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Sulfate attack in sewer pipes: Derivation of a concrete corrosion model via two-scale convergence

by

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Sulfate attack in sewer pipes: Derivation of a concrete corrosion model via two-scale convergence

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Abstract

We explore the homogenization limit and rigorously derive upscaled equations for a microscopic reaction-diffusion system modeling sulfate corrosion in sewer pipes made of concrete. The system, defined in a periodically-perforated domain, is semi-linear, partially dissipative and weakly coupled via a non-linear ordinary differential equation posed on the solid-water interface at the pore level. Firstly, we show the well-posedness of the microscopic model. We then apply homogenization techniques based on two-scale convergence for an uniformly periodic domain and derive upscaled equations together with explicit formulae for the effective diffusion coefficients and reaction constants. We use a boundary unfolding method to pass to the homogenization limit in the non-linear ordinary differential equation. Finally, besides giving its strong formulation, we also prove that the upscaled two-scale model admits a unique solution.

Key words: Sulfate corrosion of concrete, periodic homogenization, semi-linear partially dissipative system, two-scale convergence, periodic unfolding method, multiscale system.

1 Introduction

This paper treats the periodic homogenization of a semi-linear reaction-diffusion system coupled with a nonlinear differential equation arising in the modeling of the sulfuric acid attack in sewer pipes made of concrete. The concrete corrosion situation we are dealing with here strongly influences the durability of cement-based materials especially in hot environments leading to spalling of concrete and macroscopic fractures of sewer pipes. It is financially important to have a good estimate on the moment in time when such pipe systems need
to be replaced, for instance, at the level of a city like Los Angeles. To get good such practical estimates, one needs on one side easy-to-use macroscopic corrosion models to be used for a numerical forecast of corrosion, while on the other side one needs to ensure the reliability of the averaged models by allowing them to incorporate a certain amount of microstructure information. The relevant question is: How much of this oscillatory-type information is needed to get a sufficiently accurate description of the heterogeneous reality? Due to the complexity of possible shapes of the microstructure, averaging concrete materials is far more difficult than averaging metallic composites with rigorously defined well-packed structure. In this paper, we imagine our concrete piece to be made of a periodically-distributed microstructure. Based on this assumption, we provide here a rigorous justification of the formal asymptotic expansion performed by us (in [1]) for this reaction-diffusion scenario. Note that in [1] upscaled models are derived for a more general situation involving a locally-periodic distribution of perforations\(^1\). Locally periodic geometries refer to a special case of \(x\)-dependent microstructures, where, inherently, the outer normals to (microscopic) inner interfaces are dependent on both spatial slow variable, say \(x\), and fast variable, say \(y\).

In the framework of this paper, we combine two-scale convergence concepts with the periodic unfolding of interfaces to pass to the homogenization limit (i.e. to \(\varepsilon \to 0\), where \(\varepsilon\) is a small parameter linked to the relative size of the perforation) for the uniformly periodic case. Here, the outer normals to the inner interfaces are dependent only on the spatial fast variable. For more details on the mathematical modeling of sulfate corrosion of concrete, we refer the reader to [2,3] (a moving-boundary approach: numerics and formal matched asymptotics), [4] (a two-scale reaction-diffusion system modeling sulfate corrosion), as well as to [5], where a nonlinear Henry-law type transmission condition (modeling \(H_2S\) transfer across all air-water interfaces present in this sulfatation problem) is analyzed. Mathematical background on periodic homogenization can be found in e.g., [6–8], while a few relevant (remotely resembling) worked-out examples of this averaging methodology are explained, for instance, in [9–14]. It is worth noting that, since it deals with the homogenization of a linear Henry-law setting, the paper [11] is related to our approach. The major novelty here compared to [11] is that we now need to pass to the limit in a non-dissipative object, namely a nonlinear ordinary differential equation (ode). The ode is describing sulfatation reaction at the inner water-solid interface – place where corrosion localizes. This aspect makes a rigorous averaging challenging. For instance, compactness-type methods do not work in the case when the nonlinear ode is posed on \(\varepsilon\)-dependent surfaces. We circumvent this issue by "boundary unfolding" the ode. Thus we fix, as independent of \(\varepsilon\), the reaction interface similarly as in [15], and only then we pass to the limit. Alternatively, one could use varifolds (cf. e.g. [16]), since this seems to

\(^1\) The word "perforation" is seen here as a synonym for "pore" or "microstructure".
be the natural framework for the rigorous passage to the limit when both the surface measure and the oscillating sequences depend on $\epsilon$. However, we find the boundary unfolding technique easier to adapt to our scenario than the varifolds.

Note that here we approach the corrosion problem deterministically. However, we have reasons to expect that the uniform periodicity assumption can be relaxed by assuming instead a Birkhoff-type ergodicity of the microstructure shapes and positions, and hence, the natural averaging context seems to be the one offered by random fields; see ch. 1, sect. 6 in [17], ch. 8 and 9 in [18], or [19]. But, methodologically, how big is the overlap between homogenizing deterministically locally-periodic distributions of microstructures compared to working in the random fields context? We will treat these and related aspects elsewhere.

The paper is organized as follows: We start off in section 2 (and continue in section 3) with the analysis of the microscopic model. In section 4, we obtain the $\epsilon$-independent estimates needed for the passage to the limit $\epsilon \to 0$. Section 5 contains the main result of the paper: the set of the upscaled two-scal equations.

2 The microscopic model

In this section, we describe the geometry of our array of periodic microstructures and briefly indicate the most aggressive chemical reaction mechanism typically active in sewer pipes. Finally, we list the set of microscopic equations.

2.1 Basic geometry

Fig. 1 (i) shows a cross-section of a sewer pipe hosting corrosion. We assume that the geometry of the porous medium in question consists of a system of pores periodically distributed inside the three-dimensional cube $\Omega := [a, b]^3$ with $a, b \in \mathbb{R}$ and $b > a$. The exterior boundary of $\Omega$ consists of two disjoint, sufficiently smooth parts: $\Gamma^N$ - the Neumann boundary and $\Gamma^D$ - the Dirichlet boundary. The reference pore, say $Y := [0, 1]^3$, has three pairwise disjoint connected domains $Y^s$, $Y^w$ and $Y^a$ with smooth boundaries $\Gamma_{sw}$ and $\Gamma_{wa}$, as shown in Fig. 1 (iii). Moreover, $Y := \bar{Y}^s \cup \bar{Y}^w \cup \bar{Y}^a$.

Let $\epsilon$ be a sufficiently small scaling factor denoting the ratio between the characteristic length of the pore $Y$ and the characteristic length of the domain $\Omega$. Let $\chi^w$ and $\chi^a$ be the characteristic functions of the sets $Y^w$ and $Y^a$, respectively. The shifted set $Y^w_k$ is defined by

$$Y^w_k := Y + \sum_{j=0}^{3} k_j e_j \text{ for } k = (k_1, k_2, k_3) \in \mathbb{Z}^3,$$

where $e_j$ is the $j^{th}$ unit vector. The union of all shifted subsets of $Y^w_k$ multiplied
by $\varepsilon$ (and confined within $\Omega$) defines the perforated domain $\Omega^\varepsilon$, namely

$$\Omega^\varepsilon := \bigcup_{k \in \mathbb{Z}^3} \{ \varepsilon Y^w_k \mid \varepsilon Y^w_k \subset \Omega \}.$$ 

Similarly, $\Omega_1^\varepsilon$, $\Gamma_{sw}^\varepsilon$, and $\Gamma_{wa}^\varepsilon$ denote the union of the shifted subsets (of $\Omega$) $Y^s_k$, $\Gamma^w_k$, and $\Gamma^a_k$ scaled by $\varepsilon$. Since usually the concrete in sewer pipes is not completely dry, we decide to take into account a partially saturated porous material\(^2\). We assume that every pore has three distinct non-overlapping parts: a solid part (grain) which is placed in the center of the pore, the water film which surrounds the solid part, and an air layer bounding the water film and filling the space of $Y$ as shown in Fig. 1. The air connects neighboring pores to one another. The geometry defined above satisfies the following assumptions:

1. Neither solid nor water-filled parts touch the boundary of the pore.
2. All internal (air-water and water-solid) interfaces are sufficiently smooth and do not touch each other.

These geometrical restrictions imply that the pores are connected by air-filled parts only which is needed not only to give a meaning to functions defined across interfaces, but also to introduce the concept of extension as given, for instance, in [20]. Furthermore, there are no solid-air interfaces.

2.2 Description of the chemistry

There are many variants of severe attack to concrete in sewer pipes, we focus here on the most aggressive one – the sulfuric acid attack. The situation can be described briefly as follows: (The anaerobic bacteria in the flowing waste water release hydrogen sulfide gas ($H_2S$) within the air space of the pipe. These bacteria are especially active in hot environments. From the air space inside the pipe, $H_2S(g)$\(^3\) enters the pores of the concrete matrix where it diffuses and then dissolves in the pore water. The aerobic bacteria catalyze some of the $H_2S$ into sulfuric acid $H_2SO_4$. $H_2S$ molecules can move between air-filled part and water-filled part the water-air interfaces [21]. We model this microscopic

\(^2\) The solid, water and air parts corresponds to $Y^s$, $Y^w$ and $Y^a$, respectively.

\(^3\) $H_2S(g)$ and $H_2S(aq)$ refer to gaseous, and respectively, aqueous $H_2S$. 
interfacial transfer via Henry’s law \cite{22}, (see the boundary conditions at \(\Gamma_{wa}\) in (3) and (4)). \(H_2SO_4\) being an aggressive acid reacts with the solid matrix\footnote{The solid matrix is assumed here to consist of \(CaCO_3\) only. This assumption can be removed in the favor of a more complex cement chemistry.} at the solid-water interface, which is made up of cement, sand, and aggregate, and produces gypsum (i.e. \(CaSO_4 \cdot 2H_2O\)). Here we restrict our attention to a minimal set of chemical reactions mechanisms as suggested in \cite{2}, namely.

\[
\begin{align*}
10H^+ + SO_4^{2-} + \text{org. matter} & \rightarrow H_2S(aq) + 4H_2O + \text{oxid. matter} \\
H_2S(aq) + 2O_2 & \rightarrow 2H^+ + SO_4^{2-} \\
H_2S(aq) & \rightleftharpoons H_2S(g) \\
2H_2O + H^+ + SO_4^{2-} + CaCO_3 & \rightarrow CaSO_4 \cdot 2H_2O + HCO_3^-
\end{align*}
\]

We assume that reactions (1) do not interfere with the mechanics of the solid part of the pores. This is a rather strong assumption since it is known that (1) can actually produce local ruptures of the solid matrix \cite{23}. For more details on the involved cement chemistry and connections to acid corrosion, we refer the reader to \cite{24} (for a nice enumeration of the involved physicochemical mechanisms), \cite{23} (standard textbook on cement chemistry), as well as to \cite{25–27} and references cited therein. For a mathematical approach of a similar theme related to the conservation and restoration of historical monuments, we refer to the work by R. Natalini and co-workers (cf. e.g. \cite{28}).

### 2.3 Setting of the equations

The data and unknown are given by

\[
\begin{align*}
u_{10}^\varepsilon & : \Omega \rightarrow \mathbb{R}_+ - \text{initial concentration of } H_2SO_4(aq) \\
u_{20}^\varepsilon & : \Omega \rightarrow \mathbb{R}_+ - \text{initial concentration of } H_2S(aq) \\
u_{30}^\varepsilon & : \Omega \rightarrow \mathbb{R}_+ - \text{initial concentration of } H_2S(g) \\
u_{40}^\varepsilon & : \Omega \rightarrow \mathbb{R}_+ - \text{initial concentration of moisture} \\
u_{50}^\varepsilon & : \Omega \rightarrow \mathbb{R}_+ - \text{initial concentration of gypsum} \\
\end{align*}
\]

\[
\begin{align*}
u_3^D & : \Gamma_D \times (0, T) \rightarrow \mathbb{R}_+ - \text{exterior concentration (Dirichlet data) of } H_2S(g) \\
u_1^\varepsilon & : \Omega^\varepsilon \times (0, T) \rightarrow \mathbb{R} - \text{concentration of } H_2SO_4(aq) \\
u_2^\varepsilon & : \Omega^\varepsilon \times (0, T) \rightarrow \mathbb{R} - \text{concentration of } H_2S(aq) \\
u_3^\varepsilon & : \Omega^\varepsilon \times (0, T) \rightarrow \mathbb{R} - \text{concentration of } H_2S(g) \\
u_4^\varepsilon & : \Omega^\varepsilon \times (0, T) \rightarrow \mathbb{R} - \text{concentration of moisture} \\
u_5^\varepsilon & : \Gamma_{sw}^\varepsilon \times (0, T) \rightarrow \mathbb{R} - \text{concentration of gypsum}
\end{align*}
\]

All concentrations are viewed as mass concentrations. We consider the following system of mass-balance equations defined at the pore level. The mass-
balance equation for $H_2SO_4$ is
\[ \frac{\partial}{\partial t} u^\varepsilon_i + \text{div}(-d^\varepsilon_i \nabla u^\varepsilon_i) = -k^\varepsilon i u^\varepsilon_1 + k^\varepsilon 2 u^\varepsilon_2, \quad x \in \Omega^\varepsilon, \ t \in (0, T) \]
\[ u^\varepsilon_i(x, 0) = u^\varepsilon i_{10}(x), \quad x \in \Omega^\varepsilon \]
\[ -n^\varepsilon \cdot d^\varepsilon_i \nabla u^\varepsilon_i = 0, \quad x \in \Gamma_{\text{wa}}^\varepsilon, \ t \in (0, T) \]
\[ -n^\varepsilon \cdot d^\varepsilon_i \nabla u^\varepsilon_i = \varepsilon(u^\varepsilon_1, u^\varepsilon_2), \quad x \in \Gamma_{\text{sw}}^\varepsilon t \in (0, T). \] (2)

The mass-balance equation for moisture follows
\[ \frac{\partial}{\partial t} u^\varepsilon_2 + \text{div}(-d^\varepsilon_2 \nabla u^\varepsilon_2) = k^\varepsilon 2 u^\varepsilon_1 - k^\varepsilon 2 u^\varepsilon_2, \quad x \in \Omega^\varepsilon, \ t \in (0, T) \]
\[ u^\varepsilon_2(x, 0) = u^\varepsilon 2_{10}(x), \quad x \in \Omega^\varepsilon \]
\[ -n^\varepsilon \cdot d^\varepsilon_2 \nabla u^\varepsilon_2 = \varepsilon(a^\varepsilon(x)u^\varepsilon_3 - b^\varepsilon(x)u^\varepsilon_2), \quad x \in \Gamma_{\text{wa}}^\varepsilon, \ t \in (0, T) \]
\[ -n^\varepsilon \cdot d^\varepsilon_2 \nabla u^\varepsilon_2 = 0, \quad x \in \Gamma_{\text{sw}}^\varepsilon t \in (0, T). \] (3)

The mass-balance equation for $H_2S(aq)$ is given by
\[ \frac{\partial}{\partial t} u^\varepsilon_3 + \text{div}(-d^\varepsilon_3 \nabla u^\varepsilon_3) = 0, \quad x \in \Omega_1^\varepsilon, \ t \in (0, T) \]
\[ u^\varepsilon_3(x, 0) = u^\varepsilon 3_{10}(x), \quad x \in \Omega_1^\varepsilon \]
\[ -n^\varepsilon \cdot d^\varepsilon_3 \nabla u^\varepsilon_3 = 0, \quad x \in \Gamma^N, \ t \in (0, T) \]
\[ u^\varepsilon_3(x, t) = u^\varepsilon 3_D(x, t), \quad x \in \Gamma^D, \ t \in (0, T) \]
\[ -n^\varepsilon \cdot d^\varepsilon_3 \nabla u^\varepsilon_3 = -\varepsilon(a^\varepsilon(x)u^\varepsilon_3 - b^\varepsilon(x)u^\varepsilon_2), \quad x \in \Gamma_{\text{sw}}^\varepsilon, \ t \in (0, T). \] (4)

The mass-balance equation for moisture reads
\[ \frac{\partial}{\partial t} u^\varepsilon_1 + \text{div}(-d^\varepsilon_1 \nabla u^\varepsilon_1) = k^\varepsilon 1 u^\varepsilon_1, \quad x \in \Omega^\varepsilon, \ t \in (0, T) \]
\[ u^\varepsilon_1(x, 0) = u^\varepsilon 1_{10}(x), \quad x \in \Omega^\varepsilon \]
\[ -n^\varepsilon \cdot d^\varepsilon_1 \nabla u^\varepsilon_1 = 0, \quad x \in \Gamma_{\text{wa}}^\varepsilon, \ t \in (0, T) \]
\[ -n^\varepsilon \cdot d^\varepsilon_1 \nabla u^\varepsilon_1 = 0, \quad x \in \Gamma_{\text{sw}}^\varepsilon t \in (0, T). \] (5)

The mass-balance equation for the gypsum produced at the water-solid interface is
\[ \frac{\partial}{\partial t} u^\varepsilon_5 = \eta(u^\varepsilon_1, u^\varepsilon_5), \quad x \in \Gamma_{\text{sw}}^\varepsilon, \ t \in (0, T) \]
\[ u^\varepsilon_5(x, 0) = u^\varepsilon 5_{10}(x), \quad x \in \Gamma_{\text{sw}}^\varepsilon t \in (0, T). \] (6)

3 Weak formulation and basic results

We begin this section with a list of notations and function spaces. Then we indicate our working assumptions and give the weak formulation of the microscopic problem; we bring reader’s attention to the well-posedness of the system (2)–(6).

3.1 Notations and function spaces

We use $(\alpha, \beta)_{(0, T) \times \Omega^\varepsilon} := \int_0^T \int_{\Omega^\varepsilon} \alpha \beta dx dt$, $(\alpha, \beta)_{(0, T) \times \Gamma^\varepsilon} := \int_0^T \int_{\Gamma^\varepsilon} \alpha \beta \sigma dx dt$. $\langle \cdot \rangle$, $| \cdot |$ and $\| \cdot \|$ denote the dual pairing of $H^1(\Omega^\varepsilon)$ and $H^{-1}(\Omega^\varepsilon)$, the norm in $L^2(\Omega^\varepsilon)$, and the norm in $H^1(\Omega^\varepsilon)$, respectively. $\varphi^+$ and $\varphi^-$ will point out the
positive and respectively the negative part of the function $\varphi$. We denote by $C^\infty_\#(Y), H^1_\#(Y)$, and $H^{1}_\#(Y)/\mathbb{R}$, the space of infinitely differentiable functions in $\mathbb{R}^n$ that are periodic of period $Y$, the completion of $C^\infty_\#(Y)$ with respect to $H^1$–norm, and the respective quotient space, respectively. Furthermore, $H^1_{\Gamma^D}(\Omega) := \{ u \in H^1(\Omega) | u = 0 \text{ on } \Gamma^D \}$. The Sobolev space $H^2(\Omega)$ as a completion of $C^\infty(\Omega)$ is a Hilbert space equipped with a norm

$$
\| \varphi \|_{H^2(\Omega)} = \| \varphi \|_{H^1(\Omega)} + \left( \int_{\Omega} \int_{\Omega} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{n+2|\beta|}} \, dx \, dy \right)^{\frac{1}{2}}
$$

and (cf. Theorem 7.57 in [29]) the embedding $H^2(\Omega) \hookrightarrow L^2(\Omega)$ is continuous.

Since we deal with an evolution problem, we need typical Bochner spaces like $H_{\Gamma^D}$ in $\mathbb{R}$

$$
C\varepsilon
$$

inequality for $\varepsilon$–dependent hypersurfaces $\Gamma_{\varepsilon}^\alpha$: For $\varphi \varepsilon \in H^1(\Omega^\varepsilon)$, there exists a constant $C^\varepsilon$, which is independent of $\varepsilon$, such that

$$
\varepsilon |\varphi|_{2,2(\Gamma^\varepsilon)} \leq C^\varepsilon (|\varphi|_{2,2(\Omega^\varepsilon)} + \varepsilon^2 \| \nabla \varphi \|_{2,2(\Omega^\varepsilon)}).
$$

The proof of (7) is given in Lemma 3 of [30]. For a function $\varphi \varepsilon \in H^2(\Omega^\varepsilon)$ with $\beta \in (\frac{1}{2}, 1)$, the inequality (7) refines into

$$
\varepsilon |\varphi|_{2,2(\Gamma^\varepsilon)} \leq C^\varepsilon (|\varphi|_{2,2(\Omega^\varepsilon)} + \varepsilon^{2\beta} \int_\Omega \int_\Omega \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{n+2\beta}} \, dx \, dy),
$$

where $C^\varepsilon$ is again a constant independent of $\varepsilon$. For proof of (8), see [15].

To simplify the writing of some of the estimates, we employ the next set of notations:

$$
d^\varepsilon_i := \min_{[0,T] \times \Omega} |d^\varepsilon_i|, \quad d^\varepsilon_i := \min_{[0,T] \times \Omega} |d^\varepsilon_i|,
$$

$$
D_m := \max_{[0,T] \times \Omega} |\partial d^\varepsilon_m|, \quad m \in \{1, 2, 3\}, \quad k_j := \min_{[0,T] \times \Omega} |k^\varepsilon_j|, \quad j \in \{1, 2\}
$$

$$
K_j := \min_{[0,T] \times \Omega} |\partial k_j^\varepsilon|, \quad \tilde{k}_j := \min_{[0,T] \times \Omega} |\tilde{k}_j^\varepsilon|,
$$

$$
K^\varepsilon_m := \sup_{(0,T) \times \Omega} |k_m^\varepsilon|, \quad \tilde{k}_m^\varepsilon := \sup_{(0,T) \times \Omega} |\tilde{k}_m|,
$$

$$
K^\varepsilon_m := \sup_{(0,T) \times \Omega} |\partial k_m^\varepsilon|, \quad M_i := \sup_{(0,T) \times \Omega} |u_i^\varepsilon|, \quad i \in \{1, 2, 3, 4, 5\},
$$

$$
A^\infty := \sup_{(0,T) \times \Gamma_{\varepsilon}^w} |a^\varepsilon|, \quad B^\infty := \sup_{(0,T) \times \Gamma_{\varepsilon}^w} |b^\varepsilon|,
$$

$$
A^\infty := \sup_{(0,T) \times \Gamma_{\varepsilon}^w} |\partial a^\varepsilon|, \quad B^\infty := \sup_{(0,T) \times \Gamma_{\varepsilon}^w} |\partial b^\varepsilon|,
$$

$$
\tilde{a}^\infty := \sup_{(0,T) \times \Gamma_{\varepsilon}^w} |\tilde{a}|, \quad \tilde{b}^\infty := \sup_{(0,T) \times \Gamma_{\varepsilon}^w} |\tilde{b}|,
$$

$$
Q^\infty := \sup_{\eta \in (0,T) \times \Gamma_{\varepsilon}^w} |Q(s)|, \quad \tilde{\eta} := ||\eta||_{\infty}, \quad \tilde{\eta} := ||\partial \eta||_{\infty}.
$$
3.2 Assumptions on the data and parameters

We consider the following restriction on the data and parameters:

(A1) \( d_i \in L^\infty((0,T) \times Y)^{3 \times 3} \), \( \partial d_i \in L^\infty((0,T) \times Y)^{3 \times 3} \), \( \partial_t d_i \in L^\infty((0,T) \times Y)^{3 \times 3} \), \( (d_i(t,x)\xi, \xi) \geq d_{i0} \mid \xi \mid^2 \) for \( d_{i0} > 0 \), for every \( \xi \in \mathbb{R}^3 \), \( (t,x) \in (0,T) \times Y \), \( i \in \{1, 2, 3, 4\} \).

(A2) \( \eta \) is measurable w.r.t. \( t \) and \( x \) and \( \eta(\alpha, \beta) = k_3^\beta R(\alpha)Q(\beta) \), \( R \) is sub-linear and locally Lipschitz function and \( Q \) is bounded and locally Lipschitz function such that

\[
R(\alpha) = \begin{cases} 
\text{positive, if } \alpha \geq 0, \\
0, \text{ otherwise}
\end{cases} \quad Q(\beta) = \begin{cases} 
\text{positive, if } \beta < \beta_{\text{max}}, \\
0, \text{ otherwise}
\end{cases}
\]

Additionally to (A2), we sometimes assume (A2)', that is

\[
\text{(A2)'} \quad \partial_t \eta \leq \hat{\eta}.
\]

(A3) \( u^e_{i0} \in L^2(\Omega^e) \cap L^\infty(\Omega^e) \), \( i \in \{1, 2, 4\} \), \( u^e_{50} \in L^2(\Omega^e_1) \cap L^\infty(\Omega^e_1) \), \( u^e_{50} \in L^2(\Gamma^e_{sw}) \cap L^\infty(\Gamma^e_{sw}) \).

(A4) \( a^\infty M_3 = b^\infty M_2 \), \( k_1^\infty M_1 = M_4 \), \( k_1 M_1 = k_2^\infty M_2 \).

(A5) \( a, b \in C^1([0,T]; C^{0,\alpha}(\Gamma^e_{sw})) \), \( a, b \geq 0 \) in \( [0,T] \times \Gamma^e_{sw} \), \( \partial_t a, \partial_t b \in L^\infty((0,T) \times \Gamma^e_{sw}) \).

(A6) \( \partial_t u^D_3 \), \( \partial_t u^D_5 \) and \( \nabla \partial_t u^D_3 \) are bounded.

(A7) \( k_3 \in C^1([0,T]; C^{0,\alpha}(\Gamma^e_{sw})) \) and \( k_j \in C^1([0,T]; C^{0,\alpha}(\partial \Omega)) \) for any \( j \in \{1, 2\} \) and \( \alpha \in]0, 1\].

The assumptions (A1)–(A3), (A5), and (A6) are of technical nature. The first equality in (A4) points out an infinitely fast (equilibrium) Henry law, while the last two equalities remotely resemble a detailed balance in two of the involved chemical reactions.

3.3 Weak formulation of the microscopic model

Definition 1 Assume (A1) and (A3). We call the vector \( u^e = (u^e_1, u^e_2, u^e_3, u^e_4, u^e_5) \), a weak solution to (2)–(6) if \( u^e_j \in L^2(0,T; H^1(\Omega^e)) \), \( \partial_t u^e_j \in L^2(0,T; H^-1(\Omega^e)) \), \( j \in \{1, 2, 4\} \), \( u^e_3 \in L^2(0,T; H^1_{D}(\Omega^e)) \), \( \partial_t u^e_3 \in L^2(0,T; H^-1(\Omega^e_1)) \), \( u^e_5 \in L^\infty((0,T) \times \Gamma^e_{sw}) \), \( \partial_t u^e_5 \in L^\infty((0,T) \times \Gamma^e_{sw}) \) such that the following identities hold

\[
\langle \partial_t u^e_1, \varphi_1 \rangle_{(0,T) \times \Omega^e} + (d_1^{-1} \nabla u^e_1, \nabla \varphi_1)_{(0,T) \times \Omega^e} = -(k_1 u^e_1, \varphi_1)_{(0,T) \times \Omega^e} + (k_2 u^e_2, \varphi_1)_{(0,T) \times \Omega^e} + \varepsilon(\eta(u^e_1, u^e_4), \varphi_1)_{(0,T) \times \Gamma^e_{sw}}, \tag{9}
\]

\[
\langle \partial_t u^e_2, \varphi_2 \rangle_{(0,T) \times \Omega^e} + (d_2^{-1} \nabla u^e_2, \nabla \varphi_2)_{(0,T) \times \Omega^e} = (k_1^2 u^e_1, \varphi_2)_{(0,T) \times \Omega^e} - (k_2^2 u^e_2, \varphi_2)_{(0,T) \times \Omega^e} + \varepsilon(a \varepsilon u^e_3, \varphi_2)_{(0,T) \times \Gamma^e_{sw}} - \varepsilon(a \varepsilon u^e_2, \varphi_2)_{(0,T) \times \Gamma^e_{sw}}, \tag{10}
\]
\( (\partial_t u_3^\varepsilon, \varphi_3)_{(0,T)\times \Omega_1^\varepsilon} = -(d_3^\varepsilon \nabla u_3^\varepsilon), \nabla \varphi_3)_{(0,T)\times \Omega_1^\varepsilon} \\
- \varepsilon(a_3^\varepsilon u_3^\varepsilon, \varphi_3)_{(0,T)\times \Gamma_4^{ws}} + \varepsilon(a_3^\varepsilon u_3^\varepsilon, \varphi_3)_{(0,T)\times \Gamma_4^{ws}}, \)  
(11)

\( (\partial_t u_4^\varepsilon, \varphi_4)_{(0,T)\times \Omega_1^\varepsilon} = -(d_4^\varepsilon \nabla u_4^\varepsilon), \nabla \varphi_4)_{(0,T)\times \Omega_1^\varepsilon} + (k_1^\varepsilon u_1^\varepsilon, \varphi_4)_{(0,T)\times \Omega_1^\varepsilon} \)  
(12)

for all \( \varphi_j \in L^2(0,T; H^1(\Omega_1^\varepsilon)) \), \( j \in \{1, 2, 4\} \) and \( \varphi_3 \in L^2(0,T; H_1^{ws}(\Omega_1^\varepsilon)) \) altogether with the ode

\( \partial_t u_5^\varepsilon = \eta(u_1^\varepsilon, u_5^\varepsilon) \) a.e. on \((0,T) \times \Gamma_5^{ws} \)  
(13)

and the initial conditions

\[
\begin{align*}
  u_1^\varepsilon(0,x) &= u_1^{0\varepsilon}(x) \quad x \in \Omega_1^\varepsilon \text{ for all } i \in \{1, 2, 4\}, \\
  u_3^\varepsilon(0,x) &= u_3^{0\varepsilon}(x) \quad x \in \Omega_1^\varepsilon, \\
  u_5^\varepsilon(0,x) &= u_5^{0\varepsilon}(x) \quad x \in \Gamma_5^{ws}.
\end{align*}
\]
(14)

3.4 Basic results

**Lemma 2** (Positivity and \( L^\infty \)-estimates) Assume (A1)-(A6), and let \( t \in [0,T] \) be arbitrarily chosen. Then the following estimates hold:

(i) \( u_i^\varepsilon(t) \geq 0 \), \( i \in \{1, 2, 4\} \) a.e. in \( \Omega_1^\varepsilon \), \( u_3^\varepsilon(t) \geq 0 \) a.e. in \( \Omega_1^\varepsilon \) and \( u_5^\varepsilon(t) \geq 0 \) a.e. on \( \Gamma_5^{ws} \).

(ii) \( u_i^\varepsilon(t) \leq M_i \), \( i \in \{1, 2\} \), \( u_3^\varepsilon(t) \leq (t + 1)M_4 \) a.e. in \( \Omega_1^\varepsilon \), \( u_5^\varepsilon(t) \leq M_5 \) a.e. in \( \Omega_1^\varepsilon \) and \( u_5^\varepsilon(t) \leq M_5 \) a.e. on \( \Gamma_5^{ws} \).

**Proof** (i). We test (9)-(12) with \( \varphi = (-u_1^\varepsilon, -u_2^\varepsilon, -u_3^\varepsilon, -u_4^\varepsilon) \) element of the space \( [L^2(0,T; H^1(\Omega_1^\varepsilon))]^2 \times L^2(0,T; H_1^{ws}(\Omega_1^\varepsilon)) \times L^2(0,T; H^1(\Omega_1^\varepsilon)) \). We obtain the following inequality

\[
\frac{1}{2}\partial_t |u_1^\varepsilon|^2 + d_1 |\nabla u_1^\varepsilon|^2 \leq -k_1 |u_1^\varepsilon|^2 + k_2^\infty (|u_1^\varepsilon|^2 + |u_2^\varepsilon|^2) \\
- \varepsilon(\eta(u_1^\varepsilon, u_5^\varepsilon), -u_1^\varepsilon)_{\Gamma_5^{ws}}.
\]
(15)

Note that the first term on the r.h.s of (15) is negative, while the third term is zero because of (A2). We then get

\[
\partial_t |u_1^\varepsilon|^2 + 2d_1 |\nabla u_1^\varepsilon|^2 \leq k_2^\infty \left( |u_1^\varepsilon|^2 + |u_2^\varepsilon|^2 \right).
\]
(16)

On the other hand, (10) leads to

\[
\frac{1}{2}\partial_t |u_2^\varepsilon|^2 + d_2 |\nabla u_2^\varepsilon|^2 \leq \frac{k_2^\infty}{2} \left( |u_1^\varepsilon|^2 + |u_2^\varepsilon|^2 \right) \\
+ \varepsilon a^\infty (u_2^\varepsilon, u_3^\varepsilon)_{\Gamma_4^{ws}} + \varepsilon b^\infty |u_2^\varepsilon|^2_{\Gamma_4^{ws}}.
\]

By the trace inequality (7) (with \( \varepsilon < 1 \)), we get

\[
\partial_t |u_2^\varepsilon|^2 + 2(d_2 - C_s^\infty b^\infty) |\nabla u_2^\varepsilon|^2 \leq k_1^\infty \left( |u_1^\varepsilon|^2 + |u_2^\varepsilon|^2 \right) \\
+ 2C_s^\infty b^\infty |u_2^\varepsilon|^2 + 2\varepsilon a^\infty (u_2^\varepsilon, u_3^\varepsilon)_{\Gamma_4^{ws}}.
\]
(17)
(11) leads to
\[ \partial_t |u_3^\varepsilon|^2 + 2(d_3 - C^*a^\infty)|\nabla u_3^\varepsilon|^2 \leq 2\varepsilon b^\infty(u_2^\varepsilon, u_3^\varepsilon)_{\Gamma_{\varepsilon}} + 2C^*a^\infty|u_3^\varepsilon|^2, \] \hfill (18)
while from (12), we see that
\[ \partial_t |u_4^\varepsilon|^2 + 2d_4|\nabla u_4^\varepsilon|^2 \leq k_1^\infty \left( |u_1^\varepsilon|^2 + |u_5^\varepsilon|^2 \right). \] \hfill (19)

Adding up inequalities (16)-(19) gives
\[ \partial_t \sum_{i=1}^{4} |u_i^\varepsilon|^2 + 2d_1|\nabla u_1^\varepsilon|^2 + (2d_2 - C^*b^\infty)|\nabla u_2^\varepsilon|^2 \]
\[ + 2(d_3 - C^*a^\infty)|\nabla u_3^\varepsilon|^2 + 2d_4|\nabla u_4^\varepsilon|^2 \]
\[ \leq (2k_1^\infty + k_2^\infty + 2C^*b^\infty + 2C^*a^\infty) \sum_{i=1}^{4} |u_i^\varepsilon|^2 \]
\[ + 2\varepsilon(a^\infty + b^\infty)(u_2^\varepsilon, u_3^\varepsilon)_{\Gamma_{\varepsilon}}, \] \hfill (20)
and hence,
\[ \partial_t \sum_{i=1}^{4} |u_i^\varepsilon|^2 + 2d_1|\nabla u_1^\varepsilon|^2 + (2d_2 - C^*b^\infty)|\nabla u_2^\varepsilon|^2 \]
\[ + 2(d_3 - C^*a^\infty)|\nabla u_3^\varepsilon|^2 + 2d_4|\nabla u_4^\varepsilon|^2 \]
\[ \leq (2k_1^\infty + k_2^\infty + C^*(a^\infty + b^\infty)) \sum_{i=1}^{4} |u_i^\varepsilon|^2 \]
\[ + \varepsilon \left( a^\infty + b^\infty)(\delta |u_2^\varepsilon|^2_{\Gamma_{\varepsilon}} + \frac{1}{\delta} |u_3^\varepsilon|^2_{\Gamma_{\varepsilon}}) \right). \] \hfill (21)

Applying the trace inequality (7) to estimate the last term on the right side of (21), we finally get
\[ \partial_t \sum_{i=1}^{4} |u_i^\varepsilon|^2 + 2d_1|\nabla u_1^\varepsilon|^2 + (2d_2 - 2C^*b^\infty - C^*\delta(a^\infty + b^\infty))|\nabla u_2^\varepsilon|^2 \]
\[ + (2d_3 - 2C^*a^\infty - \frac{C^*}{\delta}(a^\infty + b^\infty))|\nabla u_3^\varepsilon|^2 + 2d_4|\nabla u_4^\varepsilon|^2 \]
\[ \leq C_1 \sum_{i=1}^{4} |u_i^\varepsilon|^2. \]

Thus, we have
\[ \partial_t \sum_{i=1}^{4} |u_i^\varepsilon|^2 \leq C_1 \sum_{i=1}^{4} |u_i^\varepsilon|^2. \]
where \( C_1 := 2k_1^\infty + k_2^\infty + C^*(a^\infty + b^\infty) + C^*(\delta + \frac{1}{\delta})(a^\infty + b^\infty) \) and \( \delta \) is chosen conveniently. Gronwall’s inequality together with \([u_i^\varepsilon(0)]^- = 0\) gives now the desired result. Note that (A2) ensures automatically the positivity of \( u_5^\varepsilon \).
(ii) We consider the test function

\[(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = ((u_1^\varepsilon - M_1)^+, (u_2^\varepsilon - M_2)^+, (u_3^\varepsilon - M_3)^+, (u_4^\varepsilon - (t + 1)M_4)^+).\]

Obviously, \(\varphi \in [L^2(0, T; H^1(\Omega))]^2 \times L^2(0, T; H^1_0(\Omega^\varepsilon)) \times L^2(0, T; H^1(\Omega^\varepsilon))\) is allowed as test function. We obtain from (9) that

\[
\frac{1}{2} \partial_t |(u_1^\varepsilon - M_1)^+|^2 + d_1 |\nabla (u_1^\varepsilon - M_1)^+|^2 \leq -k_1 |(u_1^\varepsilon - M_1)^+|^2 - (k_1M_1, (u_1^\varepsilon - M_1)^+)
\]

\[
+ k_2^\infty ((u_2^\varepsilon - M_1)^+, (u_1^\varepsilon - M_1)^+)^2
\]

\[
+ \varepsilon (\eta(u_1^\varepsilon, u_2^\varepsilon), (u_1^\varepsilon - M_1)^+)_\Gamma_{\psi^\varepsilon}.
\]

Relying on (A4), we get the estimate

\[
\partial_t |(u_1^\varepsilon - M_1)^+|^2 \leq k_2^\infty |(u_1^\varepsilon - M_1)^+|^2 + |(u_2^\varepsilon - M_2)^+|^2. \tag{22}
\]

(10) in combination with (A4) gives that

\[
\partial_t |(u_2^\varepsilon - M_2)^+|^2 + 2(d_2 - C^*b^\infty)|\nabla (u_2^\varepsilon - M_2)^+|^2
\]

\[
\leq k_1^\infty |(u_1^\varepsilon - M_1)^+|^2 + |(u_2^\varepsilon - M_2)^+|^2
\]

\[
+ 2C^*b^\infty |(u_2^\varepsilon - M_2)^+|^2
\]

\[
+ 2\varepsilon a^\infty ((u_2^\varepsilon - M_2)^+, (u_3^\varepsilon - M_3)^+)_\Gamma_{\psi^\varepsilon}. \tag{23}
\]

By (11), we obtain

\[
\partial_t |(u_3^\varepsilon - M_3)^+|^2 + 2(d_3 - C^*a^\infty)|\nabla (u_3^\varepsilon - M_3)^+|^2
\]

\[
\leq 2C^*a^\infty |\nabla (u_3^\varepsilon - M_3)^+|^2
\]

\[
+ 2\varepsilon b^\infty ((u_2^\varepsilon - M_2)^+, (u_3^\varepsilon - M_3)^+)_\Gamma_{\psi^\varepsilon}. \tag{24}
\]

Using again (A4), (12) yields

\[
\partial_t |(u_4^\varepsilon - (t + 1)M_4)^+|^2 \leq k_1^\infty |(u_1^\varepsilon - M_1)^+|^2 + |(u_4^\varepsilon - (t + 1)M_4)^+|^2. \tag{25}
\]

Adding up (22)–(25) side by side, we get

\[
\sum_{j=1}^3 \partial_t |(u_j^\varepsilon - M_j)^+|^2 + \partial_t |(u_4^\varepsilon - (t + 1)M_4)^+|^2 + (2d_2 - 2C^*b^\infty)|\nabla (u_2^\varepsilon - M_2)^+|^2
\]

\[
+ (2d_3 - 2C^*a^\infty)|\nabla (u_3^\varepsilon - M_3)^+|^2
\]

\[
\leq (2k_2^\infty + k_1^\infty + 2C^*a^\infty + 2C^*b^\infty)(\sum_{j=1}^3 |(u_j^\varepsilon - M_j)^+|^2
\]

\[
+ |(u_4^\varepsilon - (t + 1)M_4)^+|^2) + \varepsilon (a^\infty + b^\infty)(\delta |(u_2^\varepsilon - M_2)^+|^2)
\]

\[
+ \frac{1}{\delta} |(u_3^\varepsilon - M_3)^+|^2)_\Gamma_{\psi^\varepsilon}.
\]
We use the trace inequality (7) (with $\varepsilon < 1$) to deal with the boundary terms in (26). Then Gronwall’s inequality yields for all $t \in (0, T)$ the following estimate
\[
\begin{align*}
  u_1^j(t) &\leq M_j, \quad j \in \{1, 2, 5\} \text{ a.e. in } \Omega^s, \\
  u_5^j(t) &\leq M_3, \quad \text{a.e. in } \Omega_1^s, \\
  u_4^5 &\leq (t + 1)M_4 \text{ a.e. in } \Omega^s.
\end{align*}
\]
Furthermore, by (A2) $u_5^5$ is bounded.

**Proposition 3 (Uniqueness)** Assume (A1)–(A4). Then there exists at most one weak solution in the sense of Definition 1.

**Proof.** We assume that $u^{j, \varepsilon} = (u_1^{j, \varepsilon}, u_2^{j, \varepsilon}, u_3^{j, \varepsilon}, u_4^{j, \varepsilon}, u_5^{j, \varepsilon}), j \in \{1, 2\}$ are two distinct weak solutions in the sense of Definition 1. We set $u_i^\varepsilon := u_i^{1, \varepsilon} - u_i^{2, \varepsilon}$ for all $i \in \{1, 2, 3, 4\}$. Firstly, we deal with (15). We obtain
\[
\partial_t u_5^{1, \varepsilon} - \partial_t u_5^{2, \varepsilon} = \eta(u_1^{1, \varepsilon}, u_5^{1, \varepsilon}) - \eta(u_1^{2, \varepsilon}, u_5^{2, \varepsilon}).
\]  
(26)

Integrating (26) along $(0, T)$ and using (A2), we get
\[
|u_5^{1, \varepsilon} - u_5^{2, \varepsilon}| \leq k_3 c_{RCQ} M_1 \int_0^T |u_5^{1, \varepsilon} - u_5^{2, \varepsilon}| d\tau + k_3 c_{RCQ}^\infty \int_0^T |u_1^{1, \varepsilon} - u_1^{2, \varepsilon}| d\tau.
\]

Gronwall’s inequality implies
\[
|u_5^{1, \varepsilon}(t) - u_5^{2, \varepsilon}(t)| \leq C_2 \int_0^t |u_1^{1, \varepsilon} - u_1^{2, \varepsilon}| d\tau \quad \text{for a.e. } t \in (0, T), 
\]  
(27)

where $C_2 := k_3 c_{RCQ}^\infty (1 + C_4 t e^{C_3 t})$ and $C_3 := k_3 c_{RCQ} M_1$. We calculate
\[
\frac{1}{2} \partial_t |u_1^{\varepsilon}|^2 + d_1 \partial_t |u_2^{\varepsilon}|^2 \leq -k_1 |u_1^{\varepsilon}|^2 + k_2^\infty (u_1^{\varepsilon}, u_2^{\varepsilon}) + \varepsilon (\eta_1 - \eta_2, u_2^{\varepsilon})_{\Gamma^w}, 
\]  
(28)

where we denote $\eta_1 - \eta_2 := \eta(u_1^{1, \varepsilon}, u_5^{1, \varepsilon}) - \eta(u_1^{2, \varepsilon}, u_5^{2, \varepsilon})$. We can write
\[
\begin{align*}
  \frac{1}{2} \partial_t |u_1^{\varepsilon}|^2 + d_1 \partial_t |u_2^{\varepsilon}|^2 &\leq -k_1 |u_1^{\varepsilon}|^2 + \frac{k_2^\infty}{2} (|u_1^{\varepsilon}|^2 + |u_2^{\varepsilon}|^2) \\
  &\quad + \varepsilon C_3 (u_5^{1, \varepsilon} - u_5^{2, \varepsilon}, u_1^{\varepsilon})_{\Gamma^w} \\
  &\quad + \varepsilon k_3 c_{RCQ}^\infty (u_1^{1, \varepsilon} - u_1^{2, \varepsilon}, u_1^{\varepsilon})_{\Gamma^w}.
\end{align*}
\]  
(29)

Now, inserting (27) in (29) yields
\[
\begin{align*}
  \frac{1}{2} \partial_t |u_1^{\varepsilon}|^2 + d_1 \partial_t |u_2^{\varepsilon}|^2 &\leq -k_1 |u_1^{\varepsilon}|^2 + \frac{k_2^\infty}{2} (|u_1^{\varepsilon}|^2 + |u_2^{\varepsilon}|^2) \\
  &\quad + C_4 \varepsilon |u_1^{\varepsilon}|^2_{\Gamma^w} + \frac{\varepsilon C_3^2}{2 \delta} \int_0^t |u_1^{\varepsilon}|^2_{\Gamma^w} d\tau,
\end{align*}
\]  
(30)

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where $C_4 := k_3^\infty c_R Q^\infty + C_2^3$. Using (7), we estimate the last two terms in (30) to obtain the inequality

$$
\frac{1}{2} \partial_t |u_1^\varepsilon|^2 + d_1 |\nabla u_1^\varepsilon|^2 \leq -k_1|u_1^\varepsilon|^2 + \frac{k_2^\infty}{2} (|u_1^\varepsilon|^2 + |u_2^\varepsilon|^2) + C^* C_4 (|u_1^\varepsilon|^2 + \varepsilon^2 |\nabla u_1^\varepsilon|^2)
$$

(31)

Note that the constant $C^*$, arising from in (31), stems from (7). Rearranging now the terms, we have

$$
\partial_t |u_1^\varepsilon|^2 + (2d_1 - 2C^* C_4 \varepsilon^2) |\nabla u_1^\varepsilon|^2 + 2k_1|u_1^\varepsilon|^2 \leq (k_2^\infty + C^* C_4) (|u_1^\varepsilon|^2 + \varepsilon^2 |\nabla u_1^\varepsilon|^2)
$$

(32)

Following the same line of arguments as before, we obtain from (10) that

$$
\partial_t |u_2^\varepsilon|^2 + 2d_2 |\nabla u_2^\varepsilon|^2 \leq -2k_2 |u_2^\varepsilon|^2 + k_1^\infty (|u_1^\varepsilon|^2 + |u_2^\varepsilon|^2) + 2\varepsilon a^\infty(u_2^\varepsilon, u_2^\varepsilon)_{\Gamma^\varepsilon} + 2\varepsilon b^\infty |u_2^\varepsilon|^2_{\Gamma^\varepsilon^0},
$$

(33)

while from (11), we deduce

$$
\partial_t |u_3^\varepsilon|^2 + 2d_3 |\nabla u_3^\varepsilon|^2 \leq 2\varepsilon b^\infty(u_2^\varepsilon, u_3^\varepsilon)_{\Gamma^\varepsilon^0} + 2\varepsilon a^\infty |u_3^\varepsilon|^2_{\Gamma^\varepsilon^0}.
$$

(34)

Proceeding similarly, (12) yields

$$
\partial_t |u_4^\varepsilon|^2 + 2d_4 |\nabla u_4^\varepsilon|^2 \leq k_2^\infty (|u_1^\varepsilon|^2 + |u_4^\varepsilon|^2).
$$

(35)

Putting together (32)–(35), we get

$$
\partial_t \Sigma_{i=1}^4 |u_i^\varepsilon|^2 + (2d_1 - C^* C_4 \varepsilon^2) |\nabla u_1^\varepsilon|^2 + 2d_2 |\nabla u_2^\varepsilon|^2 + 2d_3 |\nabla u_3^\varepsilon|^2 + 2d_4 |\nabla u_4^\varepsilon|^2

+ 2k_1|u_1^\varepsilon|^2 \leq (2k_1^\infty + k_2^\infty + C^* C_2) \Sigma_{i=1}^4 |u_i^\varepsilon|^2

+ C^* C_2^3 \int_0^t (|u_1^\varepsilon|^2 + \varepsilon^2 |\nabla u_1^\varepsilon|^2) d\tau

+ 2\varepsilon b |u_2^\varepsilon|^2_{\Gamma^\varepsilon^0} + 2\varepsilon a |u_3^\varepsilon|^2_{\Gamma^\varepsilon^0}

+ \varepsilon (a^\infty + b^\infty)(\delta |u_2^\varepsilon|^2_{\Gamma^\varepsilon^0} + \frac{1}{\delta} |u_3^\varepsilon|^2_{\Gamma^\varepsilon^0}).
$$

(36)

Applying the trace inequality (7) to the boundary terms in (36), we get

$$
\partial_t \Sigma_{i=1}^4 |u_i^\varepsilon|^2 + (2d_1 - 2C^* C_4 \varepsilon^2) |\nabla u_1^\varepsilon|^2

+ (2d_2 - 2C^* b^\infty \varepsilon^2 - C^* \delta \varepsilon^2 (a^\infty + b^\infty)) |\nabla u_2^\varepsilon|^2

+ (2d_3 - 2C^* a^\infty \varepsilon^2 - \frac{C^* \varepsilon^2}{\delta} (a^\infty + b^\infty)) |\nabla u_3^\varepsilon|^2

+ 2d_4 |\nabla u_4^\varepsilon|^2 + 2k_1|u_1^\varepsilon|^2 \leq C_5 \Sigma_{i=1}^4 |u_i^\varepsilon|^2
$$
\[
\int_{t_0}^{t} \left( |u^\varepsilon_j|^2 + \varepsilon^2 |\nabla u^\varepsilon_j|^2 \right) d\tau,
\]
where \( C_\varepsilon := 2k_1^\infty + k_2^\infty + C^*C_2 + 2C^*(a^\infty + b^\infty) + C^*(a^\infty + b^\infty)(\delta + \frac{1}{2}) \). Let us choose \( \varepsilon \) and \( \delta \) such that\[\varepsilon \in \left[0, \frac{2d_1}{C_1C^*} \right], \quad \delta \in \left[\frac{C^*\varepsilon^2(a^\infty + b^\infty)}{2d_3 - C^*a^\infty\varepsilon^2}, \frac{2d_2 - C^*b^\infty\varepsilon^2}{C^*\varepsilon^2(a^\infty + b^\infty)} \right].\]

With this choice of \((\varepsilon, \delta)\), (37) takes the form
\[
\partial_t \Sigma^4_{i=1} |u^\varepsilon_j|^2 + \bar{C} |\nabla u^\varepsilon_j|^2 + \bar{C} |u^\varepsilon_j|^2 \leq C_0 (\Sigma^4_{i=1} |u^\varepsilon_j|^2 + \int_{t_0}^{t} (|u^\varepsilon_j|^2 + \varepsilon^2 |\nabla u^\varepsilon_j|^2) d\tau),
\]
where \( C_0 := 2k_1^\infty + k_2^\infty + C^*C_2 + 2C^*(a^\infty + b^\infty) + C^*C_2^2 \frac{1}{43} \) and \( \bar{C} := \min\{2d_1 - 2C^*C_2\varepsilon^2, 2k_1\} \). Taking in (37) the supremum along \( t \in (0, T) \) and applying Gronwall’s inequality, we obtain the following estimate
\[
\Sigma^4_{i=1} |u^\varepsilon_j|^2 + \bar{C} \int_{t_0}^{T} |\nabla u^\varepsilon_j|^2 dt + \bar{C} \int_{t_0}^{T} |u^\varepsilon_j|^2 dt \leq 0.
\]

Thus, the proof of Proposition 3 is completed.

**Theorem 4 (Global Existence)** Assume \((A1) - (A3)\). Then there exists at least a global-in-time weak solution in the sense of Definition 1.

**Proof.** The proof is based on the Galerkin argument. Since the proof is rather standard, and here we wish to focus on the passage to the limit \( \varepsilon \to 0 \), we omit it.

### 4 A priori estimates for microscopic solutions

This section includes the \( \varepsilon \)- independent estimates.

**Lemma 5** Assume \((A1)-(A6)\). Then the weak solution of the microscopic model (9)-(14) satisfies the following a priori bounds:

\[
\left\| u^\varepsilon_j \right\|_{L^2(0,T;H^1(\Omega^\varepsilon))} \leq C, \quad j \in \{1, 2, 3, 4\} \tag{39}
\]
\[
\left\| \nabla \partial_t u^\varepsilon_j \right\|_{L^2(0,T;L^2(\Omega^\varepsilon))} \leq C, \tag{40}
\]
\[
\left\| \partial_t u^\varepsilon_j \right\|_{L^2(0,T;L^2(\Omega^\varepsilon))} \leq C, \tag{41}
\]
\[
\left\| u^\varepsilon_3 \right\|_{L^2(0,T;H^1(\Omega^\varepsilon))} \leq C, \tag{42}
\]
\[
\left\| \nabla \partial_t u^\varepsilon_3 \right\|_{L^2(0,T;L^2(\Omega^\varepsilon))} \leq C, \tag{43}
\]
\[
\left\| \partial_t u^\varepsilon_3 \right\|_{L^2(0,T;L^2(\Omega^\varepsilon))} \leq C. \tag{44}
\]
\[ \| u^\varepsilon_1 \|_{L^\infty((0,T) \times \Gamma^v)} \leq C, \]
\[ \| \partial_t u^\varepsilon_1 \|_{L^2((0,T) \times \Gamma^v)} \leq C. \]  \hspace{1cm} (45) \hspace{1cm} (46)

In (39)–(46), the generic constant $C$ is independent of $\varepsilon$.

**Proof.** We test (9) with $\varphi_1 = u^\varepsilon_1$ to get
\[
\frac{1}{2} \partial_t |u^\varepsilon_1|^2 + d_1 |\nabla u^\varepsilon_1|^2 \leq -k_1 |u^\varepsilon_1|^2 + k_2 \infty (u^\varepsilon_1, u^\varepsilon_2) - \varepsilon(\eta, u^\varepsilon_1)_{\Gamma^v},
\leq \frac{k_2 \infty}{2} (|u^\varepsilon_1|^2 + |u^\varepsilon_2|^2) + \varepsilon k_3 \infty C_R (u^\varepsilon_1, u^\varepsilon_1)_{\Gamma^v}. \]  \hspace{1cm} (47)

After applying the trace inequality to the last term on r.h.s of (47), we get
\[
\frac{1}{2} \partial_t |u^\varepsilon_1|^2 + d_1 |\nabla u^\varepsilon_1|^2 \leq \frac{k_2 \infty}{2} (|u^\varepsilon_1|^2 + |u^\varepsilon_2|^2) + C^* k_3 \infty Q \infty c_R (|u^\varepsilon_1|^2 + \varepsilon^2 |\nabla u^\varepsilon_1|^2)_{\Gamma^v}.
\]
\[
\frac{1}{2} \partial_t |u^\varepsilon_1|^2 + (d_1 - \varepsilon^2 C^* k_3 \infty Q \infty c_R) |\nabla u^\varepsilon_1|^2 \leq C_7 (|u^\varepsilon_1|^2 + |u^\varepsilon_2|^2), \]  \hspace{1cm} (48)
where $C_7 := \frac{k_2 \infty}{2} + C^* k_3 \infty Q \infty c_R$. Taking $\varphi_2 = u^\varepsilon_2$ in (10), we get
\[
\frac{1}{2} \partial_t |u^\varepsilon_2|^2 + d_2 |\nabla u^\varepsilon_2|^2 \leq \frac{k_2 \infty}{2} (|u^\varepsilon_1|^2 + |u^\varepsilon_2|^2) - k_2 |u^\varepsilon_2|^2
+ \varepsilon a \infty (u^\varepsilon_3, u^\varepsilon_2)_{\Gamma^w} + \varepsilon b \infty |u^\varepsilon_2|^2_{\Gamma^w}.
\]

Application of the trace inequality (7) only to the last term leads to
\[
\frac{1}{2} \partial_t |u^\varepsilon_2|^2 + (d_2 - C^* a \infty \varepsilon^2) |\nabla u^\varepsilon_2|^2 \leq \frac{k_2 \infty}{2} (|u^\varepsilon_1|^2 + |u^\varepsilon_2|^2) + 2
+ \varepsilon a \infty (u^\varepsilon_3, u^\varepsilon_2)_{\Gamma^w}. \]  \hspace{1cm} (49)

We choose $\varphi_3 = u^\varepsilon_3$ as a test function in (11) to calculate
\[
\frac{1}{2} \partial_t |u^\varepsilon_3|^2 + (d_3 - C^* a \infty \varepsilon^2) |\nabla u^\varepsilon_3|^2 \leq \varepsilon b \infty (u^\varepsilon_3, u^\varepsilon_2)_{\Gamma^w} + C^* a \infty |u^\varepsilon_3|^2. \]  \hspace{1cm} (50)

Setting $\varphi_4 = u^\varepsilon_3$ in (12), we are led to
\[
\frac{1}{2} \partial_t |u^\varepsilon_3|^2 + d_4 |\nabla u^\varepsilon_3|^2 \leq \frac{k_2 \infty}{2} (|u^\varepsilon_1|^2 + |u^\varepsilon_3|^2). \]  \hspace{1cm} (51)

Putting together (48)-(51), we obtain
\[
\frac{1}{2} \sum_{i=1}^{4} \partial_t |u^\varepsilon|^2 + (d_1 - \varepsilon^2 C^* k_3 \infty Q \infty c_R) |\nabla u^\varepsilon_1|^2 + d_4 |\nabla u^\varepsilon_4|^2
+ (d_2 - C^* b \infty \varepsilon^2) |\nabla u^\varepsilon_2|^2 + (d_3 - C^* a \infty \varepsilon^2) |\nabla u^\varepsilon_3|^2
\leq (k_1 \infty + \frac{k_2 \infty}{2} + C^* b \infty + C^* a \infty) \Sigma_{i=1}^{4} |u^\varepsilon_i|^2
+ \varepsilon (a \infty + b \infty) (u^\varepsilon_3, u^\varepsilon_2)_{\Gamma^w}. \]  \hspace{1cm} (52)
Combing Young’s inequality and the trace inequality to the boundary term, (52) turns out to be

\[
\frac{1}{2} \sum_{i=1}^{4} \partial_{t}|u_{i}^{\varepsilon}|^2 + (d_{1} - \varepsilon^2 C^* k_{3}^{\infty} Q^{\infty} c_{R}) \|
abla u_{1}^{\varepsilon}\|^2 \\
+ (d_{2} - C^* b \varepsilon^2 - \frac{C^* \varepsilon^2}{2} (a^{\infty} + b^{\infty})) \|
abla u_{2}^{\varepsilon}\|^2 \\
+ (d_{3} - C^* a \varepsilon^2 - \frac{C^* \varepsilon^2}{2\delta} (a^{\infty} + b^{\infty})) \|
abla u_{3}^{\varepsilon}\|^2 + d_{4} \|
abla u_{4}^{\varepsilon}\|^2 \\
\leq (k_{1}^{\infty} + \frac{k_{2}^{\infty}}{2} + C^* (a^{\infty} + b^{\infty}) (\delta + \frac{1}{\delta})) \sum_{i=1}^{4} |u_{i}^{\varepsilon}|^2.
\]

Choosing \(\varepsilon\) small enough and \(\delta\) conveniently such that the coefficients of the terms involving \(\|
abla u_{i}^{\varepsilon}\|^2\) and \(\|
abla u_{j}^{\varepsilon}\|^2\) stay positive, we are led to

\[
\sum_{i=1}^{4} \partial_{t}|u_{i}^{\varepsilon}|^2 + d_{1} \|
abla u_{1}^{\varepsilon}\|^2 + d_{2} \|
abla u_{2}^{\varepsilon}\|^2 + d_{3} \|
abla u_{3}^{\varepsilon}\|^2 + 2d_{4} \|
abla u_{4}^{\varepsilon}\|^2 \leq C_{7} \sum_{i=1}^{4} |u_{i}^{\varepsilon}|^2,
\]

where

\[
d_{1}^{'} := 2(d_{1} - \varepsilon^2 C^* k_{3}^{\infty} Q^{\infty} c_{R}),
\]

\[
d_{2}^{'} := 2(d_{2} - C^* b \varepsilon^2 - \frac{C^* \varepsilon^2}{2} (a^{\infty} + b^{\infty})),
\]

\[
d_{3}^{'} := 2(d_{3} - C^* a \varepsilon^2 - \frac{C^* \varepsilon^2}{2\delta} (a^{\infty} + b^{\infty})),
\]

while the constant \(C\) is given by

\[
C_{8} := 2k_{1}^{\infty} + \frac{k_{2}^{\infty}}{2} + C^* a^{\infty} + C^* b^{\infty} + C^* (a^{\infty} + b^{\infty}) (\delta + \frac{1}{\delta}).
\]

Summarizing, we have

\[
\sum_{i=1}^{4} \partial_{t}|u_{i}^{\varepsilon}|^2 + d_{0} \sum_{j=1}^{3} \|
abla u_{j}^{\varepsilon}\|^2 + d_{0} \|
abla u_{3}^{\varepsilon}\|^2 \leq C \sum_{i=1}^{4} |u_{i}^{\varepsilon}|^2,
\]

where \(d_{0} := min \{d_{1}^{'} , d_{2}^{'} , d_{3}^{'} , d_{4}^{'} \}\). By Gronwall’s inequality, we have

\[
\sum_{i=1}^{4} |u_{i}^{\varepsilon}|^2 \leq C \sum_{i=1}^{4} |u_{i}(0)|^2,
\]

and hence,

\[
\|
 u_{j}^{\varepsilon}\|_{L^2(0,T;L^2(\Omega^{\varepsilon}))} \leq C \text{ for all } i \in \{1, 2, 4\} \text{ and } \|
 u_{3}^{\varepsilon}\|_{L^2(0,T;L^2(\Omega_{1}^{\varepsilon}))} \leq C,
\]

where \(C\) depends on initial data and model parameters but is independent of \(\varepsilon\). Integrating (53) along \((0, T)\), we get

\[
\|
 u_{j}^{\varepsilon}\|_{L^2(0,T;H^1(\Omega^{\varepsilon}))} \leq C, \ j \in \{1, 2, 4\},
\]

\[
\|
 u_{3}^{\varepsilon}\|_{L^2(0,T;H^1(\Omega_{1}^{\varepsilon}))} \leq C.
\]
With the help of (A2) together with the boundedness of \( u_1^\varepsilon \), we conclude from (13) that 
\[
\| u_5^\varepsilon \|_{L^\infty((0,T)\times\Gamma_{sw}^w)} \leq C.
\]
Multiplying (13) by \( \partial_t u_5^\varepsilon \) and using (A2), we get 
\[
\| \partial_t u_5^\varepsilon \|_{L^2((0,T)\times\Gamma_{sw}^w)} \leq C.
\]

Now, we focus on obtaining \( \varepsilon \)-independent estimates on the time derivative of the concentrations. Firstly, we choose \( \varphi_1 = \partial_t u_1^\varepsilon \) and get 
\[
\begin{align*}
\int_0^t \int_{\Omega^\varepsilon} \partial_t u_1^\varepsilon \partial_t u_1^\varepsilon dx d\tau &+ \int_0^t \int_{\Omega^\varepsilon} d_1^\varepsilon \nabla u_1^\varepsilon \nabla \partial_t u_1^\varepsilon dx d\tau \\
&= -\int_0^t \int_{\Omega^\varepsilon} k_1^\varepsilon \partial_t u_1^\varepsilon \partial_t u_1^\varepsilon dx d\tau + \int_0^t \int_{\Omega^\varepsilon} k_2^\varepsilon \partial_t u_1^\varepsilon \partial_t u_1^\varepsilon dx d\tau \\
&\quad - \varepsilon \int_0^t \int_{\Gamma_{sw}^w} \eta \partial_t u_1^\varepsilon d\sigma_x d\tau.
\end{align*}
\]
Consequently, it holds 
\[
\begin{align*}
\int_0^t \int_{\Omega^\varepsilon} |\partial_t u_1^\varepsilon|^2 dx d\tau &+ \int_0^t \int_{\Omega^\varepsilon} \left( \frac{1}{2} \partial_t (d_1^\varepsilon |\nabla u_1^\varepsilon|^2) - (\partial_t d_i^\varepsilon) |\nabla u_1^\varepsilon|^2 \right) dx d\tau \\
&\leq -\frac{k_1}{2} \int_0^t \int_{\Omega^\varepsilon} \partial_t |u_1^\varepsilon|^2 dx d\tau \\
&\quad + \frac{k_2}{2} \int_0^t \int_{\Omega^\varepsilon} \left( \frac{1}{\delta} |u_2^\varepsilon|^2 + \delta |\partial_t u_1^\varepsilon|^2 \right) dx d\tau \\
&\quad - \varepsilon \int_0^t \int_{\Gamma_{sw}^w} (|\partial_t \eta u_1^\varepsilon| - (\partial_t \eta) u_1^\varepsilon) d\sigma_x d\tau,
\end{align*}
\]
\[
(1 - \frac{k_2 \delta}{2}) \int_0^t \int_{\Omega^\varepsilon} |\partial_t u_1^\varepsilon|^2 dx d\tau \leq D_1 \int_0^t \int_{\Omega^\varepsilon} |\nabla u_1^\varepsilon|^2 dx d\tau \\
+ \frac{d_1}{2} \int_{\Omega^\varepsilon} |\nabla u_{10}|^2 dx + \frac{k_2}{2 \delta} \int_0^t \int_{\Omega^\varepsilon} |u_2^\varepsilon|^2 dx d\tau \\
+ \frac{\varepsilon}{2} \int_{\Gamma_{sw}^w} (|\eta|^2 + |u_1^\varepsilon|^2 + |\eta(0)|^2 + |u_1^\varepsilon(0)|^2) dx d\tau \\
+ \frac{\varepsilon}{2} \int_0^t \int_{\Gamma_{sw}^w} (|\partial_t \eta|^2 + |u_1^\varepsilon|^2) d\sigma_x d\tau,
\]
where \( \eta(0) := \eta(u_1^\varepsilon(0), u_5^\varepsilon(0)) \). Applying (7) and recalling (55), we have 
\[
\int_0^t \int_{\Omega^\varepsilon} |\partial_t u_1^\varepsilon|^2 dx d\tau \leq C_9,
\]
where 
\[
C_9 := D_1 \int_0^t \int_{\Omega^\varepsilon} |\nabla u_1^\varepsilon|^2 dx d\tau + \frac{k_1}{2} \int_{\Omega^\varepsilon} |u_1^\varepsilon(0)|^2 dx + \frac{d_1}{2} \int_{\Omega^\varepsilon} |\nabla u_{10}|^2 dx.
\]
By (7) and (55), we get
\begin{align*}
+ \frac{k_2^\infty}{2\delta} \int_0^t \int_{\Omega^c} |u_2^\varepsilon|^2 dxd\tau + \varepsilon \int_0^t \int_{\Gamma_w^{\varepsilon}} (|\eta|^2 + |\eta(0)|^2 + |\dot{\eta}|^2)
\end{align*}
\begin{align*}
+ \frac{C^*}{2} \int_0^t \int_{\Omega^c} (|u_1^\varepsilon|^2 + \varepsilon^2 |\nabla u_1^\varepsilon|^2) dxd\tau + \frac{C^*}{2} \int_0^t (|u_1^\varepsilon|^2 + \varepsilon^2 |\nabla u_1^\varepsilon|^2) dxd\tau,
\end{align*}
and \( \delta \in \left]0, \frac{2}{k_2^\infty}\right[. \) Testing (10) with \( \varphi_2 = \partial_t u_2^\varepsilon \) gives
\begin{align*}
\int_0^t \int_{\Omega^c} |\partial_t u_2^\varepsilon|^2 dxd\tau + \int_0^t \int_{\Omega^c} \left( \frac{1}{2} \partial_t (d_2^\varepsilon |\nabla u_2^\varepsilon|^2) - (\partial_t d_2^\varepsilon) |\nabla u_2^\varepsilon|^2 \right) dxd\tau
\end{align*}
\begin{align*}
\leq -\frac{k_2}{2} \int_0^t \int_{\Omega^c} \partial_t |u_2^\varepsilon|^2 dxd\tau + \frac{k_1^\infty}{2} \int_0^t \int_{\Omega^c} \left( \frac{1}{\delta} |u_1^\varepsilon|^2 \right) dxd\tau
\end{align*}
\begin{align*}
+ \delta |\partial_t u_2^\varepsilon|^2 dxd\tau + \frac{\varepsilon a^\infty}{2} \int_0^t \int_{\Gamma_w^{\varepsilon}} \left( |u_3|_2^2 + |\partial_t u_3^\varepsilon|^2 \right) d\sigma_x d\tau
\end{align*}
\begin{align*}
+ \frac{\varepsilon b^\infty}{2} \int_0^t \int_{\Gamma_w^{\varepsilon}} \partial_t |u_2^\varepsilon|^2 d\sigma_x d\tau,
\end{align*}
and hence,
\begin{align*}
\int_0^t \int_{\Omega^c} |\partial_t u_2^\varepsilon|^2 dxd\tau + \frac{d_2^\infty}{2} \int_0^t \int_{\Omega^c} |\nabla u_2^\varepsilon|^2 dxd\tau
\end{align*}
\begin{align*}
\leq \frac{d_2^\infty}{2} \int_0^t \int_{\Omega^c} |\nabla u_2^\varepsilon(0)|^2 dx + D_2 \int_0^t \int_{\Omega^c} |\nabla u_2^\varepsilon|^2 dxd\tau
\end{align*}
\begin{align*}
+ \frac{k_1^\infty}{2} \int_0^t \int_{\Omega^c} \left( \frac{1}{\delta} |u_1^\varepsilon|^2 + \delta |\partial_t u_2^\varepsilon|^2 \right) dxd\tau
\end{align*}
\begin{align*}
+ \frac{C^* a^\infty}{2} \int_0^t \int_{\Omega^c} \left( |u_3|_2^2 + \varepsilon^2 |\nabla u_3^\varepsilon|^2 + \varepsilon^2 |\nabla \partial_t u_3^\varepsilon|^2 \right) dxd\tau
\end{align*}
\begin{align*}
+ \frac{\varepsilon b^\infty}{2} \int_0^t \int_{\Gamma_w^{\varepsilon}} \left( |u_2^\varepsilon|^2 - |u_2^\varepsilon(0)|^2 \right) d\sigma_x.
\end{align*}
By (7) and (55), we get
\begin{align*}
\left( 1 - \frac{C^* a^\infty}{2} - \frac{k_1^\infty \delta}{2} \right) \int_0^t \int_{\Omega^c} |\partial_t u_2^\varepsilon|^2 dxd\tau \leq C_{10} \left( 1 + \varepsilon^2 \int_0^t \int_{\Omega^c} |\nabla \partial_t u_2^\varepsilon|^2 dxd\tau \right).
\end{align*}
Consequently, choosing \( \delta \in \left]0, \frac{2 - C^* a^\infty}{k_1^\infty}\right[ \), we are led to
\begin{align*}
\int_0^t \int_{\Omega^c} |\partial_t u_2^\varepsilon|^2 dxd\tau \leq C_{10} (1 + \varepsilon^2) \int_0^t \int_{\Omega^c} |\nabla \partial_t u_2^\varepsilon|^2 dxd\tau, \quad (59)
\end{align*}
where
\begin{align*}
C_{10} := D_2 \int_0^t \int_{\Omega^c} |\nabla u_1^\varepsilon|^2 dxd\tau + \frac{d_2^\infty}{2} \int_0^t \int_{\Omega^c} |\nabla u_2^\varepsilon(0)|^2 dx + \frac{k_1^\infty}{2\delta} \int_0^t \int_{\Omega^c} |u_2^\varepsilon|^2 dxd\tau
\end{align*}
\begin{align*}
+ \frac{C^* b^\infty}{2} \int_0^t \int_{\Omega^c} \left( |u_2^\varepsilon|^2 + \varepsilon^2 |\nabla u_2^\varepsilon|^2 + |u_2^\varepsilon(0)|^2 + \varepsilon^2 |\nabla u_2^\varepsilon(0)|^2 \right) dx
\end{align*}
\begin{align*}
+ \frac{C^* a^\infty}{2} \int_0^t \int_{\Omega^c} \left( |u_3|^2 + \varepsilon^2 |\nabla u_3^\varepsilon|^2 \right) dxd\tau.
\end{align*}
The initial data \( u^{\epsilon}_{30} \) holding in \( \Omega_1^\epsilon \) and the Dirichlet data \( u^{D}_{3} \) acting on the exterior boundary of \( \Omega_1^\epsilon \) are considered here as restrictions of the respective functions defined on whole of \( \overline{\Omega} \). Testing now (11) with \( \varphi_3 = \partial_t(u^\epsilon_3 - u^D_3) \) leads to

\[
\int_0^t \int_{\Omega^\epsilon} |\partial_t u^\epsilon_3|^2 \, dx \, dt + \frac{d_3}{2} \int_0^t \int_{\Omega^\epsilon} |\nabla u^\epsilon_3|^2 \, dx \, dt \\
\leq \frac{d_3}{2} \int_0^t \int_{\Omega^\epsilon} |\nabla u^\epsilon_3(0)|^2 \, dx + \frac{1}{2}(|\partial_t u^\epsilon_3|^2 + |\partial_t u^D_3|^2) \\
+ D_3 \int_0^t \int_{\Omega^\epsilon} |\nabla u^\epsilon_3|^2 + \frac{d_3^2}{2} \int_0^t \int_{\Omega^\epsilon} (|\nabla u^\epsilon_3|^2 + |\nabla \partial_t u^D_3|^2) \\
+ \frac{\varepsilon a^\infty}{\delta} \int_0^t \int_{\Gamma^\epsilon} |u^\epsilon_3|^2 + \frac{\varepsilon \delta}{2} (a^\infty + a^\infty) \int_0^t \int_{\Gamma^\epsilon} |\partial_t u^\epsilon_3|^2 \\
+ \frac{\varepsilon b^\infty}{\delta} \int_0^t \int_{\Gamma^\epsilon} |\partial_t u^\epsilon_3|^2 + \frac{\varepsilon b^\infty}{2} \int_0^t \int_{\Gamma^\epsilon} |u^\epsilon_2|^2.
\]

Using (7) and (A6), we obtain

\[
\int_0^t \int_{\Omega^\epsilon} |\partial_t u^\epsilon_3|^2 \, dx \, dt \leq C_{11}(1 + \varepsilon^2 \delta) \int_0^t \int_{\Omega^\epsilon} |\nabla \partial_t u^\epsilon_3|^2 \, dx \, dt,
\]

where \( \delta \in \left[0, \frac{2}{\varepsilon \gamma(a^\infty + b^\infty)} \right] \) and

\[
C_{11} := D_3 \int_0^t \int_{\Omega^\epsilon} |\nabla u^\epsilon_3|^2 \, dx \, dt + \frac{d_3}{2} \int_0^t \int_{\Omega^\epsilon} |\nabla u^\epsilon_3(0)|^2 \, dx + \frac{1}{2\delta} \int_0^t \int_{\Omega^\epsilon} |\nabla u^D_3|^2 \\
+ \frac{d_3^2}{2} \int_0^t \int_{\Omega^\epsilon} (|\nabla u^\epsilon_3|^2 + |\nabla \partial_t u^D_3|^2) + \frac{C_* a^\infty}{\delta} \int_0^t \int_{\Omega^\epsilon} (|u^\epsilon_3|^2 + \varepsilon^2 |\nabla u^\epsilon_3|^2) \\
+ \frac{C_* b^\infty}{\delta} \int_0^t \int_{\Omega^\epsilon} (|u^\epsilon_2|^2 + \varepsilon^2 |\nabla u^\epsilon_2|^2) \, dx \, dt \\
+ \frac{C_* (a^\infty + b^\infty)}{2} \int_0^t \int_{\Omega^\epsilon} (|\partial_t u^D_3|^2 + \varepsilon^2 |\nabla \partial_t u^D_3|^2) \, dx \, dt.
\]

From (12), we get

\[
\int_0^t \int_{\Omega^\epsilon} |\partial_t u^\epsilon_3|^2 \, dx \, dt \leq C_{12}.
\]

In order to estimate (59) and (60), we proceed first with differentiating (10) with respect to time and then testing the result with \( \partial_t u^\epsilon_2 \). Consequently, we derive

\[
\frac{1}{2} \int_0^t |\partial_t u^\epsilon_2|^2 \, dx + d_2 \int_0^t \int_{\Omega^\epsilon} |\nabla \partial_t u^\epsilon_2|^2 \, dx \, dt
\]
Using (7), it yields
\[
\frac{1}{2} \int_{\Omega_t} |\partial_t u_2^\varepsilon|^2 dx + D_3 \int_0^t \int_{\Omega_x} |\nabla \partial_t u_3^\varepsilon|^2 dx d\tau
+ \int_0^t \int_{\Omega_x} \left( \frac{1}{2} (\partial_t d_3 |\nabla u_3^\varepsilon|^2 - (\partial_t \partial_t d_3) |\nabla u_3^\varepsilon|^2) \right) dx d\tau
\leq \frac{D_3}{2} \int_0^t \int_{\Omega_x} |\nabla u_3^\varepsilon|^2 dx d\tau + \frac{d_3^\infty}{2} \int_0^t \int_{\Omega_x} |\nabla \partial_t u_3^\varepsilon|^2 dx d\tau
+ \frac{d_3^\infty + D_3}{2} \int_0^t \int_{\Gamma_y^a} |\nabla \partial_t u_3^D|^2 dx d\tau + \frac{\varepsilon A^\infty}{2} \int_0^t \int_{\Gamma_y^a} |\partial_t u_3^\varepsilon|^2 dx d\tau
+ \frac{\varepsilon a^\infty}{2} \int_0^t \int_{\Gamma_y^a} (|\partial_t u_3^\varepsilon|^2 + |\partial_t u_3^D|^2) dx d\tau
+ \frac{\varepsilon B^\infty}{2} \int_0^t \int_{\Gamma_y^a} (|u_2^\varepsilon|^2 + |\partial_t u_3^\varepsilon|^2 + |u_2^\varepsilon|^2 + |\partial_t u_3^D|^2) dx d\tau
\]

Using (7) to deal with the boundary terms, we obtain
\[
\frac{1}{2} \int_{\Omega_t} |\partial_t u_3^\varepsilon|^2 dx + \frac{1}{2} \int_0^t \int_{\Gamma_y^a} (|\partial_t u_3^\varepsilon|^2 + \delta |\partial_t u_3^\varepsilon|^2 + |\partial_t u_2^\varepsilon|^2 + |\partial_t u_3^D|^2) dx d\tau
\]
\[ \leq C_{14} + C_{15} \int_0^t \int_{\Omega^\varepsilon} |\partial_t u_3^\varepsilon|^2 dx d\tau \quad (64) \]
\[ + C^* b^\infty \int_0^t \int_{\Omega^\varepsilon} (|\partial_t u_3^\varepsilon|^2 + \varepsilon^2 |\nabla \partial_t u_3^\varepsilon|^2) dx d\tau \quad (65) \]

Adding (63) and (64) and using (59) and (60) to get the desired result.

4.1 Extension step

Since we deal here with an oscillating system posed in a perforated domain, the natural next step is to extend all concentrations to the whole \( \Omega \). We do this by following a two-steps procedure: In Step 1, we rely on the standard extension results indicated in section 4.2 to extend all active concentrations \( u_\ell^\varepsilon (\ell \in \{1, \ldots, 4\}) \) to \( \Omega \). In step 2, we unfold the ode for \( u_5^\varepsilon \) such that the unfolded concentration is defined on the fixed boundary \( \Gamma \); see section 5.1.

4.2 Extension lemmas

Since all the concentrations are defined in \( \Omega^\varepsilon \) and \( \Omega_1^\varepsilon \), to get macroscopic equations we need to extend them into \( \Omega \).

**Remark 6** Take \( \varphi^\varepsilon \in L^2(0, T; H^1(\Omega^\varepsilon)) \). Note that since our microscopic geometry is sufficiently regular, we can speak in terms of extensions. Recall the linearity of the extension operator

\[ \mathcal{P}^\varepsilon : L^2(0, T; H^1(\Omega^\varepsilon)) \rightarrow L^2(0, T; H^1(\Omega)) \]

defined by \( \mathcal{P}^\varepsilon \varphi^\varepsilon = \tilde{\varphi}^\varepsilon \). To keep notation simple, we denote the extension \( \tilde{\varphi}^\varepsilon \) again by \( \varphi^\varepsilon \).

**Lemma 7** (Extension) Consider the geometry described in Section 2.1. There exists an extension \( \tilde{u}^\varepsilon \) of \( u^\varepsilon \) such that

1. \( \| \tilde{u}^\varepsilon \|_{L^2(Y)} \leq \tilde{C} \| u^\varepsilon \|_{L^2(Y^w)} \), for \( u^\varepsilon \in L^2(Y^w) \)
2. \( \| \nabla \tilde{u}^\varepsilon \|_{L^2(Y)} \leq \tilde{C} \| \nabla u^\varepsilon \|_{L^2(Y^w)} \), for \( \nabla u^\varepsilon \in L^2(Y^w) \)
3. \( \| \tilde{u}^\varepsilon \|_{H^1(\Omega)} \leq \tilde{C} \| u^\varepsilon \|_{H^1(\Omega^\varepsilon)} \), for \( u^\varepsilon \in H^1(\Omega^\varepsilon) \)

**Proof.** For the proof of this Lemma, see Section 2 in [20] or compare Lemma 5, p.214 in [30].

**Definition 8** (Two-scale convergence cf. [31,32]) Let \( \{u^\varepsilon\} \) be a sequence of functions in \( L^2((0, T) \times \Omega) \) (\( \Omega \) being an open set of \( \mathbb{R}^N \)) where \( \varepsilon \) being a
sequence of strictly positive numbers that tends to zero. \( \{u^\varepsilon\} \) is said to two-scale converge to a unique function \( u_0(t,x,y) \in L^2((0,T) \times \Omega \times Y) \) if and only if for any \( \psi \in C_0^\infty((0,T) \times \Omega \times Y) \), we have

\[
\lim_{\varepsilon \to 0} \int_0^T \int_\Omega u^\varepsilon \psi(t,x,\frac{x}{\varepsilon}) \, dx \, dt = \int_\Omega \int_Y u_0(t,x,y) \psi(t,x,y) \, dy \, dx \, dt.
\]

We denote (66) by \( u^\varepsilon \rightharpoonup u_0 \).

**Theorem 9**

(i) From each bounded sequence \( \{u^\varepsilon\} \) in \( L^2((0,T) \times \Omega) \), one can extract a subsequence which two-scale converges to \( u_0(t,x,y) \in L^2((0,T) \times \Omega \times Y) \).

(ii) Let \( \{u^\varepsilon\} \) be a bounded sequence in \( H^1((0,T) \times \Omega) \), which converges weakly to a limit function \( u_0(t,x,y) \in H^1((0,T) \times \Omega \times Y) \). Then there exists \( \tilde{u} \in L^2(0; H^1_\#(Y)/\mathbb{R}) \) such that up to a subsequence \( \{u^\varepsilon\} \) two-scale converges to \( u_0(t,x,y) \) and \( \nabla u^\varepsilon \rightharpoonup \nabla_x \tilde{u} + \nabla_y \tilde{u} \).

(iii) Let \( \{u^\varepsilon\} \) and \( \{\varepsilon \nabla u^\varepsilon\} \) be bounded sequences in \( L^2((0,T) \times \Omega) \), then there exists \( u_0 \in L^2((0,T) \times \Omega; H^1_\#(Y)) \) such that up to a subsequence \( u^\varepsilon \) and \( \varepsilon \nabla u^\varepsilon \) two-scale converge to \( u_0(t,x,y) \) and \( \nabla_y u_0(t,x,y) \) respectively.

**Definition 10** (Two-scale convergence for \( \varepsilon \)-periodic hypersurfaces [33]) A sequence of functions \( \{u^\varepsilon\} \) in \( L^2((0,T) \times \Gamma_\varepsilon) \) is said to two-scale converge to a limit \( u_0 \in L^2((0,T) \times \Omega \times \Gamma) \) if and only if for any \( \psi \in C_0^\infty((0,T) \times \Omega, C^\infty_0(\Gamma)) \) we have

\[
\lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_{\Gamma_\varepsilon} u^\varepsilon \psi(t,x,\frac{x}{\varepsilon}) \, d\sigma_x \, dt = \int_\Omega \int_\Gamma u_0(t,x,y) \psi(t,x,y) \, d\sigma_y \, dx \, dt.
\]

**Theorem 11**

(i) From each bounded sequence \( \{u^\varepsilon\} \) in \( L^2((0,T) \times \Gamma_\varepsilon) \), one can extract a subsequence \( u^\varepsilon \) which two-scale converges to a function \( u_0 \in L^2((0,T) \times \Omega \times \Gamma) \).

(ii) If a sequence of functions \( \{u^\varepsilon\} \) is bounded in \( L^\infty((0,T) \times \Gamma_\varepsilon) \), then \( u^\varepsilon \) two-scale converges to a function \( u_0 \in L^\infty((0,T) \times \Omega \times \Gamma) \).

**Proof.** For proof of (i), see [33] and the one for (ii), see [15].

**Lemma 12** Assume the hypotheses of Lemma 5 and Lemma 7 to hold. The a priori estimates lead to the following convergence results:

(a) \( u_i^\varepsilon \rightharpoonup u_i \) in \( L^2(0,T; H^1(\Omega)) \) for all \( i \in \{1,2,3,4\} \),

(b) \( u_i^\varepsilon \rightharpoonup u_i \) in \( L^\infty((0,T) \times \Omega) \),

(c) \( \partial_t u_i^\varepsilon \rightharpoonup \partial_t u_i \) in \( L^2((0,T) \times \Omega) \),

(d) \( u_i^\varepsilon \rightharpoonup u_i \) in \( L^2(0,T; H^3(\Omega)) \) for \( \frac{1}{2} < \beta < 1 \), also \( \| u_i^\varepsilon - u_i \|_{L^2((0,T) \times \Gamma_\varepsilon)} \to 0 \) as \( \varepsilon \to 0 \),

(e) \( u_i^\varepsilon \rightharpoonup u_i, \nabla u_i^\varepsilon \rightharpoonup \nabla_x u_i + \nabla_y u_{i1}, u_{i1} \in L^2((0,T) \times \Omega; H^1_\#(Y)/\mathbb{R}) \),

(f) \( u_5^\varepsilon \rightharpoonup u_5 \) and \( u_5 \in L^\infty((0,T) \times \Omega \times \Gamma^\text{sw}) \),

(g) \( \partial_t u_5^\varepsilon \rightharpoonup \partial_t u_5 \), and \( u_5 \in L^2((0,T) \times \Omega \times \Gamma^\text{sw}) \).
Proof. (a) and (b) are obtained as a direct consequence of the fact that \( u_i^\varepsilon \) is bounded in \( L^2(0, T; H^1(\Omega)) \cap L^\infty((0, T) \times \Omega) \); up to a subsequence (still denoted by \( u_i^\varepsilon \) \( u_i^\varepsilon \) converges weakly to \( u_i \) in \( L^2(0, T; H^1(\Omega)) \cap L^\infty((0, T) \times \Omega) \). A similar argument gives (c). To get (d), we use the compact embedding \( H^\beta(\Omega) \hookrightarrow H^\beta'(\Omega) \), for \( \beta \in (\frac{1}{2}, 1) \) and \( 0 < \beta < \beta' \leq 1 \) (since \( \Omega \) has Lipschitz boundary). We have

\[
W := \{ u_i \in L^2(0, T; H^1(\Omega)) \text{ and } \partial_t u_i \in L^2((0, T) \times \Omega) \text{ for all } i \in \{1, 2, 3, 4\} \}
\]

For a fixed \( \varepsilon \), \( W \) is compactly embedded in \( L^2(0, T; H^\beta(\Omega)) \) by the Lions-Aubin Lemma; cf. e.g. [34]. Using the trace inequality (8)

\[
\| u_i^\varepsilon - u_i \|_{L^2((0,T)\times \Gamma^\varepsilon)} \leq C_0^* \| u_i^\varepsilon - u_i \|_{L^2(0,T;H^\beta(\Omega))},
\]

\[
\leq C \| u_i^\varepsilon - u_i \|_{L^2(0,T;H^\beta(\Omega))},
\]

where \( \| u_i^\varepsilon - u_i \|_{L^2(0,T;H^\beta(\Omega))} \to 0 \) as \( \varepsilon \to 0 \). To investigate (e), (f) and (g), we use the notion of two-scale convergence as indicated in Definition 8 and 10. Since \( u_i^\varepsilon \) are bounded in \( L^2(0, T; H^1(\Omega)) \), up to a subsequence \( u_i^\varepsilon \rightharpoonup u_i \) in \( L^2((0, T) \times \Omega \times Y) \), and \( \nabla u_i^\varepsilon \overset{2}{\to} \nabla_x u_i + \nabla_y \tilde{u}_i \), \( \tilde{u}_i \in L^2((0, T) \times \Omega; H^1_#(Y)/\mathbb{R}) \). By Theorem 11, \( u_5^\varepsilon \in L^\infty((0,T) \times \Omega \times \Gamma) \) converges two-scale to \( u_5 \) in the same space and \( \partial_t u_5^\varepsilon \) converges two-scale to \( \partial_t u_5 \) in \( L^2((0, T) \times \Omega \times \Gamma) \). Due to the presence of the non-linear reaction rate on the interface \( \Gamma^\varepsilon_{sw} \), the convergences listed in Lemma 12 are still not sufficient to pass to the limit \( \varepsilon \to 0 \) in the microscopic model. To be more precise, we can pass to \( \varepsilon \to 0 \) in the pde’s, but not in the ode.

4.3 Cell problems

In order to be able to formulate the upscaled equations, we define two classes of cell problems very much in the spirit of [9]. One class of problems will refer to the water-filled parts of the pore, while the second class will refer to the air-filled part of the pores.

**Definition 13 (Cell problems)** The cell problems in water-filled part are given by

\[
-\nabla_y \cdot (D_\ell(t,y)\nabla_y \chi_i) = \sum_{k=1}^{3} \partial_{x_k} D_{\ell_{ki}}(t,y), \text{ in } Y^w,
\]

\[
-D_\ell(t,y) \frac{\partial \chi_i}{\partial n} = \sum_{k=1}^{3} D_{\ell_{ki}}(t,y)n_k \text{ on } \Gamma^{sw},
\]

\[
-D_\ell(t,y) \frac{\partial \chi_i}{\partial n} = \sum_{k=1}^{3} D_{\ell_{ki}}(t,y)n_k \text{ on } \Gamma^{wa},
\]
for all $i, \ell \in \{1, 2, 4\}$ and $\chi_i$ are $Y$-periodic in $y$. The cell problems in air-filled part are given by

$$-\nabla_y (D_3(t, y) \nabla_y \zeta_i) = \sum_{k=1}^{3} \partial_{y_k} D_{3k_i}(t, y), \text{ in } Y^a,$$

$$-D_3(t, y) \frac{\partial \zeta_i}{\partial n} = \sum_{k=1}^{3} D_{3k_i}(t, y)n_k \text{ on } \Gamma^{wa},$$

$$-D_3(t, y) \frac{\partial \zeta_i}{\partial n} = \sum_{k=1}^{3} D_{3k_i}(t, y)n_k \text{ on } \partial Y^a - \Gamma^{wa},$$

for all $i \in \{1, 2, 3\}$ and $\zeta_i$ are $Y$-periodic in $y$.

5 Two-scale limit equations

Theorem 14 The sequences of the solutions of the weak formulation (9)-(13) converges to the weak solution $u_i, i \in \{1, 2, 3, 4, 5\}$ as $\varepsilon \to 0$ such that $u_i \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \cap L^\infty((0, T) \times \Omega)$ and $u_5 \in H^1(0, T; L^2(\Omega \times \Gamma)) \cap L^\infty((0, T) \times \Omega \times \Gamma))$. The weak formulation of the two-scale limit equations is given by

$$\int_{0}^{T} \int_{\Omega} \partial_t u_i(t, x) \phi_i(t, x) dx dt + \int_{0}^{T} \int_{\Omega} \phi_i(t) \nabla u_i(t, x) \nabla \phi_i dx dt \quad (67)$$

$$= \int_{0}^{T} \int_{\Omega} F_i(u) \phi_i dx dt \text{ for all } i \in \{1, 2, 3, 4\},$$

where

$$F_1(u) := -\tilde{k}_1(t)u_1(t, x) + \tilde{k}_2(t)u_2(t, x)$$

$$- \frac{1}{|Y|} \int_{\Gamma} k_3(t, y) R(u_1(t, x))Q(u_5(t, x, y)) d\sigma_y,$$

$$F_2(u) := \tilde{k}_1(t)u_1(t, x) - \tilde{k}_2(t)u_2(t, x) + \tilde{a}(t)u_3(t, x) - \tilde{b}(t)u_2(t, x),$$

$$F_3(u) := -\tilde{a}(t)u_3(t, x) + \tilde{b}(t)u_2(t, x),$$

$$F_4(u) := \tilde{k}_1(t)u_1(t, x),$$

with the initial values $u_i(0, x) = u_{i0}(x)$ for $x \in \Omega$, and

$$\int_{0}^{T} \int_{\Omega \times \Gamma} \partial_t u_5(t, x, y) \phi_5(t, x, y) dt dx d\sigma_y \quad (68)$$

$$= \int_{0}^{T} \int_{\Omega \times \Gamma} k_3(t, y) R(u_1(t, x))Q(u_5(t, x, y)) \phi_5(t, x, y) dt dx d\sigma_y,$$

with $u_5(0, x, y) = u_{50}(x, y)$ for $x \in \Omega$, $y \in \Gamma^{wa}$. Also $\phi := (\phi_1, \phi_2, \phi_3, \phi_4) \in [C^\infty((0, T) \times \Omega)]^4$, $\psi := (\psi_1, \psi_2, \psi_3, \psi_4) \in [C^\infty((0, T) \times \Omega); C^\infty_\#(Y)]^4$. 

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\( \tilde{k}_j(t) := \frac{1}{|Y|} \int_Y k_j(t, y) dy, \ j \in \{1, 2\}, \) (69)

\( \tilde{a}(t) := \frac{1}{|\Gamma|} \int_{\Gamma_w} a(t, y) d\sigma_y, \) (70)

\( \tilde{b}(t) := \frac{1}{|\Gamma|} \int_{\Gamma_w} b(t, y) d\sigma_y, \) (71)

\[ \tilde{d}_{ij} := \sum_{k=1}^{3} \int_Y (d_{ij}(t, y) + d_{ik}(t, y))(d_{ij}(t, y) + \delta_{ni} \partial_{y_k} \chi_j + \delta_{3i} \partial_{y_k} \varsigma_j) dy, \] (72)

\( \ell \in \{1, 2, 3\}, n \in \{1, 2, 4\}\]

with \( \chi_j, \varsigma_j \) being solutions of the cell problems defined in Definition 13, while \( \delta \) denotes here the Kronecker’s symbol.

Proof. We apply two-scale convergence techniques together with Lemma 12 to get macroscopic equations. We take test functions incorporating the following oscillating behavior

\[ \varphi_i(t, x) = \phi_i(t, x) + \varepsilon \psi_i(t, x, \frac{x}{\varepsilon}), \phi_i \in C^\infty((0, T) \times \Omega), \phi_i \in C^\infty((0, T) \times \Omega; C^\infty(Y)), i \in \{1, 2, 3, 4\}. \]

Applying two-scale convergence yields

\[ |Y| \int_0^T \int_\Omega \partial_t u_i \phi_i(t, x) dx dt + \int_0^T \int_\Omega d_i(t, y)(\nabla_x u_i(t, x) + \nabla_y \tilde{u}_i(t, y)) (\nabla_x \phi_i(t, x) + \nabla_y \psi_i(t, x, \frac{x}{\varepsilon})) dy dx dt = \int_0^T \int_\Omega f_i(u) \phi_i(t, x) dx dt. \] (73)

Using Lemma 12, we have

\[ \int_0^T \int_\Omega f_1(u) \phi_1(t, x) dx dt = - \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_{\varepsilon}} k_1^\varepsilon u_1^\varepsilon(\phi_1(t, x) + \varepsilon \psi_1(t, x, \frac{x}{\varepsilon})) dx dt + \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_{\varepsilon}} k_2^\varepsilon u_2^\varepsilon(\phi_1(t, x) + \varepsilon \psi_1(t, x, \frac{x}{\varepsilon})) dx dt - \lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_{\Gamma_{\varepsilon}} \eta(R(u_1^\varepsilon), Q(u_2^\varepsilon)) (\phi_1(t, x) + \varepsilon \psi_1(t, x, \frac{x}{\varepsilon})) d\sigma_x dt. \]

Using Lemma 12, we have

\[ \int_0^T \int_\Omega f_1(u) \phi_1(t, x) dx dt = - \int_0^T \int_Y k_1(t, y) u_1(t, x) \phi_1(t, x) dy dx dt + \int_0^T \int_Y k_2(t, y) u_2(t, x) \phi_1(t, x) dy dx dt - \lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_{\Gamma_{\varepsilon}} \partial_t u_2^\varepsilon(\phi_1(t, x) + \varepsilon \psi_1(t, x, \frac{x}{\varepsilon})) d\sigma_x dt. \]
\[ \int_0^T \int_{\Omega} f_1(u_1(t,x)) \phi_1(t,x) \, dx \, dt = -|Y| \int_0^T \int_{\Omega} \tilde{k}_1(t) u_1(t,x) \phi_1(t,x) \, dx \, dt \]
\[ + |Y| \int_0^T \int_{\Omega} \tilde{k}_2(t) u_2(t,x) \phi_1(t,x) \, dx \, dt \]
\[ - \int_0^T \int_{\Omega} \partial_t u_5 \phi_1(t,x) \, ds \, dx \, dt. \]

\[ \int_0^T \int_{\Omega} f_2(u_2(t,x)) \phi_2(t,x) \, dx \, dt = \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_\varepsilon} k_1^{\varepsilon}(t) u_2^{\varepsilon}(t,x) + \varepsilon \psi_2(t,x,x/\varepsilon) \, dx \, dt \]
\[ - \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_\varepsilon} k_2^{\varepsilon}(t) u_2^{\varepsilon}(t,x) + \varepsilon \psi_2(t,x,x/\varepsilon) \, dx \, dt \]
\[ + \lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_{\Gamma_\varepsilon} a(z) u_2^{\varepsilon}(t,x) + \varepsilon \psi_2(t,x,x/\varepsilon) \, ds \, dx \, dt \]
\[ - \lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_{\Gamma_\varepsilon} b(z) u_2^{\varepsilon}(t,x) + \varepsilon \psi_2(t,x,x/\varepsilon) \, ds \, dx \, dt. \]

\[ \int_0^T \int_{\Omega} f_3(u_3(t,x)) \phi_3(t,x) \, dx \, dt = \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_\varepsilon} \tilde{k}_1(t) u_3(t,x) \phi_3(t,x) \, dx \, dt \]
\[ - \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_\varepsilon} \tilde{k}_2(t) u_2(t,x) \phi_3(t,x) \, dx \, dt \]
\[ + \lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_{\Gamma_\varepsilon} \tilde{a}(t) u_3(t,x) \phi_3(t,x) \, dx \, dt \]
\[ - \lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_{\Gamma_\varepsilon} \tilde{b}(t) u_2(t,x) \phi_3(t,x) \, dx \, dt. \]

We also have
\[ \int_0^T \int_{\Omega} f_4(u_4(t,x)) \phi_4(t,x) \, dx \, dt = \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_\varepsilon} \tilde{k}_1(t) u_1(t,x) \phi_4(t,x) \, dx \, dt \]
\[ - \lim_{\varepsilon \to 0} \int_0^T \int_{\Omega_\varepsilon} \tilde{k}_2(t) u_2(t,x) \phi_4(t,x) \, dx \, dt \]
\[ + \lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_{\Gamma_\varepsilon} \tilde{a}(t) u_4(t,x) \phi_4(t,x) \, dx \, dt \]
\[ - \lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_{\Gamma_\varepsilon} \tilde{b}(t) u_2(t,x) \phi_4(t,x) \, dx \, dt. \]

We set \( \phi_i = 0, i \in \{1, 2, 3, 4\} \) in (73) to calculate the expression of the known function \( \tilde{u}_1 \) and obtain
\[ \int_0^T \int_Y d(t,y)(\nabla_x u_{i}(t,x) + \nabla_y \tilde{u}_i(t,x,y))\nabla_y \psi_i(t,x,x/\varepsilon) \, dy \, dx \, dt = 0, \text{ for all } \psi_i. \]
Since $\tilde{u}_1$ depends linearly on $\nabla x u_1$, it can be defined as

$$\tilde{u}_i := \sum_{j=1}^{3} \partial_{x_j} u_i (\delta_n \chi_j(t,y) + \delta_3 \varsigma_j(t,y)) \text{ for } n \in \{1, 2, 4\}$$

where the function $\chi_j, \varsigma_j$ are the unique solutions of the cell problems defined in Definition 13. Setting $\psi_i = 0$ in (73), we get

$$\int_0^T \int_\Omega \sum_{j,k=1}^{3} d_{ijk}(t,y) \partial_{x_k} u_i(t,x)$$

$$+ \sum_{m=1}^{3} (\delta_n \partial_{y_m} \chi_m + \delta_3 \partial_{y_m} \varsigma_m) \partial_{x_m} u_i(t,x) \partial_{x_j} \phi_k(t,x) d\text{y} d\text{x} d\text{t}$$

$$= |Y| \int_0^T \int_\Omega \sum_{j,k=1}^{3} \tilde{d}_{ijk} \partial_{x_k} u_i(t,x) \partial_{x_j} \phi_i(t,x) d\text{x} d\text{t}.$$  

Hence, the coefficients (entering the effective diffusion tensor) are given by

$$\tilde{d}_{ijk} := \frac{1}{|Y|} \sum_{k=1}^{3} \int_Y (d_{bij}(t,y) + d_{bik}(t,y) (\delta_n \partial_{y_b} \chi_j + \delta_3 \partial_{y_b} \varsigma_j) ) d\text{y}. \quad (74)$$

$i \in \{1, 2, 3, 4\}, n \in \{1, 2, 4\}$ and $j, k \in \{1, 2, 3\}$.  

5.1 Passing to the limit $\varepsilon \to 0$ in (13)

It is not yet possible to pass to the limit $\varepsilon \to 0$ with the convergence results stated in Lemma 12. To overcome this difficulty, we use the notion of periodic unfolding. It is worth mentioning that there is an intimate link between the two-scale convergence and weak convergence of the unfolded sequences; see [35,15]. The key idea is: Instead of getting strong convergence for $u_5^\varepsilon$, obtain strong convergence for the periodic unfolding of $u_5^\varepsilon$.

**Definition 15** For $\varepsilon > 0$, the boundary unfolding of a measurable function $\varphi$ posed on oscillating surface $\Gamma_\varepsilon$ is defined by

$$T_\varepsilon^b \varphi(x,y) = \varphi(\varepsilon y + \varepsilon k), \quad y \in \Gamma, x \in \Omega$$

where $k := [\frac{x}{\varepsilon}]$ denotes the unique integer combination $\Sigma_{j=1}^{3} k_j e_j$ of the periods such that $x - [\frac{x}{\varepsilon}]$ belongs to $Y$. Note that the oscillation due to the perforations are shifted into the second variable $y$ which belongs to fixed surface $\Gamma$.

**Lemma 16** If $u_\varepsilon$ converges two-scale to $u$ and $T_\varepsilon^b u_\varepsilon$ converges weakly to $u^*$ in $L^2((0,T) \times \Omega; L^2_b(\Gamma))$, then $u = u^*$ a.e. in $(0,T) \times \Omega \times \Gamma$.

**Proof.** The proof details for this statement can be found in Lemma 4.6 of [15].

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Lemma 17 If \( \varphi \in L^2((0, T) \times \Gamma^\varepsilon) \), then the following identity holds

\[
\frac{1}{|Y|} \left\| T_b^\varepsilon \varphi \right\|_{L^2((0, T) \times \Omega \times \Gamma)} = \varepsilon \left\| \varphi \right\|_{L^2((0, T) \times \Gamma^\varepsilon)}.
\]

Proof. Consider

\[
\frac{1}{|Y|} \left\| T_b^\varepsilon \varphi \right\|_{L^2(\Omega \times \Gamma)}^2 = \frac{1}{|Y|} \int_{\Omega \times \Gamma} |T_b^\varepsilon \varphi|^2 \, d\sigma_y = \frac{1}{|Y|} \int_{\Omega \times \Gamma} T_b^\varepsilon \varphi^2 \, d\sigma_y,
\]

\[
eq \frac{1}{|Y|} \sum_{k=1}^3 \int_{\varepsilon_k(y+k)} T_b^\varepsilon \varphi^2 \, d\sigma_y = \frac{1}{|Y|} \sum_{k=1}^3 \int_{\varepsilon_k(y+k)} \varphi^2 \, d\sigma_y,
\]

Changing variable \( z = \varepsilon(y+k) \), where \( k = \lfloor \frac{z}{\varepsilon} \rfloor \), we get

\[
\frac{1}{|Y|} \left\| T_b^\varepsilon \varphi \right\|_{L^2(\Omega \times \Gamma)}^2 = \sum_{k=1}^3 \varepsilon^3 \int_{\varepsilon_k(y+k)} \varphi^2 \, d\sigma_y = \sum_{k=1}^3 \varepsilon \int_{\varepsilon_k(y+k)} \varphi^2 \, d\sigma_y = \varepsilon \int_{\Gamma} \varphi^2 \, d\sigma_z.
\]

This completes the proof of (17).

Lemma 18 If \( \varphi \in L^2(\Omega) \), then \( T_b^\varepsilon \varphi \to \varphi \) as \( \varepsilon \to 0 \) strongly in \( L^2(\Omega \times \Gamma) \).

Proof. See in [36,37] for proof details.

Using the boundary unfolding operator \( T_b^\varepsilon \), we unfold the ode (13). Changing the variable, \( x = \varepsilon y + \varepsilon k \) (for \( x \in \Gamma_{\varepsilon_{w}} \)) to the fixed domain \((0, T) \times \Omega \times \Gamma\), we have

\[
\partial_t T_b^\varepsilon u_5^\varepsilon(t, x, y) = \eta(T_b^\varepsilon u_1^\varepsilon(t, x, y), T_b^\varepsilon u_5^\varepsilon(t, x, y)).
\]

(75)

In the remainder of this section, we prove that \( T_b^\varepsilon u_5^\varepsilon \) converges strongly to \( u_5 \) in \( L^2(\Omega \times \Gamma) \). From the two-scale convergence of \( u_5^\varepsilon \), we obtain weak convergence of \( T^\varepsilon u_5^\varepsilon \) to \( u_5 \) in \( L^\infty((0, T) \times \Omega; L^2_{\text{per}}(\Gamma)) \). We start with showing that \( \{T_b^\varepsilon u_5^\varepsilon\} \) is a Cauchy sequence in \( L^2(\Omega \times \Gamma) \). To this end, we choose \( m, n \in \mathbb{N} \) with \( n > m \) arbitrary. Writing down (75) for the two different choices of \( \varepsilon \) (i.e. \( \varepsilon_i = \varepsilon_n \) and \( \varepsilon_i = \varepsilon_m \)), we obtain after subtracting the corresponding equations that

\[
\partial_t \int_{\Omega \times \Gamma} |T_b^\varepsilon u_5^\varepsilon|^2 - |T_b^\varepsilon u_5^\varepsilon|^2 \, d\sigma_y dx
\]

\[
= \int_{\Omega \times \Gamma} |k_3^\varepsilon R(T_b^\varepsilon u_1^\varepsilon)Q(T_b^\varepsilon u_5^\varepsilon) - k_3^\varepsilon R(T_b^\varepsilon u_5^\varepsilon)Q(T_b^\varepsilon u_5^\varepsilon)| (T_b^\varepsilon u_5^\varepsilon)
\]

\[
- \int_{\Omega \times \Gamma} |T_b^\varepsilon u_5^\varepsilon|^2 dx \, d\sigma_y dx,
\]

\[
\leq k_3^\infty c_R \left( \frac{Q^\infty}{2} + c_Q \sup_{\Omega \times \Gamma} |T_b^\varepsilon u_1^\varepsilon| \right) \int_{\Omega \times \Gamma} |T_b^\varepsilon u_5^\varepsilon|^2 \, d\sigma_y dx
\]

\[
+ \frac{k_3^\infty c_R Q^\infty}{2} \int_{\Omega \times \Gamma} |T_b^\varepsilon u_5^\varepsilon|^2 \, d\sigma_y dx.
\]

(76)
To get (76), we have used the uniform boundedness of $T_b^{\epsilon n}u_1^{\epsilon n}$. We consider now
\[
\int_{\Omega \times \Gamma} |T_b^{\epsilon n}u_1^{\epsilon n} - T_b^{\epsilon m}u_1^{\epsilon m}|^2 d\sigma_y dx
\leq \int_{\Omega \times \Gamma} (|T_b^{\epsilon n}u_1^{\epsilon n} - T_b^{\epsilon m}u_1|^2 + |T_b^{\epsilon n}u_1 - u_1|^2) d\sigma_y dx
+ \int_{\Omega \times \Gamma} (|T_b^{\epsilon m}u_1 - u_1|^2 + |T_b^{\epsilon m}u_1^{\epsilon m} - T_b^{\epsilon m}u_1|^2) d\sigma_y dx. \tag{77}
\]
Since $u_1$ is constant w.r.t. $y$, we have that $T_b^{\epsilon m}u_1 \to u_1$ strongly in $L^2((0,T) \times \Omega \times \Gamma)$ as $\epsilon \to 0$. From Lemma 17, we conclude that
\[
\int_{\Omega \times \Gamma} |T_b^{\epsilon n}u_1^{\epsilon n} - T_b^{\epsilon m}u_1| d\sigma_y dx \leq C \epsilon \int_{\Gamma^w} |u_1^{\epsilon n} - u_1|^2 d\sigma_y dx \leq \epsilon C.
\]
(77) turns out to be
\[
\int_{\Omega \times \Gamma} |T_b^{\epsilon n}u_1^{\epsilon n} - T_b^{\epsilon m}u_1^{\epsilon m}|^2 d\sigma_y dx \leq C(\epsilon_n + \epsilon_m),
\]
while (76) becomes
\[
\partial_t \int_{\Omega \times \Gamma} |T_b^{\epsilon n}u_1^{\epsilon n} - T_b^{\epsilon m}u_1^{\epsilon m}|^2 d\sigma_y dx \leq C_{15} \int_{\Omega \times \Gamma} |T_b^{\epsilon n}u_5^{\epsilon n} - T_b^{\epsilon m}u_5^{\epsilon m}|^2 d\sigma_y dx + \frac{C_{16}}{n},
\]
where $C_{15} := k_{\epsilon n}^\omega c_R (\frac{Q}{2} + c_Q \sup_{\Omega _x \Gamma} |T_b^{\epsilon n}u_1^{\epsilon n}|)$ and $C_{16} := \frac{k_{\epsilon n}^\omega c_R Q}{2} C$. The Gronwall’s inequality gives
\[
\| T_b^{\epsilon n}u_5^{\epsilon n} - T_b^{\epsilon m}u_5^{\epsilon m} \|_{L^2(\Omega \times \Gamma)} \leq \frac{C_{16}}{n}. \tag{78}
\]
By (78), $\{T_b^{\epsilon}u_5^{\epsilon}\}$ is a Cauchy sequence. Now, we take the two-scale limit in the ode (75) to get
\[
\lim_{\epsilon \to 0} \int_0^T \int_{\Gamma^w} \partial_t T_b^{\epsilon}u_5^{\epsilon} \phi_1(t, x, \frac{x}{\epsilon}) d\sigma_x dt = \lim_{\epsilon \to 0} \int_0^T \int_{\Gamma^w} \eta(T_b^{\epsilon}u_1^{\epsilon}, T_b^{\epsilon}u_5^{\epsilon}) \phi_1(t, x, \frac{x}{\epsilon}) d\sigma_x dt.
\]
Consequently, we have
\[
\int_0^T \int_{\Omega \times \Gamma^w} \partial_u \phi_5(t, x, y) d\sigma_y dt
\leq \lim_{\epsilon \to 0} \int_0^T \int_{\Gamma^w} T_b^{\epsilon}k_{\epsilon}^{\frac{\epsilon}{3}} R(T_b^{\epsilon}u_1^{\epsilon}) Q(u_5^{\epsilon}) \phi_5(t, x, \frac{x}{\epsilon}) d\sigma_x dt,
\]
\[
= \lim_{\epsilon \to 0} \int_0^T \int_{\Gamma^w} T_b^{\epsilon}k_{\epsilon}^{\frac{\epsilon}{3}} R(T_b^{\epsilon}u_1^{\epsilon}) Q(u_5) \phi_5(t, x, \frac{x}{\epsilon}) d\sigma_x dt
\]
\[
+ \lim_{\epsilon \to 0} \int_0^T \int_{\Gamma^w} T_b^{\epsilon}k_{\epsilon}^{\frac{\epsilon}{3}} R(T_b^{\epsilon}u_1^{\epsilon}) (Q(T_b^{\epsilon}u_5^{\epsilon}) - Q(u_5)) \phi_5(t, x, \frac{x}{\epsilon}) d\sigma_x dt. \tag{79}
\]
By (A2) and the strong convergence of $u^\varepsilon_1$, the first term on the right hand side of (79) converges two-scale to
\[ \int_0^T \int_\Omega \int_{\Gamma_\varepsilon} k_3(t,y) R(u_1) Q(u_5) \phi_5(t,x,y) d\sigma_y dx dt, \]
while the second integral of (79)
\[ \varepsilon \int_0^T \int_{\Gamma_\varepsilon} T_0^\varepsilon k_3^\varepsilon R(T_0^\varepsilon u_1^\varepsilon)(Q(T_0^\varepsilon u_5^\varepsilon) - Q(u_5^\varepsilon)) \phi_5(t,x,x^\varepsilon) d\sigma_x dt \]
\[ \leq \varepsilon \left( \int_0^T \int_{\Gamma_\varepsilon} |T_0^\varepsilon k_3^\varepsilon R(T_0^\varepsilon u_1^\varepsilon)\phi_5(t,x,x^\varepsilon)|^2 d\sigma_x dt \right)^{\frac{1}{2}} \cdot \left( \int_0^T \int_{\Gamma_\varepsilon} (Q(T_0^\varepsilon u_5^\varepsilon) - Q(u_5^\varepsilon))^2 d\sigma_x dt \right)^{\frac{1}{2}}, \]
\[ \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \]

At this point, we have used again (A2) in combination with the strong convergence of $T_0^\varepsilon u_5^\varepsilon$. So, as result of passing to the limit $\varepsilon \rightarrow 0$ in (13) we get (68).

It is worth noting that the weak solution to the two-scale model inherits a.e. the positivity and boundedness properties from the corresponding properties of the weak solution of the microscopic model. Now, it only remains to ensure the uniqueness of weak solutions to the upscaled model.

**Lemma 19** (Uniqueness of solutions of (67)-(68) Assume (A1)-(A6). There exists at most one weak solution to the two-scale limit problem (67) and (68). Proof. Suppose there are two weak solutions to the two-scale limit problem $(u_1^j, u_2^j, u_3^j, u_4^j, u_5^j)$ with $j \in \{1, 2\}$. We denote $u_\ell = u_1^j - u_2^j$, $\ell \in \{1, 2, 3, 4\}$ and choose as test function $\phi_\ell = u_\ell$. After straightforward calculations, we have from (68)
\[ |u_1^j - u_2^j| \leq C \int_0^t |u_1^j - u_2^j| d\tau. \] (80)

Take $\phi_1 = u_1$ in (67) to obtain
\[ \frac{1}{2} \int_0^t \int_\Omega \partial_t |u_1|^2 dx dt + \bar{d}_1 \int_0^t \int_\Omega |\nabla u_1|^2 dx dt \]
\[ \leq -\bar{k}_1 \int_0^t \int_\Omega \partial_t |u_1|^2 dx dt + \frac{\bar{k}_2^\infty}{2} \int_0^t \int_\Omega (|u_1|^2 + |u_2|^2) dx dt \]
\[ + \frac{k_3^\infty}{|Y|} c_R c_Q M_1 \int_0^t \int_{\Omega \times \Gamma_\varepsilon} (u_1 - u_2) u_1 dx d\sigma_y dt \]
\[ + \frac{k_3^\infty}{|Y|} c_R Q^\infty \int_0^t \int_{\Omega \times \Gamma_\varepsilon} |u_1|^2 dx d\sigma_y dt. \] (81)

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Using (80) together with the trace inequality for fixed domains, see section 5.5 Theorem 1 in [38] and also the fact that $u_1$ is independent of $y$ in (81), we get

$$
\int_0^t \int_\Omega |\partial_t u_1|^2 dx dt + (2\tilde{\delta}_1 - k_3^\infty c_R C^*(\delta M_1 + Q^\infty)) \int_0^t \int_\Omega |\nabla u_1|^2 dx d\tau \\
+ 2\tilde{k}_1 \int_0^t \int_\Omega |\partial_t u_1|^2 dx d\tau \\
\leq (\tilde{k}_2^\infty + k_3^\infty c_R C^*(\delta M_1 + Q^\infty)) \int_0^t \int_\Omega (|u_1|^2 + |u_2|^2) dx d\tau \\
+ k_3^\infty \delta c_R c_Q M_1 C^* \int_0^t \int_\Omega \int_0^\tau (|u_1|^2 + |\nabla u_1|^2) ds dxd\tau.
$$

For suitable choice of $\delta \in ]0, \frac{2d_1-k_3^\infty c_R C^* Q^\infty}{k_3^\infty c_R C^* M_1 - [}[$, we have

$$
\int_0^T \int_\Omega |\partial_t u_1|^2 dx dt + \tilde{\delta}_1 \int_0^T \int_\Omega |\nabla u_1|^2 dx d\tau + 2\tilde{k}_1 \int_0^T \int_\Omega |\partial_t u_1|^2 dx d\tau \\
\leq (\tilde{k}_2^\infty + k_3^\infty c_R C^*(\delta M_1 + Q^\infty)) \int_0^T \int_\Omega (|u_1|^2 + |u_2|^2) dx d\tau \\
+ \frac{k_3^\infty}{\delta} c_R c_Q M_1 C^* \int_0^T \int_\Omega \int_0^\tau (|u_1|^2 + |\nabla u_1|^2) ds dxd\tau. \quad (82)
$$

Take $\phi_2 = u_2$ in (67), we get

$$
\frac{1}{2} \int_0^t \int_\Omega |\partial_t u_2|^2 dx d\tau + \tilde{a}_2 \int_0^t \int_\Omega |\nabla u_2|^2 dx d\tau \\
\leq -\tilde{k}_2 \int_0^t \int_\Omega |\partial_t u_2|^2 dx d\tau + \frac{\tilde{k}_2^\infty}{2} \int_0^t \int_\Omega (|u_1|^2 + |u_2|^2) dx d\tau \\
+ \tilde{a}_2 \int_0^t \int_\Omega u_2 u_3 dx dt - \tilde{b}_2 \int_0^T \int_\Omega |u_2|^2 dx d\tau.
$$

$$
\int_0^t \int_\Omega |\partial_t u_2|^2 dx d\tau + \tilde{a}_2 \int_0^t \int_\Omega |\nabla u_2|^2 dx d\tau \\
\leq (\tilde{k}_2^\infty + \tilde{a}_2^\infty) \int_0^t \int_\Omega (|u_1|^2 + |u_2|^2 + |u_3|^2) dx d\tau. \quad (83)
$$

Similarly, we obtain from (67)

$$
\int_0^t \int_\Omega |\partial_t u_3|^2 dx d\tau + \tilde{a}_3 \int_0^t \int_\Omega |\nabla u_3|^2 dx d\tau \leq \tilde{b}_3^\infty \int_0^t \int_\Omega (|u_2|^2 + |u_3|^2) dx dt \quad (84)
$$

$$
\int_0^t \int_\Omega |\partial_t u_4|^2 dx d\tau + \tilde{a}_4 \int_0^t \int_\Omega |\nabla u_4|^2 dx d\tau \leq \tilde{k}_4^\infty \int_0^t \int_\Omega (|u_1|^2 + |u_3|^2) dx dt \quad (85)
$$

Adding side by side (82)-(85) and applying Gronwall’s inequality to the corresponding result, we receive
\[ \Sigma_{i=1}^4 \int_{\Omega} |u_i|^2 dx + \hat{d} \Sigma_{i=1}^4 \int_0^t \int_{\Omega} |\nabla u_i|^2 dxd\tau + \hat{d} \int_0^t \int_{\Omega} |u_1|^2 dxd\tau \leq 0. \]  

In (86), we have \( \hat{d} := \min\{\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \tilde{d}_4, \tilde{k}_1\} > 0 \). Taking in (87) supremum over \((0, T)\), we obtain

\[ \Sigma_{i=1}^4 \int_{\Omega} |u_i|^2 dx + \hat{d} \Sigma_{i=1}^4 \int_0^T \int_{\Omega} |\nabla u_i|^2 dxd\tau \leq 0, \]  

which concludes the proof of the Lemma.

**Lemma 20** (Strong formulation of the two-scale limit equations) Assume the hypothesis of Lemma 12 to hold. Then the strong formulation of the two-scale limit equations (for all \( t \in (0, T) \)) reads

\[ \partial_t u_1(t, x) + \nabla \cdot (-\tilde{d}_1 \nabla u_1(t, x)) = -\tilde{k}_1(t)u_1(t, x) + \tilde{k}_2(t)u_2(t, x) \]
\[ - \frac{1}{|Y|} \int_{\Gamma_{sw}} k_3(t, y) R(u_1(t, x)) Q(u_5(t, x, y)) d\sigma_y, \quad x \in \Omega \]
\[ n \cdot (-\tilde{d}_1 \nabla u_1(t, x)) = 0, \quad x \in \partial\Omega \]

\[ \partial_t u_2(t, x) + \nabla \cdot (-\tilde{d}_2 \nabla u_2(t, x)) = \tilde{k}_1(t)u_1(t, x) - \tilde{k}_2(t)u_2(t, x) \]
\[ + \tilde{a}(t)u_3(t, x) - \tilde{b}(t)u_2(t, x), \quad x \in \Omega, \]
\[ u_2(0, x) = u_{20}(x), \quad x \in \bar{\Omega}, \]

\[ n \cdot (-\tilde{d}_2 \nabla u_2(t, x)) = 0, \quad x \in \partial\Omega, \]

\[ \partial_t u_3(t, x) + \nabla \cdot (-\tilde{d}_3 \nabla u_3(t, x)) = -\tilde{a}(t)u_3(t, x) + \tilde{b}(t)u_2(t, x), \quad x \in \Omega, \]
\[ u_3(0, x) = u_{30}(x), \quad x \in \bar{\Omega}, \]
\[ u_3(t, x) = u_3^D(x), \quad x \in \Gamma^D, \]

\[ n \cdot (-\tilde{d}_3 \nabla u_3(t, x)) = 0, \quad x \in \Gamma^N \]

\[ \partial_t u_4(t, x) + \nabla \cdot (-\tilde{d}_4 \nabla u_4(t, x)) = \tilde{k}_1(t)u_1(t, x), \quad x \in \Omega, \]
\[ u_4(0, x) = u_{40}(x), \quad x \in \bar{\Omega}, \]

\[ n \cdot (-\tilde{d}_4 \nabla u_4(t, x)) = 0, \quad x \in \partial\Omega, \]

\[ \partial_t u_5(t, x, y) = k_3(t, y) R(u_1(t, x)) Q(u_5(t, x, y)), \quad x \in \Omega, y \in \Gamma_{sw}, \]
\[ u_5(0, x, y) = u_{50}(x, y), \quad x \in \bar{\Omega}, y \in \Gamma_{sw}, \]

where \( \tilde{d}_i, i \in \{1, 2, 3, 4\} \) and \( \tilde{k}_j, j \in \{1, 2\} \) are defined in Theorem 14.

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References


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