0$, the extrema of $\psi_{\pm}(y)$ occur at $y = 4, 6$. By smoothness of $\psi_{\pm}(y)$ (as a function of $(a, y) \in (0, 1) \times [4, 6]$), we have that $\arg \max_y(\psi_+(y))$ and $\arg \max_y(\psi_-(y))$ are constant as a function of a in the intervals $(0, a_0)$ and $(a_0, 1)$ with a possible jump at $a = a_0$. An analysis for small $a > 0$ and small $1-a > 0$ then shows that

$$\psi_+(y) \text{ is maximal at } y = 4 \ (x = -1), \quad 0 < a < a_0 \quad (24)$$

$$\psi_+(y) \text{ is maximal at } y = 6 \ (x = +1), \quad a_0 < a < 1 \quad (25)$$

$$\psi_-(y) \text{ is minimal at } y = 6 \ (x = +1), \quad 0 < a < 1. \quad (26)$$

C. Computation of Extreme Eigenvalues and Condition Numbers

It follows from (15), (18), (24)–(26) and some simplifications that

$$\begin{aligned} 6a^2\lambda_{\max} &= \psi_+(4) \\ &= 4 \left(a^2 - a + 1 + \sqrt{(a^2 - a + 1)^2 - a^2} \right) \\ &\quad 0 < a < a_0 \end{aligned} \quad (27)$$

$$\begin{aligned} 6a^2\lambda_{\max} &= \psi_+(6) \\ &= 6 \left(a^2 - a + \frac{2}{3} + \sqrt{(a^2 - a + \frac{2}{3})^2 - \frac{1}{3}a^2} \right) \\ &\quad a_0 < a < 1 \end{aligned} \quad (28)$$

$$\begin{aligned} 6a^2\lambda_{\min} &= \psi_-(6) \\ &= 6 \left(a^2 - a + \frac{2}{3} - \sqrt{(a^2 - a + \frac{2}{3})^2 - \frac{1}{3}a^2} \right) \\ &\quad 0 < a < 1. \end{aligned} \quad (29)$$

We thus obtain that

$$\begin{aligned} \frac{\lambda_{\max}}{\lambda_{\min}} &= 2 \left(\frac{a^2 - a + 1}{a} + \sqrt{\left(\frac{a^2 - a + 1}{a} \right)^2 - 1} \right) \\ &\quad \cdot \left(\frac{a^2 - a + 2/3}{a} + \sqrt{\left(\frac{a^2 - a + 2/3}{a} \right)^2 - \frac{1}{3}} \right) \end{aligned} \quad (30)$$

for $0 < a < a_0$, and

$$\frac{\lambda_{\max}}{\lambda_{\min}} = 3 \left(\frac{a^2 - a + 2/3}{a} + \sqrt{\left(\frac{a^2 - a + 2/3}{a} \right)^2 - \frac{1}{3}} \right)^2 \quad (31)$$

for $a_0 < a < 1$. The function $(a^2 - a + 1)/a$ decreases for $0 < a \leq 1$, and the function $(a^2 - a + 2/3)/a$ decreases for $0 < a \leq \sqrt{2/3} = 0.816496581$ and increases for $a \geq \sqrt{2/3}$. Note that $\sqrt{2/3} > a_0$. Hence, (30) decreases for $0 < a \leq a_0$, and (31) decreases for $a_0 < a < \sqrt{2/3}$ and increases for $a \geq \sqrt{2/3}$. It follows that the minimum of $\lambda_{\max}/\lambda_{\min}$ is assumed at $a = \sqrt{2/3}$, yielding the minimum condition number

$$\begin{aligned} \alpha_{\min} &= \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} \\ &= \sqrt{3} \left(2\sqrt{\frac{2}{3}} - 1 + \sqrt{\left(2\sqrt{\frac{2}{3}} - 1 \right)^2 - \frac{1}{3}} \right) \\ &= 1.54586606. \end{aligned} \quad (32)$$

For comparison, we have from (15) at $a = 1$

$$6\lambda_{\pm}(x) = 4 \pm \sqrt{2+2x} \quad (33)$$

so that

$$\begin{aligned} \lambda_{\max} &= 6, \quad \lambda_{\min} = 2 \\ \alpha &= \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} = \sqrt{3} = 1.73205081. \end{aligned} \quad (34)$$

See Fig. 2 for a graphical representation of the condition number as a function of the interlace parameter α .

IV. CONCLUSIONS

In this correspondence, we have argued that from the point of view of stability, reconstructions in the discrete and continuous domains are essentially different. This has been verified through the example of deinterlacing using a bilinear interpolator.

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Generalized Transfer Function Estimation Using Evolutionary Spectral Deblurring

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Abstract—We present a method for estimating the generalized transfer function (GTF) of a time-varying filter from a time–frequency representation (TFR) of its output. This method uses the fact that many TFR's can be written as blurred versions of the GTF. The approach minimizes the error between the TFR found from the data and that found by blurring the GTF. The problem as such has many solutions. We, therefore, additionally constrain it to minimize the distance between the GTF-based spectrum and the autoterms of the Wigner distribution, suppressing the cross terms using an appropriate signal dependent mask function. To illustrate the performance of the proposed procedure, we apply it to the spectral representation of speech signals and to signal masking.

Index Terms—Bilinear distributions, deblurring, deconvolution, evolutionary spectrum, generalized transfer function, time–frequency analysis, time–frequency representations.

I. INTRODUCTION

According to the Wold–Cramer representation [14], [17], a nonstationary signal $x(n)$ can be considered to be the output of a causal LTV system, with impulse response $h(n, m)$, and driven by stationary white noise $e(n)$ so that

$$x(n) = \sum_{m=-\infty}^n h(n, m)e(m). \quad (1)$$

Replacing $e(n)$ in (1) by its spectral representation [17], we have that

$$x(n) = \int_{-\pi}^{\pi} H(n, \omega) e^{j\omega n} dZ(\omega) \quad (2)$$

where $H(n, \omega)$ is the generalized transfer function (GTF) [7], [16] on the unit circle, and $Z(\omega)$ is an orthogonal increment process. The Wold–Cramer evolutionary spectrum $S_{ES}(n, \omega)$ of $x(n)$ [14], [17]

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is then obtained by considering the time-dependent variance of $x(n)$, yielding

$$S_{ES}(n, \omega) = |H(n, \omega)|^2. \quad (3)$$

Detka *et al.* [5] have shown that the spectrogram [16] and Cohen's class of bilinear distributions [3], [4] are related to the evolutionary spectrum (ES) and the generalized transfer function. In fact, replacing (2) into the general form of the bilinear distributions [4]

$$S_{BD}(n, \omega) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} W[l-n, k] x^*(l-k) x(l+k) e^{-j\omega 2k} \quad (4)$$

we find that these distributions are related to the generalized transfer function by

$$E\{S_{BD}(n, \omega)\} = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} W[l-n, k] \int_{-\pi}^{\pi} H^*(l-k, \lambda) H(l+k, \lambda) e^{-j(\omega-\lambda)2k} d\lambda \quad (5)$$

where $E\{\cdot\}$ is the expected value operator, and $W[\cdot, \cdot]$ is a weighting function with finite support. Equation (5) can be expressed as the blurring relation

$$E\{S_{BD}(n, \omega)\} = \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} S_{ES}(k, \lambda) G(n-k, \omega-\lambda; H) d\lambda \quad (6)$$

where the blurring function

$$G(n-k, \omega-\lambda; H) = \sum_{k'=-\infty}^{\infty} \left[\frac{W[k+k'-n, k'] H(2k'+k, \lambda)}{H(k, \lambda)} e^{-j(\omega-\lambda)2k} \right] \quad (7)$$

not only depends on $W[\cdot, \cdot]$ but also on $H(\cdot, \cdot)$. In this correspondence, we exploit the relation in (5) to estimate the GTF and, consequently, the ES using a deconvolution approach. The deconvolution problem of interest consists of the following: Given a signal $x(n)$ and a bilinear distribution $E\{S_{BD}(n, \omega)\}$, we wish to obtain an $H(n, \omega)$ such that when blurred according to (5), it results in the given bilinear distribution.

Pitton *et al.* [15], who derived a blurring relation similar to (6) using the spectrogram and the ES, circumvent the dependence of the blurring function on the GTF by assuming that the process under consideration is stationary within the spectrogram window. While such an assumption simplifies the blurring function, it can result in an incomplete deblurring. In our method, we do not assume stationarity and also differ with current deconvolution techniques [21], [15] in that a) we can choose any TFR to deblur, and b) we estimate the generalized transfer function $H(n, \omega)$ rather than the TFR. The generalized transfer function is used not only to obtain positive estimates of the deblurred TFR using (3) but for signal reconstruction, masking, and filtering of nonstationary signals as well.

II. ESTIMATING THE GENERALIZED TRANSFER FUNCTION

In this section, we present a deconvolution procedure to estimate the generalized transfer function. We first consider unconstrained deconvolution and observe that, whereas the resulting estimate of $H(n, \omega)$ has some interesting properties, it does not display changes in frequency with respect to time. To reduce the space of solutions, we then perform a constrained deconvolution.