Two theorems on lattice expansions

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Two Theorems on Lattice Expansions

I. Daubechies, Member, IEEE, and A. J. E. M. Janssen, Senior Member, IEEE

Abstract—It is shown that there is a trade-off between the smoothness and decay properties of the dual functions, occurring in the lattice expansion problem. More precisely, it is shown that if \( g \) and \( g \) are dual, then 1) at least one of \( H^{1/2}g \) and \( H^{1/2}g \) is not in \( L^2(\mathbb{R}) \), 2) at least one of \( Hg \) and \( g \) is not in \( L^2(\mathbb{R}) \). Here, \( H \) is the operator \(-1/(4\pi^2)d^2/(dt^2) + t^2\). The first result is a generalization of a theorem first stated by Balian and independently by Low, which was recently rigorously proved by Cilman and Semmes; a new, much shorter proof was very recently given by Battle. Battle suggests a theorem proved by Coifman and Semmes; a new, much shorter proof in smoothness and decay properties of the dual functions, occurring in the lattice expansion problem. More precisely, it is shown that if \( g \) is of type (i), but our result is stronger in the sense that certain basics, and time-frequency localization.

The problem of determining the coefficients \( c_{nm} \) in (1.1) became tractable notably through the work of Zak on solid-state physics related problems [1], [20], [21], of Bastiaans on optical signal description [4], [5], and of Janssen who gave rigorous proofs of existence and convergence of the expansions (1.1) in \( L^2(\mathbb{R}) \) and other spaces of (generalized) functions [11]–[13]. In this context, Daubechies and Grossmann [7] exploited the notion of frame that indicates a set of functions \( g_{nm} \) as in (1.2) such that for some \( A > 0, B > 0, \)

\[ A\|f\|^2 \leq \sum_{n,m} |(f, g_{nm})|^2 \leq B\|f\|^2, \tag{1.3} \]

for all \( f \in L^2(\mathbb{R}) \). For \( \|\cdot\| \) and \((\cdot, \cdot)\) denote ordinary norm and inner product in \( L^2(\mathbb{R}) \). It is amply demonstrated in the comprehensive [8] that for numerically reliable expansion of \( f \) as in (1.1), one needs to consider \( g \) such that the set \( g_{nm} \) constitutes a frame. An even more desirable case occurs when the constants \( A \) and \( B \) are equal. Then the \( g_{nm} \)'s are said to constitute a tight frame, and the \( g_{nm} \)'s are orthogonal.

The procedure of finding the coefficients \( c_{nm} \) in (1.1) is as follows. Consider the mapping \( Z \) defined for \( f \in L^2(\mathbb{R}) \) by

\[ (Zf)(\tau, \Omega) = \sum_{k=-\infty}^{\infty} \frac{f(z + k)e^{-2\pi i k \Omega}}{\sqrt{2\pi}}, \quad \tau, \Omega \in \mathbb{R}. \tag{1.4} \]

This mapping has several names, such as Gel'fand mapping [10], [19], Weil–Brezin mapping [18], Zak transform [13], [14], while it seems that Gauss was already aware of some of its properties [18]. We shall call \( Z \) the Zak transform, since Zak seems to be the first one to exploit the transform systematically in the context of completeness and expansion problems. For a survey of the numerous properties of the Zak transform we refer to [14]. The relevant property of \( Z \) for the expansion problem is that

\[ (Zg_{nm})(\tau, \Omega) = e^{-2\pi i n \tau + 2\pi i m \Omega}(Zg)(\tau, \Omega). \tag{1.5} \]

Hence, we have, at least formally,

\[ c_{nm} = \frac{\int e^{2\pi i m \tau - 2\pi i k \Omega}(Zf)(\tau, \Omega)}{(Zg_{nm})(\tau, \Omega)} \, d\Omega. \tag{1.6} \]

In (1.6), the integration is over any unit square \((Zf)/Zg\) is periodic with period 1 \((\tau, \Omega)\) and its variables). To introduce the notion of dual function we note the property that \( Z \) is a Hilbert space isomorphism between \( L^2(\mathbb{R}) \) and the set of all functions \( F(\tau, \Omega) \) such that

\[ F(\tau, \Omega + 1) = F(\tau, \Omega), \quad F(\tau + 1, \Omega) = e^{2\pi i \Omega} F(\tau, \Omega). \tag{1.7} \]

In the latter set of functions the inner product of an \( F \) and \( G \) satisfying the (quasi-) periodicity relations in (1.7) is given by

\[ (F, G) = \int F(\tau, \Omega)G^*(\tau, \Omega) \, d\tau \, d\Omega, \tag{1.8} \]

I. CONSIDER in this note expansions of the type

\[ f(t) \sim \sum_{n,m} c_{nm}g_{nm}(t), \quad t \in \mathbb{R}, \tag{1.1} \]

where \( f \in L^2(\mathbb{R}) \),

\[ g_{nm}(t) = e^{-2\pi i n t}g(t + n), \quad t \in \mathbb{R}, n, m \in \mathbb{Z}, \tag{1.2} \]

\( g \in L^2(\mathbb{R}) \) is a fixed function of time, usually well concentrated in time and frequency, and \( \omega \) is a fixed real number \( \neq 0 \). For \( \omega < 1 \), many choices for \( g \) lead to convergent expansions for all \( f \in L^2(\mathbb{R}) \), [8]. In this paper, we restrict ourselves to the case \( \omega = 1 \), corresponding to lattices with the largest possible mesh size.

In (1.1), the coefficients \( c_{nm} \) depend on both \( f \) and \( g \). Problems related to the present one were considered (for the case of Gaussian \( g \)) by Von Neumann in a quantum mechanical context [16], by Gabor in the context of efficient data transmission [9], by Perelomov [17], Bargmann, Butera, Girardello, and Klauder [3], and by Bacry, Grossmann, and Zak [1], who all gave completeness properties of the set of \( g_{nm} \)'s. The problem of determining the coefficients \( c_{nm} \) in the expansion (1.1) became tractable notably through the work of Zak on solid-state physics related problems [1], [20], [21], of Bastiaans on optical signal description [4], [5], and of Janssen who gave rigorous proofs of existence and
where the integral is over any unit square and the asterisk denotes complex conjugation. In particular, for any $f_1, f_2 \in L^2(\mathbb{R})$, we have

$$ (f_1, f_2) = (Z f_1, Z f_2). \quad (1.9) $$

Now, the function $1/(Zg)^*$ satisfies the relations (1.7), and if it is square integrable over a unit square, there is a unique $\tilde{g} \in L^2(\mathbb{R})$ such that

$$ Z \tilde{g} = \frac{1}{(Zg)^*}. \quad (1.10) $$

This $\tilde{g}$ is called the dual function, and $\tilde{g}_{nm}$ constitute the dual frame. We observe that $\tilde{g} = g$, and that

$$ (g, \tilde{g}_{nm}) = \delta_{n0} \delta_{m0} (\tilde{g}, g_{nm}), \quad (1.11) $$

with $\delta$ the Kronecker function.

It follows from (1.5), (1.6), and (1.10) that the coefficients $c_{nm}$ can be expressed as

$$ c_{nm} = (f, \tilde{g}_{nm}). \quad (1.12) $$

Furthermore, the conditions of being a frame and a tight frame can be expressed in terms of the Zak transform as

$$ \text{ess sup} |Zg| < \infty, \quad \text{ess sup} |Zg| > 0 \quad (1.13) $$

and

$$ \text{ess sup} |Zg| = \text{ess inf} |Zg| < \infty, \quad (1.13) $$

respectively, (see [8]).

The two main theorems of this paper read as follows. With the notation, $H = -\frac{1}{2} \frac{d^2}{dx^2} + t^2$ for the Hermite operator, we have the following.

**Theorem I**: If $g$ and $\tilde{g}$ are dual functions in the sense previously explained, then $H^{1/2}g$ and $H^{1/2}\tilde{g}$ cannot both be in $L^2(\mathbb{R})$.

**Theorem II**: Under the same assumptions, $H g$ and $\tilde{g}$ cannot both be in $L^2(\mathbb{R})$.

Hence, in a sense, $g$ and $\tilde{g}$ cannot both be smooth and rapidly decaying. That such results can be expected is seen as follows. Assume that $g$ is such that $Zg$ is continuous; this holds when $g$ is continuous and decays sufficiently rapidly, e.g., like $1/(1 + |t|^\alpha)$ with $\alpha > 1$. It is a curious property of the Zak transform that then $Zg$ has at least one zero in the unit square [1], [13]. Hence, $\text{ess sup} |Zg| = \infty$. And when $Zg$ is continuously differentiable, we even have that $1/Zg$ is not square integrable over the unit square.

Note that nevertheless Theorems I and II are nontrivial since $H^{1/2}g \in L^2(\mathbb{R})$ does not imply that $Zg$ is continuous, and $H g \in L^2(\mathbb{R})$ does not imply that $Zg$ is continuously differentiable.

Recently, some results like ours have been proved. Balian [2] and Low [15] both argued that at least one of $tg(t)$ and $g'(t)$ is not in $L^2(\mathbb{R})$ when $g_{nm}$ constitutes a tight frame. Their argument was made rigorous and extended by Coifman and Semmes to include the case of nontight frames; this is presented by Daubechies in [8]. Finally, an independent, more elegant, proof of the Balian–Low result was given by Battle in [6].

To see what the novelty of the present paper is, we give some further preliminary remarks. The conditions

a) $H^{1/2}g \in L^2(\mathbb{R})$,

b) $tg(t) \in L^2(\mathbb{R})$, $g'(t) \in L^2(\mathbb{R})$,

c) $\frac{d^2g}{d\tau^2} \in L^2(S)$, $\frac{d^2g}{d\tau^2} \in L^2(S)$, $\frac{d^2g}{d\tau^2} \in L^2(S)$,

are equivalent.

That a) and b) are equivalent is a standard fact; that b) and c) are equivalent follows from (2.1) and (2.9). Similarly, the conditions

e) $H_g \in L^2(\mathbb{R})$,

f) $\frac{d^2g}{d\tau^2}(t) \in L^2(\mathbb{R})$, $\frac{d^2g}{d\tau^2}(t) \in L^2(\mathbb{R})$,

g) $\frac{d^2g}{d\tau^2} \in L^2(S)$, $\frac{d^2g}{d\tau^2} \in L^2(S)$, $\frac{d^2g}{d\tau^2} \in L^2(S)$

(i.e., $Zg \in W^{2,1}(S)$),

are equivalent. Now, when $g_{nm}$ constitutes a frame and $Zg \in W^{2,1}(S)$, it follows from $\text{ess inf} |Zg| > 0$ and

$$ \frac{\partial}{\partial \tau} \left( \frac{1}{Zg} \right) = -\left( \frac{1}{Zg} \right)^2 \frac{\partial Zg}{\partial \tau}, $$

$$ \frac{\partial}{\partial \Omega} \left( \frac{1}{Zg} \right) = -\left( \frac{1}{Zg} \right)^2 \frac{\partial Zg}{\partial \Omega} \quad (1.15) $$

that

$$ \frac{1}{(Zg)^*} = Z\tilde{g} \in W^{2,1}(S). $$

That is, Theorem I implies the Coifman–Semmes result, and, a fortiori, the Balian–Low–Battle result. The Theorem II is entirely new as far as we know. While our proof of Theorem I uses a little trick that can be found in Battle’s paper, the proof of Theorem II is based on the two facts that

a) when $Zg \in W^{2,2}(S)$ then $Zg$ is continuous and has a zero in $S$,

b) when $G \in W^{2,2}(S)$ has a zero then $1/G \notin L^2(S)$.

We were unable to find the result (b) in the literature, and it may be of some independent interest.

Theorems I and II may be viewed as no-go theorems, excluding the possibility of numerically stable expansions of type (1.1) with respect to $g_{nm}$ in (1.2) with $\omega = 1$, which are well-localized in both time and frequency. This can be avoided by using expansions with tighter lattices, corresponding to the choice $\omega < 1$, [8]. It is well known that the dual function $\tilde{g}$ has many singular features [4], [5] if $g$ is Gaussian. Our Theorem II generalizes the result in [13] that $\tilde{g}$ is not square integrable.

**II. PROOF OF THEOREM I**

As explained in Section I, we must take a $g \in L^2(\mathbb{R})$ with $Zg, Z\tilde{g} \in W^{2,1}(S)$ and show that this leads to a contradiction. Denote

$$ (Qg)(t) = tg(t), \quad (Pg)(t) = \frac{1}{2\pi t} g'(t), \quad \text{etc.} \quad (2.1) $$

As in Battle’s proof, we shall show that

$$ (Qg, P\tilde{g}) = (Pg, Q\tilde{g}) \quad (2.2) $$
by assumption, all four functions involved in (2.2) are in $L^2(\mathbb{R})$. This implies that $(g,g) = 0$, which is absurd by (1.11).

To demonstrate (2.2), we need the auxiliary results

\begin{align}
(Qg, Pg) &= \sum_{n,m} (Qg, g_{nm})(g_{nm}, Pg), \\
(Pg, Qg) &= \sum_{n,m} (Pg, g_{nm})(g_{nm}, Qg).
\end{align}

(2.3)

(2.4)

In [6], the validity of the expansions (2.3), (2.4) was implicitly assumed (and not proved as is done here). The relation (2.3) follows from the fact, to be proved below, that $ZQg/Zg, ZPg/Zg \in L^2(S)$. Indeed, it then follows from $Zg \cdot (Zg)^* = 1$ that

$$(Qg, Pg) = (ZQg, ZPg) = (ZQg/Zg, ZPg/Zg).$$

(2.5)

The right-hand side of (2.5) equals the right-hand side of (2.3), since $(Qg, g_{nm})$ and $(Pg, g_{nm})$ are the Fourier coefficients of $ZQg/Zg$ and $ZPg/Zg$ by (1.6) and (1.12). Similarly, $ZPg/Zg, ZQg/Zg \in L^2(S)$ implies that (2.4) holds.

We shall show now that $ZQg/Zg, ZPg/Zg \in L^2(S)$. We have

\begin{align}
\left(\frac{1}{Zg}\right)^2 \frac{\partial Zg}{\partial \Omega} = -\frac{\partial}{\partial \Omega} \left(\frac{1}{Zg}\right) \in L^2(S).
\end{align}

(2.6)

It follows from the Cauchy–Schwarz inequality that

$$\left| \frac{1}{Zg} \frac{\partial Zg}{\partial \Omega} \right| \leq \left| \frac{1}{Zg} \right|^{1/2} \left| \frac{\partial Zg}{\partial \Omega} \right|^{1/2} \in L^2(S),$$

(2.7)

and, similarly,

$$\left| \frac{1}{Zg} \frac{\partial Zg}{\partial \tau} \right| \in L^2(S).$$

(2.8)

Since

\begin{align}
ZQg &= \frac{1}{2\pi i} \frac{\partial Zg}{\partial \Omega} + \tau (Zg)(\tau, \Omega), \\
ZPg &= \frac{1}{2\pi i} \frac{\partial Zg}{\partial \tau},
\end{align}

(2.9)

it follows that $ZQg/Zg, ZPg/Zg \in L^2(S)$, as claimed.

To show (2.2), it suffices to prove that

\begin{align}
(Qg, g_{nm}) &= (g_{n,-m}, Qg), \\
(Pg, g_{nm}) &= (g_{n,-m}, Pg).
\end{align}

(2.10)

We have

\begin{align}
(Qg, g_{nm}) &= \int t^d (g(t) \bar{g}(t)^* + t^d (g(t) \bar{g}(t)^*) dt \\
&= (t-n)g(t-n) + g_{n,-m} \bar{g} \cdot (g_{n,-m}, \bar{g}).
\end{align}

(2.11)

Together with (1.11), this implies the first part of (2.10). The second part of (2.10) follows from the first part by noting that, with $F$ the Fourier transform,

$$FP = QF, \quad Fg_{nm} = (Fg)_{-n,-m}. \quad (2.12)$$

and the fact that $Fg$ and $Fg$ are dual. This establishes (2.2).

We conclude the proof of Theorem I by showing that (2.2) implies that $(g, g) = 0$. We have by assumption

$$\int \frac{dt}{t^d (g(t) \bar{g}(t)^*)} = t^d (g(t) \bar{g}(t)^*) \in L^1(\mathbb{R}).$$

(2.13)

The right-hand side function in (2.13) equals

$$-2\pi i (Qg \cdot (Pg)^* - P \cdot (Qg)^*).$$

(2.14)

We now have, for all $a < b$

$$\int_a^b \frac{dt}{t^d (g(t) \bar{g}(t)^*)} dt = t^d (g(t) \bar{g}(t)^*).$$

(2.15)

When $a \to -\infty, b \to \infty$, the left-hand side of (2.15) tends to 0 by (2.2), (2.13) and (2.14), and the integral on the right-hand side tends to $(g, g)$. Hence,

$$\lim_{b \to \infty, a \to -\infty} t^d (g(t) \bar{g}(t)^*) = 0.$$ 

(2.16)

exists as well and equals 0 since $t^d (g(t) \bar{g}(t)^*) \in L^1(\mathbb{R})$. Therefore, $(g, g) = 0$, and the proof of Theorem I is complete.

III. PROOF OF THEOREM II

As already explained at the end of Section I, it is sufficient to show the following result.

Proposition: Assume $G \in W^{2,2}(S)$, where $S = [-i, i] \times [-i, i]$, and $G(0,0) = 0$. Then, $1/G \not\in L^2(S)$.

Proof: For notational convenience we write $x = (\tau, \Omega) \in \mathbb{R}^2$, and we denote by $\cdot$ and the Euclidean norm and inner product in $\mathbb{R}^2$, respectively. Let $0 < r < \frac{1}{2}$. We (re)define $G(x)$ for $|x| \geq r$ such that the resulting function, again denoted by $G$, is in $W^{2,2}(\mathbb{R}^2)$. When $G(\xi), \xi \in \mathbb{R}^2$ is the Fourier transform if $G$, we have

$$\int \left(1 + |\xi|^2 \right)^2 |G(\xi)|^2 d\xi < \infty.$$ 

(3.1)

Now, by Fourier inversion,

$$G(x) = G(x) - G(0) = \int (e^{2\pi i x \cdot \xi} - 1) \hat{G}(\xi) d\xi.$$ 

(3.2)

Hence, by the Cauchy–Schwarz inequality,

$$|G(x)|^2 \leq \int \left(1 + |\xi|^2 \right)^2 |\hat{G}(\xi)|^2 d\xi.$$ 

(3.3)

We have for the first integral $I_1$ in (3.3)

$$I_1 = 4 \int \frac{\sin^2 \pi x \cdot \xi}{(1 + |\xi|^2)^2} d\xi = 4 \int \frac{\sin^2 \pi x \cdot \xi}{(1 + |\xi|^2)^2} d\xi.$$ 

(3.4)

The first integral $I_2$ in (3.4) satisfies

$$I_2 \leq \pi^2 |x|^2 \int \frac{|\xi|^2}{(1 + |\xi|^2)^2} d\xi \leq \pi^2 |x|^2 \log \left(1 + \frac{1}{|x|^2} \right).$$ 

(3.5)
The second integral $I_3$ in (3.4) satisfies

$$I_3 \leq \int \frac{d\xi}{|\xi|^{2}} \frac{\pi|\xi|^2}{1 + |\xi|^2} = \pi. \tag{3.6}$$

Hence, by (3.1), (3.3), (3.5), and (3.6).

$$|G(x)|^2 = 0 \left( \frac{|x|^2 \log \frac{1}{|x|^2}}{1 + |x|^2} \right), \quad |x| \leq \frac{1}{2}. \tag{3.7}$$

Since

$$\int_{|x|^{2} \leq \frac{1}{2}} \frac{dx}{|x|^2 \log \frac{1}{|x|^2}} = \infty, \tag{3.8}$$

the proposition follows.

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