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A partial synchronization theorem

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When synchronization sets in, coupled systems oscillate in a coherent way. It is possible to observe also some intermediate regimes characterized by incomplete synchrony which are referred to as partial synchronization. The paper focuses on analysis of partial synchronization in networks of linearly coupled oscillators. © 2008 American Institute of Physics. [DOI: 10.1063/1.2959145]

A complex network of coupled oscillators can exhibit a phenomenon called partial synchronization or clusterization. This phenomenon is characterized by coherent behavior of some oscillators forming the clusters, while between the clusters there might be no apparent agreement. The partial synchronization occurs in networks possessing symmetry in the couplings. The paper studies the conditions leading to partial synchronization.

I. INTRODUCTION

Synchronization of dynamical systems is a topic that attracted attention during the last two–three decades. A large number of examples of synchronization in nature can be found in Refs. 1–3, and references therein. A particular interest in this field is the so-called partial synchronization, or clustering, that is characterized by an agreement between several nodes of the networks, see, e.g., Refs. 4–8.

If a given network of oscillators has a topology with some symmetries, this network can exhibit clustering phenomena that are characterized by existence and stability of invariant manifolds in the network. An approach initiated in Ref. 4 and developed further by Belykh, Belykh, and co-workers, see, e.g., Refs. 5–7, is based on a graph-theoretical approach to find and characterize those manifolds with subsequent application of the direct Lyapunov method to prove its stability. Another approach which was initiated in Ref. 9 is more algebraic by nature and is aimed at simultaneous study of existence and stability of those manifolds in a unified framework.

The goal of this publication is to further develop this algebraic approach to avoid a commutation condition imposed in Ref. 9. For networks of complex topology this condition can be conservative as illustrated in this paper.

This paper follows the same lines as our previous work;⁹ however to make the presentation self-contained all necessary background material is presented here in a compact way. We refer to Ref. 9 where the reader can find some explanations of the definitions that will be used in this publication.

The paper is organized as follows: The problem statement is explained in Sec. II, where the network dynamics is outlined. In Sec. III the association between the symmetry and the linear invariant manifolds of the network is discussed. Section IV begins with some background material

from control theory, after which we propose a proof of asymptotic stability of a compact subset of a specified linear invariant manifold. Section V contains an illustrative example.

Throughout the paper we use the following notations: I_k denotes the $k \times k$ identity matrix. The Euclidean norm in $\mathbb{R}^n$ is denoted simply as $|\cdot|$, $|x|^2 = x^T x$, where T defines transposition. The notation $\text{col}(x_1, \dots, x_n)$ stands for the column vector composed of the elements $x_1, \dots, x_n$. This notation will also be used in case where the components x_i are vectors too. A function $V: X \rightarrow \mathbb{R}_+$ defined on a subset X of $\mathbb{R}^n$, $0 \in X$ is positive definite if $V(x) > 0$ for all $x \in X \setminus \{0\}$ and $V(0) = 0$. It is radially unbounded (if $X = \mathbb{R}^n$) or proper if $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$. If a quadratic form $x^T P x$ with a symmetric matrix $P = P^T$ is positive definite, then the matrix P is called positive definite. For positive definite matrices we use the notation $P > 0$; moreover $P > Q$ means that the matrix $P - Q$ is positive definite. For matrices A and B the notation $A \otimes B$ (the Kronecker product) stands for the matrix composed of submatrices $A_{ij}B$, i.e.,

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nn}B \end{pmatrix}$$

where A_{ij} , $i, j = 1 \dots n$, stands for the ij th entry of the $n \times n$ matrix A .

II. PROBLEM STATEMENT

Consider k identical systems of the form

$$\dot{x}_j = f(x_j) + Bu_j, \quad y_j = Cx_j, \quad (1)$$

where f is a smooth vector field, $j = 1, \dots, k$, $x_j(t) \in \mathbb{R}^n$ is the state of the j th system, $u_j(t) \in \mathbb{R}^m$ and $y_j(t) \in \mathbb{R}^m$ are, respectively, the input and the output of the j th system, and B, C are constant matrices of appropriate dimension. We assume that matrix CB is similar to a positive definite matrix, and the k systems are interconnected through mutual linear output coupling,

$$u_j = -\gamma_{j1}(y_j - y_1) - \gamma_{j2}(y_j - y_2) - \cdots - \gamma_{jk}(y_j - y_k), \quad (2)$$

where γ_{ij} are constants. With no loss of generality we assume in the sequel that CB is a positive definite matrix.

Define the $k \times k$ matrix Γ as

$$\Gamma = \begin{pmatrix} \sum_{i=2}^k \gamma_{1i} & -\gamma_{12} & \cdots & -\gamma_{1k} \\ -\gamma_{21} & \sum_{i=1, i \neq 2}^k \gamma_{2i} & \cdots & -\gamma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{k1} & -\gamma_{k2} & \cdots & \sum_{i=1}^{k-1} \gamma_{ki} \end{pmatrix} \quad (3)$$

where all row sums are zero. With definition (3), the collection of k systems (1) with feedback (2) can be rewritten in the more compact form

$$\dot{x} = F(x) + (I_k \otimes B)u, \quad y = (I_k \otimes C)x, \quad (4)$$

with the feedback given by

$$u = -(\Gamma \otimes I_m)y, \quad (5)$$

where we denoted $x = \text{col}(x_1, \dots, x_k)$, $F(x) = \text{col}[f(x_1), \dots, f(x_k)] \in \mathbb{R}^{kn}$, $y = \text{col}(y_1, \dots, y_k)$, and $u = \text{col}(u_1, \dots, u_k) \in \mathbb{R}^{km}$.

All main points have now been introduced in order to formulate a clear problem statement. Can we exploit symmetry in the network to identify its linear invariant manifolds, and benefit from a representation of the system as (2,3), and/or (4,5), typical for control purposes, in order to give conditions that guarantee stability of some selected partial (or the full) synchronized states?

III. SYMMETRIES AND INVARIANT MANIFOLDS

If a given network possesses a certain symmetry, this symmetry must be present in matrix Γ . In particular, the network may contain some repeating patterns, when considering the arrangements of constants γ_{ij} , hence the permutation of some elements will leave the network unchanged. The matrix representation of a permutation σ of the set $\{1, 2, \dots, k\}$ is a permutation matrix $\Pi \in \mathbb{R}^{k \times k}$. Permutation matrices are orthogonal, i.e., $\Pi^T \Pi = I_k$, and they form a group with respect to the multiplication, so for any two permutation matrices Π_i, Π_j of the same size, $\Pi_i \Pi_j$ is a permutation matrix too.

Rewrite the dynamics of (4,5) in the closed loop form

$$\dot{x} = F(x) + Gx, \quad (6)$$

where $G = -(I_k \otimes B)(\Gamma \otimes I_m)(I_k \otimes C) \in \mathbb{R}^{kn \times kn}$, that can be simplified as $G = -\Gamma \otimes BC$. Let us recall here that given a dynamical system as Eq. (6), the linear manifold $\mathcal{A}_M = \{x \in \mathbb{R}^{kn} : Mx = 0\}$, with $M \in \mathbb{R}^{kn \times kn}$, is invariant if $M\dot{x} = 0$ whenever $Mx = 0$. That is, if at a certain time t_0 a trajectory is on the manifold, $x(t_0) \in \mathcal{A}_M$, then it will remain there for all time, $x(t) \in \mathcal{A}_M$ for all t . The problem can be summarized in the following terms: given G and $F(\cdot)$ find a solution M to

$$MF[x(t_0)] + MGx(t_0) = 0 \quad (7)$$

for all $x(t_0)$ for which $Mx(t_0) = 0$. A natural way to solve Eq. (7) is to exploit the symmetry of the network.

In representation (6), we can establish conditions to identify those permutations that leave a given network invariant. To this end we will establish conditions that guarantee that the set

$$\ker(I_{kn} - \Pi \otimes I_n)$$

is invariant.

Let $\Sigma = \Pi \otimes I_n$ for simplicity, and assume that at time $t_0 x(t_0)$ satisfies $(I_{kn} - \Sigma)x(t_0) = 0$. Consider Eq. (6), and suppose that there is a solution X of the following system of linear equations:

$$(I_k - \Pi)\Gamma = X(I_k - \Pi). \quad (8)$$

Since Π is a permutation matrix, it also follows that $\Sigma F(x) = F(\Sigma x)$. If we multiply both sides of Eq. (6) by $I_{kn} - \Sigma$, we obtain, at time t_0 ,

$$(I_{kn} - \Sigma)\dot{x}(t_0) = F[x(t_0)] - F[\Sigma x(t_0)] + (I_k \otimes B)(X \otimes I_m)(I_k \otimes C)(I_{kn} - \Sigma)x(t_0) = 0$$

because we assumed $(I_{kn} - \Sigma)x(t_0) = 0$. Therefore, $(I_{kn} - \Sigma)x(t) = 0$ for all t , and we can reformulate this result as:

Lemma 1: Given a permutation matrix Π such that Eq. (8) has a solution X , the set

$$\ker(I_{kn} - \Pi \otimes I_n) \quad (9)$$

is a linear invariant manifold for system (6).

IV. STABILITY ANALYSIS

A. Semipassivity

Consider systems of the form

$$\dot{x} = f(x) + Bu, \quad y = Cx, \quad (10)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^m$ is the output, the matrices B, C are of corresponding dimensions, f is smooth enough to ensure existence and uniqueness of solutions for admissible $u(t)$. Suppose it is possible to find a scalar nonnegative function V defined on $\mathbb{R}^n$, whose derivative satisfies, along the solutions of Eq. (10), the inequality

$$\dot{V}(x, u) \leq y^T u - H(x), \quad (11)$$

where function $H: \mathbb{R}^n \rightarrow \mathbb{R}$ is non-negative outside some ball,

$$\exists \rho > 0, \quad \forall |x| \geq \rho \Rightarrow H(x) \geq \varrho(|x|) \quad (12)$$

for some continuous non-negative function ϱ defined for $|x| \geq \rho$. In this case system (10) is called semipassive. This notion was introduced in Ref. 11, while in Ref. 12 an equivalent notion was called quasipassivity. If function H is positive outside some ball, i.e., Eq. (12) holds for some continuous positive function ϱ , then system (10) is said to be strictly semipassive. In brief, a semipassive system behaves like a passive system for sufficiently large $|x|$.

It is important to observe that the dissipation inequality (11) can be rewritten in an equivalent way as follows:

$$\frac{\partial V}{\partial x} f(x) \leq -H(x), \quad \frac{\partial V}{\partial x} B = x^T C^T.$$

Suppose that system (10) is strictly semipassive and the storage function V satisfying the dissipation inequality (11) is radially unbounded, that is $V(x) \rightarrow \infty$ when $|x| \rightarrow \infty$, then any feedback $u = \phi(y)$ satisfying the inequality

$$y^T \phi(y) \leq 0 \quad (13)$$

makes the closed loop system ultimately bounded. This statement can be proven just by considering the storage function V as a Lyapunov function candidate.

Therefore, the following result is valid:

Lemma 2: *If system (1) is strictly semipassive with radially unbounded storage functions and the symmetrized matrix $\Gamma + \Gamma^T$ is positive semidefinite, then all solutions of the coupled systems (1) and (2) exist for all $t \geq 0$ and are ultimately bounded.*

The technical proof of this statement and more general related results can be found in Ref. 11.

B. Convergent systems

Consider a dynamical system of the form

$$\dot{z} = q[z, w(t)], \quad (14)$$

with $z \in \mathbb{R}^l$, driven by the external signal $w(t)$ taking values from some compact set. This system is said to be convergent¹³ (see also Ref. 14) if for any bounded signal $w(t)$ defined on the whole time interval $(-\infty, +\infty)$ there is a unique bounded, globally asymptotically stable solution $\bar{z}(t)$ defined on the same interval $(-\infty, +\infty)$, from which it follows that

$$\lim_{t \rightarrow \infty} |z(t) - \bar{z}(t)| = 0, \quad (15)$$

for all initial conditions. In systems of this type the limit mode is solely determined by the external excitation $w(t)$, not by the initial conditions of z . From the existence of a unique mode $\bar{z}(t)$, it obviously descends that two identical copies of convergent system z_1 and z_2 , Eq. (14) must synchronize, that is, if Eq. (15) holds,

$$\lim_{t \rightarrow \infty} |z_1(t) - z_2(t)| = 0$$

holds as well. Convergence is then closely related to synchronization, hence it is important to find conditions ensuring it. Recently, an importance of the concept of convergent systems was recognized in control community with a potential application to observer design. In Ref. 15, a bit stronger notion was called incremental global asymptotic stability (δ GAS); therein the necessary and sufficient conditions for δ GAS were formulated in terms of the existence of Lyapunov functions. We present here a slight modification of a sufficient condition obtained by Demidovich:¹³ if there is a positive definite symmetric $l \times l$ matrix P such that all eigenvalues $\lambda_i(Q)$ of the symmetric matrix

$$Q(z, w) = \frac{1}{2} \left[P \left(\frac{\partial q}{\partial z}(z, w) \right) + \left(\frac{\partial q}{\partial z}(z, w) \right)^T P \right] \quad (16)$$

are negative and separated from zero, i.e., there is $\delta > 0$ such that

$$\lambda_i(Q) \leq -\delta < 0, \quad (17)$$

with $i = 1, \dots, l$ for all $z, w \in \mathbb{R}^l$, then system (14) is convergent, and there exists a quadratic function $W(\xi) = \xi^T P \xi$ satisfying

$$\frac{\partial W(z_1 - z_2)}{\partial \xi} [q(z_1, w) - q(z_2, w)] \leq -\alpha |z_1 - z_2|^2, \quad (18)$$

for some $\alpha > 0$. This condition is a slight modification of the Demidovich theorem on convergent systems in the case $P = I_l$.

C. On global asymptotic stability of the partial synchronization manifolds

A permutation matrix Π satisfying Eq. (8) for some X defines a linear invariant manifold of system (6), given by Eq. (9). This expression stands for a set of linear equations of the form

$$x_i - x_j = 0 \quad (19)$$

for some i and j that can be read off from the nonzero elements of the Π matrix under consideration. Therefore, we can identify a particular manifold associated with a particular matrix Π by the correspondent set $\mathcal{I}_\Pi$ of pairs i, j for which Eq. (19) holds.

In this section we are going to investigate asymptotic stability of partial synchronization as asymptotic stability of sets. In order to find a Lyapunov function which proves stability of the partial synchronization manifold, one can seek a Lyapunov function candidate as a sum of two functions, the first one dependent on the input-output relations of systems (1) and the second one dependent on the way the systems interact via coupling. The best way to carry this out is to find a globally defined coordinate change that allows us to exploit minimum phaseness.

Let us first differentiate y_j ,

$$\dot{y}_j = Cf(x_j) + CBu_j.$$

Then, choosing some $n-m$ coordinates z_j , complementary to y_j , it is possible to rewrite system (1) in the form

$$\dot{z}_j = q(z_j, y_j), \quad \dot{y}_j = a(z_j, y_j) + CBu_j, \quad (20)$$

where $z_j \in \mathbb{R}^{n-m}$, and q and a are some vector functions. It is important to emphasize that the coordinate change $x_j \mapsto \text{col}(z_j, y_j)$ can be linear, if CB is nonsingular, and that, owing to the linearity of the input-output relations, this transformation is globally defined. This transformation is explicitly computed in, for example, Ref. 16. As the reader may expect, for more complicated input-output relations, this coordinate transformation may not be globally defined. Conditions on the existence of this normal form can be found in Refs. 10 and 17, for example. In the equation for z_j in Eq. (20), y_j acts as a forcing input, hence we can apply properties of conver-

gent systems, if matrix $Q(z, w)$ defined for q in Eq. (20) has negative eigenvalues, separated from zero.

The purpose of this section is to prove the following theorem:

Theorem 1: Suppose that

- (i) Each free system (1) is strictly semipassive with respect to the input u_j and output y_j with a radially unbounded storage function.
- (ii) There exists a positive definite matrix P such that Eq. (17) holds with some $\delta > 0$ for the matrix Q defined as in Eq. (16) for q as in Eq. (20).
- (iii) The symmetrized matrix $\Gamma + \Gamma^T$ is positive semidefinite.
- (iv) There is a $k \times k$ matrix solution X of the following linear equation $(I_k - \Pi)\Gamma = X(I_k - \Pi)$.
- (v) CB is positive definite.

Let λ' be the minimal eigenvalue of $\frac{1}{2}(X + X^T)$ under the restriction that the eigenvectors of $\frac{1}{2}(X + X^T)$ are taken from the set $\text{range}(I_k - \Pi)$.

Then all solutions of network (4, 5) are ultimately bounded and there exists a positive $\bar{\lambda}$ such that if $\lambda' > \bar{\lambda}$ the set $\ker(I_{kn} - \Pi \otimes I_n)$ contains a globally asymptotically stable compact subset.

We sketch the proof of Theorem 1. To make the presentation more transparent we omitted some standard technical details which can be found in similar proofs, of related results, presented in Refs. 11 and 18. Our approach is inspired by the results on feedback-passive systems as presented in Ref. 19. In the proof we are mostly focused on the approach to find the Lyapunov function guaranteeing stability of the partial synchronization mode. As we previously introduced the notation $y = \text{col}(y_1, \dots, y_k)$, let us denote with $z \in \mathbb{R}^{km}$ the vector $\text{col}(z_1, \dots, z_k)$. Since the derivative of z -variables in Eq. (20) does not depend on the coupling, while the derivative of y -variables does, we can search for a Lyapunov function in the form

$$V(z, y) = V_1(z) + V_2(y).$$

Let us start with function V_1 . According to assumption (iii) there is a positive definite radially unbounded function $W(\zeta)$ defined on $\mathbb{R}^{n-m}$ which satisfies the partial differential inequality (18) for all $z_i, z_j \in \mathbb{R}^{n-m}$, $w \in \mathbb{R}^m$. Then we construct the function V_1 as

$$V_1(z) = \sum_{(i,j) \in \mathcal{I}_\Pi} W(z_i - z_j).$$

Along the solutions of the closed loop system, the derivative of $V_1(z)$ satisfies

$$\begin{aligned} \dot{V}_1(z, y) &= \sum_{(i,j) \in \mathcal{I}_\Pi} \frac{\partial W(z_i - z_j)}{\partial \zeta} [q(z_i, y_i) - q(z_j, y_j)] \\ &\leq -\alpha \sum_{(i,j) \in \mathcal{I}_\Pi} |z_i - z_j|^2 \\ &\quad + \sum_{(i,j) \in \mathcal{I}_\Pi} \frac{\partial W(z_i - z_j)}{\partial \zeta} [q(z_j, y_i) - q(z_j, y_j)]. \end{aligned}$$

The next step is to find the second part of the Lyapunov function, i.e., function V_2 . It is clear that if $x \in \ker(I_{kn} - \Pi \otimes I_n)$, then necessarily $y \in \ker(I_{km} - \Pi \otimes I_m)$. So, on this invariant manifold, the quantity $\xi(y) = (I_{km} - \Pi \otimes I_m)y$ is identically zero. We can therefore construct function V_2 as

$$\begin{aligned} V_2(y) &= \frac{1}{2} |\xi(y)|^2 \\ &= \frac{1}{2} y^T (I_{km} - \Pi \otimes I_m)^T (I_{km} - \Pi \otimes I_m) y \\ &= \frac{1}{2} \sum_{(i,j) \in \mathcal{I}_\Pi} |y_i - y_j|^2, \end{aligned}$$

which is positive definite with respect to ξ , and zero on the set $\ker(I_{km} - \Pi \otimes I_m)$. Differentiating V_2 gives

$$\dot{V}_2(z, y) = \sum_{j=1}^k \frac{\partial V_2(y_j)}{\partial y_j} a(z_j, y_j) - U(y)$$

where using assumption iv,

$$U(y) = \frac{1}{2} y^T (I_{km} - \Pi \otimes I_m)^T [(X + X^T) \otimes CB] (I_{km} - \Pi \otimes I_m) y.$$

It follows that

$$U(y) \geq \lambda' \beta y^T (I_{km} - \Pi \otimes I_m)^T (I_{km} - \Pi \otimes I_m) y,$$

where β is the minimal eigenvalue of matrix CB and λ' is the minimal eigenvalue of $\frac{1}{2}(X + X^T)$ under the restriction that the eigenvectors of $\frac{1}{2}(X + X^T)$ are taken from the set $\text{range}(I_k - \Pi)$.

We proceed now to evaluate the derivative of V . From the previous intermediate results it follows that

$$\begin{aligned} \dot{V}(z, y) &\leq -\alpha \sum_{(i,j) \in \mathcal{I}_\Pi} |z_i - z_j|^2 - 2\lambda' \beta V_2(y) \\ &\quad + \sum_{j=1}^k \frac{\partial V_2(y_j)}{\partial y_j} a(z_j, y_j) \\ &\quad + \sum_{(i,j) \in \mathcal{I}_\Pi} \frac{\partial W(z_i - z_j)}{\partial \zeta} [q(z_j, y_i) - q(z_j, y_j)]. \end{aligned}$$

Note that for any compact set Ω there exist some positive numbers C_1, C_2, C_3 such that the following estimates are valid on Ω :

$$\begin{aligned} &\left| \sum_{j=1}^k \frac{\partial V_2(y_j)}{\partial y_j} a(z_j, y_j) \right| \\ &= \left| \sum_{(i,j) \in \mathcal{I}_\Pi} (y_i - y_j)^T [a(z_i, y_i) - a(z_j, y_j)] \right| \\ &\leq \left| \sum_{(i,j) \in \mathcal{I}_\Pi} (y_i - y_j)^T [a(z_i, y_i) - a(z_i, y_j)] \right| \\ &\quad + \left| \sum_{(i,j) \in \mathcal{I}_\Pi} (y_i - y_j)^T [a(z_i, y_j) - a(z_j, y_j)] \right| \\ &\leq C_1 V_2(y) + C_2 \sum_{(i,j) \in \mathcal{I}_\Pi} |z_i - z_j| |y_i - y_j| \end{aligned}$$

and

FIG. 1. (Color online) A network of eight coupled identical systems with symmetric coupling.

$$\left| \sum_{(i,j) \in \mathcal{I}_{\Pi}} \frac{\partial W(z_i - z_j)}{\partial \xi} [q(z_j, y_i) - q(z_i, y_j)] \right| \leq C_3 \sum_{(i,j) \in \mathcal{I}_{\Pi}} |z_i - z_j| \cdot |y_i - y_j|.$$

Now we are going to use strict semipassivity of the systems forming the diffusive network. Recall that strict semipassivity implies ultimate boundedness of all the solutions, that is, all the solutions approach in a finite time some compact set Ω which can be chosen independently on Γ . On this compact set the derivative of V is a quadratic form with respect to $|z_i - z_j|$ and $|y_i - y_j|$. It is clear then that, if the value of λ' is large enough (that is, λ' is greater than a positive computable threshold $\bar{\lambda}$), due to Eq. (18), the derivative of $V(z, y)$ is nonpositive on this set. After some algebra, an explicit formula for $\bar{\lambda}$ is derived as

$$\bar{\lambda} = \frac{1}{\beta} \left[\frac{C_1}{2} + \frac{(C_2 + C_3)^2}{4\alpha} \right]. \quad (21)$$

This argument proves that the set $\ker(I_{kn} - \Pi \otimes I_n)$ contains a globally asymptotically stable compact subset for $\lambda' > \bar{\lambda}$. ■

V. EXAMPLE

In this section we consider an example of a network of diffusively coupled systems depicted in Fig. 1. Matrix Γ in this case is defined as

$$\Gamma = \begin{bmatrix} \gamma_1 & k_1 & 0 & k_1 & k_2 & k_4 & 0 & 0 \\ k_1 & \gamma_1 & k_1 & 0 & k_4 & k_2 & 0 & 0 \\ 0 & k_1 & \gamma_1 & k_1 & 0 & 0 & k_2 & k_4 \\ k_1 & 0 & k_1 & \gamma_1 & 0 & 0 & k_4 & k_2 \\ k_2 & k_4 & 0 & 0 & \gamma_2 & k_3 & 0 & k_3 \\ k_4 & k_2 & 0 & 0 & k_3 & \gamma_2 & k_3 & 0 \\ 0 & 0 & k_2 & k_4 & 0 & k_3 & \gamma_2 & k_3 \\ 0 & 0 & k_4 & k_2 & k_3 & 0 & k_3 & \gamma_2 \end{bmatrix}$$

with

$$\gamma_1 = -2k_1 - k_2 - k_4, \quad \gamma_2 = -k_2 - 2k_3 - k_4.$$

Consider the following permutation matrix:

$$\Pi = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The set $\ker(I_8 - \Pi)$ is given by $x_1 = x_2 = x_3 = x_4$, $x_5 = x_6 = x_7 = x_8$. As one can easily verify matrices Π and Γ do not commute and therefore the theorem presented in Ref. 9 cannot be applied to study stability of this invariant set. At the same time there is a solution X to Eq. (8) for the given Γ and Π (it was verified numerically for different k_1, k_2, k_3, k_4 using singular value decomposition) and thus Theorem 1 presented in this paper can be utilized to investigate stability of this invariant manifold.

To apply the partial synchronization theorem the other conditions of the theorem should be verified. They depend on the input-output properties of the individual dynamics forming the network. Suppose the individual dynamics are given by the Lorenz system

$$\dot{x}_{j,1} = \sigma(x_{j,2} - x_{j,1}) + u_j, \quad \dot{x}_{j,2} = rx_{j,1} - x_{j,2} - x_{j,1}x_{j,3},$$

$$\dot{x}_{j,3} = -bx_{j,3} + x_{j,1}x_{j,2},$$

with $\sigma, r, b > 0$. The Lorenz system with input u_j and output $y_j = x_{j,1}$ is strictly semipassive. To prove this statement, consider the following storage function candidate:

$$V(x_{j,1}, x_{j,2}, x_{j,3}) = \frac{1}{2}[(x_{j,1})^2 + (x_{j,2})^2 + (x_{j,3} - \sigma - r)^2].$$

Calculating the derivative of this function along the solutions of the system yields

$$\dot{V}(x_{j,1}, x_{j,2}, x_{j,3}, u) = x_{j,1}u - H(x_{j,1}, x_{j,2}, x_{j,3}),$$

where

$$H(x_{j,1}, x_{j,2}, x_{j,3}) = \sigma(x_{j,1})^2 + (x_{j,2})^2 + b \left(x_{j,3} - \frac{\sigma + r}{2} \right)^2 - b \frac{(\sigma + r)^2}{4}.$$

It is easy to see that the condition $H \leq 0$ determines an ellipsoid in $\mathbb{R}^3$. This fact proves strict semipassivity of system (22) with input u_j and output $y_j = x_{j,1}$. Hence, in a diffusive network of any number of Lorenz systems with outputs $y_j = x_{j,1}$ and inputs u_j , all solutions are ultimately bounded.

If we think of output y_j as a driving input for the remaining part (q -subsystem in the theorem conditions) of the Lorenz system, we have

$$\dot{x}_{j,2} = -x_{j,2} + ry_j - y_j x_{j,3}, \quad \dot{x}_{j,3} = -bx_{j,3} + y_j x_{j,2},$$

which is convergent. Applying Demidovich's result we see that, using $P = I_2$, matrix $Q(x_j, y_j)$ in Eq. (16) is given by

$$Q(x_j, y_j) = \text{diag}(-1 - b).$$

Now one can apply the theorem to conclude that if the parameters of the coupling matrix are appropriately chosen, the network of coupled Lorenz systems possesses partially synchronous modes.

There are other systems that satisfy conditions imposed on the input-output properties of individual dynamics of the network. This is the case if one takes, for example the Lorenz or Hindmarsh–Rose system, see Refs. 9 and 20 for details.

VI. CONCLUSION

In this paper we have demonstrated an approach, based on the second Lyapunov method, to study partial synchronization regimes in a network of linearly coupled identical dynamical systems. We presented a theorem that allows us to cope with more general networks than those presented in our previous work.⁹ For relatively simple ringlike networks the corresponding symmetry group is isomorphic to the powers of unity and the analysis of such systems can be performed by the result of Ref. 9. For more complex networks, commutativity of the corresponding symmetry group can be an issue, as illustrated by the example in this paper. To classify all permutations that satisfy Eq. (8) for a given topology one can use a special property of matrix $\Gamma + \Gamma^T$ (all row sums are zeros) and the seminal Birkhoff–von Neumann theorem on doubly stochastic matrices. This result allows us to represent a doubly stochastic matrix as a convex combination of permutation matrices. Since the partial synchronization mode is defined by another permutation matrix, the theorem due to Birkhoff and von Neumann can be a very useful tool in the analysis of partial synchronization in complex networks. That can constitute an interesting topic for future research.

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