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On the generation of mean fields by small-scale electron magnetohydrodynamic turbulence

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The problem of the generation of mean magnetic fields by small-scale turbulence within the framework of electron magnetohydrodynamics (EMHD) is considered. Two EMHD models are investigated, a two and one-half dimensional ($2\frac{1}{2}$ D) model in which the magnetic field has all three spatial components but, due to a strong external field, depends only on two coordinates, and a fully three-dimensional (3D) model with an imposed stationary and homogeneous magnetic field. It is shown that in the case of $2\frac{1}{2}$ D turbulence two possible mechanisms are responsible for the generation of mean magnetic fields. The first one is similar to the α -effect in the MHD dynamo problem and is due to a nonzero helicity of the turbulence. The second one is related to the anisotropy of the turbulence, which can give rise to negative dissipation (resistivity, viscosity) of the mean field. The influence of electron inertia on the above effects is analyzed. Inertia results in a qualitative modification of the helicity effects and may lead to a change in sign of the turbulent viscosity. The criteria for the generation of mean magnetic fields are obtained. In the case of the 3D model, the generation of large-scale helicons by the small-scale helicon turbulence is studied within the framework of the adiabatic approximation. A closed set of equations for the evolution of both the magnetic field of the large-scale helicon and of the generalized action of the small-scale turbulence is obtained. The criterion for the resonant instability of a large-scale helicon due to its interaction with small-scale helicon turbulence is obtained. © 2004 American Institute of Physics.

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I. INTRODUCTION

In the course of the last two or three decades, extensive studies of the generation of large-scale structures by small-scale turbulence have been carried out for several media, such as conducting, nonconducting fluids, and ferromagnetics (see, e.g., Ref. 1 and the bibliography therein). There are two basic mechanisms on which such a turbulent generation is based.

The first one is typical for *helical* (i.e., gyrotropic) turbulence and can be called the turbulent dynamo mechanism. By helical turbulence we mean turbulence with a broken parity invariance, so that its correlation tensor $\langle a_i^t(\mathbf{x}, t) a_j^t(\mathbf{x}', t') \rangle$ depends on a nonzero net helicity $\langle \mathbf{a}^t \cdot \text{curl} \mathbf{a}^t \rangle$. Here, $\mathbf{a}^t$ is the vector field that describes the turbulence and the angular brackets denote a statistical averaging over the turbulent ensemble. This helical mechanism is essentially three-dimensional (or at least $2\frac{1}{2}$ D) and does not exist in a purely 2D configuration. From a physical point of view it is expected that such turbulence will arise in systems where mirror symmetry is broken due to a variety of causes, such as the presence of external force fields with pseudovector properties like a magnetic field or a Coriolis force.

The possibility of the generation of large-scale magnetic fields in a conducting medium by a helical turbulent velocity field has been pointed out in the pioneering paper Ref. 2, in which it was reported that a homogeneous, isotropic tur-

lent velocity field with nonzero helicity results in the so-called α -effect on the mean magnetic field

$$\frac{\partial \langle \mathbf{B} \rangle}{\partial t} = \alpha \text{curl} \langle \mathbf{B} \rangle + \eta \nabla^2 \langle \mathbf{B} \rangle, \quad (1)$$

where $\eta \propto \langle (\mathbf{v}^t)^2 \rangle > 0$ is the turbulent resistivity (or magnetic viscosity) and $\alpha \propto \langle \mathbf{v}^t \cdot \text{curl} \mathbf{v}^t \rangle$ corresponds to helicity. This α -effect is due to the helicity of the turbulence and leads to an exponential growth of the large-scale magnetic field $\langle \mathbf{B} \rangle$, irrespective of the sign of α . The sign of the helicity $\langle \mathbf{B} \rangle \cdot \text{curl} \langle \mathbf{B} \rangle$ of the mean field is determined by that of the helicity of the turbulence.

The occurrence of the α -effect in nonconducting fluids, described by standard hydrodynamical equations, has been studied in Refs. 3–7.

In the first of these papers Krause and Rüdiger have shown that, unlike in the case of a conducting fluid described by magnetohydrodynamics (MHD), a homogeneous isotropic incompressible hydrodynamic turbulence with only nonzero helicity does not result into an α -effect on the mean velocity. This conclusion is known as the *exclusion theorem*. This theorem has been proven in the quasi-linear approximation which neglects correlation functions of higher than second order. In subsequent papers it has been shown that additional symmetry-breaking physical properties of the system allow to overcome this exclusion and to release a mechanism

which makes it possible to pump energy from the helical turbulence into the large-scale motions of a nonconducting fluid. Within this context fluid compressibility,^{4,7} inhomogeneous regular flow,⁵ a gravity force and temperature gradients,⁶ and anisotropy of the turbulence⁸ have been analyzed. All these additional factors result in the suppression of small-scale motions in helical turbulence and in the transfer of energy to larger scales.

A more favorable situation for the generation of large-scale structures occurs in electron magnetohydrodynamics (EMHD), which describes fast, small-scale motions of the electron fluid in a background of fixed ions. In incompressible EMHD, the displacement current may be neglected and a direct relationship between the magnetic field $\mathbf{B}$ and the fluid velocity exists. It has been shown in Ref. 9 that 3D homogeneous, isotropic, but helical turbulence results in an effect on large scales that is similar to the α -effect in MHD. The basic result we refer to is presented in Ref. 9

$$\frac{\partial \langle \mathbf{B} \rangle}{\partial t} = \text{curl } \alpha \nabla^2 \langle \mathbf{B} \rangle - \mu \nabla^4 \langle \mathbf{B} \rangle, \quad (2)$$

where the turbulent coefficients α and μ are defined by the helical and isotropic parts of the correlation tensor, respectively. Like in MHD, the isotropic part of the turbulent spectrum leads to magnetic field diffusion, while the helicity of the turbulence results in an α -effect that leads to a spontaneous amplification of the mean magnetic field. We will give a more detailed discussion of (2) in Sec. III and will use it as a benchmark for other results.

A second mechanism for the generation of large-scale structures exists that does not involve helicity but is related with the contributions from the mirror-symmetric parts of the correlation tensor. In general this contribution results into turbulent dissipation of the large-scale motions in the form of viscosity and resistivity (magnetic viscosity). Under some conditions the associated diffusion constants can be negative which means that turbulent energy is pumped into large-scale perturbations. This mechanism occurs in the case of 2D turbulence which is often characterized by an energy flow towards large scales (inverse energy cascade) and where the turbulence results into negative viscosity on large-scale motions. Instabilities due to negative viscosity have been discussed for several physical problems, see, e.g., Refs. 10–12. Within the framework of the Navier–Stokes equation, it has been shown¹⁰ that a two-dimensional, time-independent parallel periodic flow, a particular example of which is the Kolmogorov flow, is unstable with respect to large-scale perturbations transverse to the basic flow. This instability is due to negative viscosity (see Ref. 10 and references therein).

In two recent studies^{11,12} the effect of negative viscosity on large-scale fields has been analyzed for the Charney–Hasegawa–Mima equation^{13,14} and its simplified version—the geostrophic equation. These equations are the simplest models used to describe both drift waves in the magnetized plasma and Rossby waves in the ocean and the atmosphere of planets.

The first of these papers¹¹ has been motivated by experiments on the transitions between regimes of low and high

plasma confinement (L – H transitions) in tokamaks. It has been proposed that turbulent mechanisms could be responsible for the generation of radial electric fields and sheared poloidal flows that are observed in experiments. The theory is based on the geostrophic equation without dissipation and the derivation of the evolution equation of the large-scale perturbations due to their interaction with small-scale turbulence is carried out in the quasi-linear approximation. It is shown that the contribution of the adiabatic electron response is essential and, in the case of homogeneous isotropic turbulence, gives rise to a negative viscosity effect on the mean field. This does not occur in 2D incompressible hydrodynamics described by the Euler equation.¹⁵ The second paper¹² has been motivated by atmospheric and oceanic applications and deals with the 2D Charney equation supplemented by molecular viscosity and an external force. The small-scale turbulent field is assumed to be sufficiently weak, so that the Reynolds number of the small-scale motions is small and terms that are quadratic with respect to the amplitude of small-scale motions can be neglected in the equations for the turbulent field. This is equivalent to a quasi-linear approximation. Again the negative viscosity effect that has been found is due to the same additional term, which corresponds here to fluid compressibility. The 2D model is in some sense the extreme limiting case of the anisotropic 3D model, in which the gradients of the field variables along the preferred direction are infinitely small compared to the gradients in the perpendicular plane. Such an anisotropy can be caused, e.g., by the presence of an external magnetic field. It seems quite natural to assume that the turbulence possesses different correlation properties in the directions along the external magnetic field and in the perpendicular plane. Hence, negative dissipation will play a significant role in 3D anisotropic turbulent systems. In this paper we will show that this also holds for turbulence in EMHD.

Weak turbulence is very characteristic for plasmas. This kind of turbulence is by its nature wave turbulence and is usually considered as an ensemble of weakly interacting, random waves described by the corresponding dispersion relation. Such an approach to turbulence in plasmas is justified when wave dispersion prevails over nonlinear effects.¹⁶ The problem of the interaction of waves with very different scales has been first investigated in Ref. 17, where the interaction between high-frequency, small-scale Langmuir waves and low-frequency, large-scale acoustic waves has been studied. In this case the adiabatic approximation can be used. Here, the term adiabatic means that the high-frequency, small-scale modes propagate in a slowly varying weakly inhomogeneous medium. The change in the parameters of the medium is due to the presence of the low-frequency, large-scale mode. The latter is obviously influenced by the high-frequency modes, and this effect is taken into account by averaging over the fast oscillations. In the approach suggested in Ref. 17 the small-scale waves are described by the wave kinetic equation for the occupation number in the phase space of coordinates (due to their interaction with the low-frequency, large-scale mode) and wave-vectors, and thus are considered as quasi-particles. On the other hand, the low-frequency, large-scale perturbations are described by equa-

tions of hydrodynamic type, which are averaged over the fast time- and space-scales of the small-scale waves. In recent years the approach of Ref. 17 has been successfully applied in the study of the generation of zonal flows and streamers in tokamak plasmas by small-scale plasma turbulence, in the context of the explanation of L to H transitions in tokamak experiments. Different kinds of plasma turbulence of drift type have been investigated, drift turbulence,^{18,19} ion-temperature gradient driven turbulence,²⁰ and drift-Alvén turbulence.^{21,22} It has been shown in these papers that in general the system involving the small-scale turbulence and the large scale zonal perturbation possesses some invariant value depending only on the small-scale waves, the generalized action invariant, which is equivalent to the occupation number, but usually does not coincide with it.

This paper is a contribution to the investigation of the turbulent generation of large-scale structures in helical and anisotropic media. Little attention has been paid up to now to the turbulent generation of waves and the emphasis has been on the generation of nonpropagating structures. Also, not much attention has been paid to the problem of generation of structures in EMHD. In order to fill this gap, we concentrate in this paper on standard incompressible EMHD, where a direct dependence between the magnetic field $\mathbf{B}$ and the electron fluid velocity $\mathbf{v}$ exists. This paper is organized as follows. The model of EMHD and the conditions under which it is applicable are discussed in Sec. II. In Sec. III the problem of the generation of large-scale perturbations by small-scale EMHD turbulence is considered within the framework of a $2\frac{1}{2}$ D model. The latter model is used to describe the electron dynamics in a strong, external magnetic field. We consider both basic mechanisms mentioned above, helicity and anisotropy, and discuss the effect of electron inertia on the turbulent coefficients. In Sec. IV, the spontaneous amplification of large-scale helicons by small-scale helicon turbulence is analyzed within the framework of a 3D EMHD model with an externally imposed homogeneous magnetic field. We use the approach suggested in Ref. 17 and restrict our analysis to the quasi-linear regime. This is justified when the amplitudes of the small-scale helicons are so small that the associated non-linear frequency shift is negligible compared to their highly dispersive frequencies. Generally speaking, the frequencies of the small-scale helicons are high if the background magnetic field is strong, and therefore, the quasi-linear approach is justified. The main results presented in this paper are summarized in Sec. V, where also an outlook is given on the possible directions of future research on the turbulent generation of mean magnetic fields by EMHD turbulence.

II. ELECTRON MAGNETOHYDRODYNAMICS (EMHD)

Electron magnetohydrodynamics describes plasma motions with spatial scales $L < c/\omega_{pi}$ and frequencies $\omega_{Bi}, \omega_{pi} < \omega < \omega_{Be}, \omega_{pe}$, where $\omega_{pe,i}$ and $\omega_{Be,i}$ are the Langmuir frequencies and gyrofrequencies of electrons and ions, respectively. In these ranges of length and time scales, the electrons may be described by fluid equations; the ions form an immobile, charge neutralizing background. We assume that the equilibrium plasma density is homogeneous, $n = n_0 = \text{const}$,

and restrict ourselves to the case of incompressible EMHD, where the density remains unperturbed. This approach is justified if²³

$$\frac{B_0}{4\pi en_0 l_0} \ll 1, \quad (3)$$

where B_0 is a typical value of the magnetic field, and l_0 a typical scale length of the phenomena involved. This inequality can also be written as $d_e^2 \omega_{Be}^2 / \omega_{pe}^2 \ll 1$, $d_e^2 = c^2 / \omega_{pe}^2 l_0^2$ being the normalized electron inertial skin depth.

In this standard, incompressible EMHD model, the electron fluid motion is completely determined by the evolution of the magnetic field, and described by the well-known equation (see, e.g., Refs. 24 and 25)

$$\frac{\partial}{\partial t} (\mathbf{B} - d_e^2 \text{curl } \mathbf{v}) = \text{curl} [\mathbf{v} \times (\mathbf{B} - d_e^2 \text{curl } \mathbf{v})], \quad (4)$$

where $\mathbf{v} = -\text{curl } \mathbf{B}$. Here, we have normalized the spatial variables to a typical spatial scale l_0 , time to the so-called helicon time $t_0 = \omega_{Be}^{-1} (l_0 \omega_{pe} / c)^2$, the magnetic field strength to B_0 , and the velocity to l_0 / t_0 .

Equation (4) means that the curl of the generalized momentum of the electron fluid, i.e., the generalized vorticity $\mathbf{B} - d_e^2 \text{curl } \mathbf{v}$, is frozen into the electron fluid (see, e.g., Refs. 25 and 26).

The plasma may be embedded in a uniform, homogeneous, background magnetic field $\mathbf{B}_0$. In this case the EMHD equation describes the helicon branch of plasma oscillations with dispersion relation (in dimensional form)

$$\omega = |\cos \theta| \frac{\omega_{Be} k^2 c^2}{\omega_{pe}^2 (1 + k^2 c^2 / \omega_{pe}^2)}, \quad \cos \theta = \frac{\mathbf{k} \cdot \mathbf{B}_0}{kB_0}, \quad (5)$$

where k and ω are the wave number and the frequency, respectively. At this point it is appropriate to underline the difference between the generation of large-scale vortices in incompressible hydrodynamics and of mean-fields in EMHD by small-scale turbulence. The right hand side of Eq. (4) can be written as $\text{curl } \nabla \cdot (-\mathbf{B}\mathbf{B} + d_e^2 \nabla \mathbf{v})$. The first term is the magnetic stress tensor and the second one is the Reynolds stress tensor. Both tensors are symmetric. When electron inertia may be neglected, i.e., in the limit $d_e \rightarrow 0$, Eq. (4) is known as Ohm's law of an ideal plasma and describes the conservation of magnetic flux with the motion of the electron fluid. In the opposite limit where one may formally take $\mathbf{B} \rightarrow 0$, Eq. (4) becomes the hydrodynamic Euler equation that describes the conservation of fluid vorticity. In spite of these similarities, the actual equations in these limits are quite different with respect to which quantity is advanced in time or, more fundamentally, to which quantity is conserved. In the limit of ideal hydrodynamics ($\mathbf{B} \rightarrow 0$), the curl of the velocity field is frozen into the velocity field, while in an ideal magnetic fluid ($d_e \rightarrow 0$) the magnetic field is frozen into its curl. As a result incompressible hydrodynamics and EMHD possess very different properties in their ability to produce turbulent effects on the mean fields. In incompressible hydrodynamics the helicity of the isotropic and homogeneous turbulence itself does not lead to an α -like effect, while it does in EMHD.

III. THE GENERATION OF LARGE-SCALE FIELDS IN 2½D ELECTRON MAGNETOHYDRODYNAMICS

The spontaneous amplification of large-scale perturbations by helical EMHD turbulence was first considered approximately twenty years ago.⁹ In that treatment a 3D model of EMHD perturbations with spatial scales $l > d_e$ was considered. The problem of the behavior of mean fields with spatial scale L and time scale T in the presence of homogeneous turbulent fluctuations with scales l and τ , was analyzed in the two-scale approximation $l \ll L, \tau \ll T$. The turbulence was assumed to be strong, such that, in spite of the disturbance of the turbulent field by the mean field, the relaxation processes restore the stationarity on a time scale of the order of the correlation time τ . In the case of isotropic EMHD turbulence with broken parity invariance (nonzero net helicity) Eq. (2) for the evolution of the mean field was obtained. In that equation the turbulent coefficients α and μ are defined in terms of the helical and isotropic parts of the spectral tensor, respectively,

$$\mu \propto \langle \mathbf{B}^2 \rangle, \quad \alpha \propto \langle \mathbf{B} \cdot \text{curl} \mathbf{B} \rangle. \quad (6)$$

The isotropic part of the turbulent spectrum leads to magnetic field diffusion on a typical time scale L^4/μ , while the chirality of the turbulence results into the growth of the mean field on a time scale $L^3/|\alpha|$.

The applicability of the approach used in Ref. 9 requires that the conditions $d_e < l \ll L < d_i$ are satisfied. Actually, d_e enters the EMHD equations in the combination d_e^2/l^2 , and therefore the electron inertia effects can be neglected when $d_e^2 \ll l^2$. Taking into account that $d_i/d_e = (m_i/m_e)^{1/2} \approx 40$, one concludes that the length-scale of the large-scale perturbation should be of the order of d_i . It means that the theory presented in Ref. 9 is on the limits of applicability of EMHD for the large-scale perturbation.

Here we consider a similar problem of turbulent generation of the mean fields in the framework of 2½D EMHD. Such a model contains many aspects of a fully 3D turbulent model and at the same time possesses some additional essential features due to its intrinsically anisotropic nature. Besides the energy, 2½D EMHD contains two sets of quadratic invariants. One of these is the volume integral of the square of the generalized momentum, the other one is the generalized helicity. The energy has a direct cascade for length scales larger as well as smaller than the inertial skin depth d_e , while the other invariants have an inverse cascade for large and a direct cascade for small scale-lengths.²⁷

A. 2½D EMHD model

When the external magnetic field is dominant, the electron dynamics is effectively restricted to two dimensions and the magnetic field depends only on two spatial coordinates x and y . In this case the magnetic field can be represented in the form

$$\mathbf{B} = \nabla A \times \mathbf{e}_z + B \mathbf{e}_z \equiv \text{curl}(A \mathbf{e}_z) + B \mathbf{e}_z, \quad (7)$$

where both A and B depend on (x, y, t) . The z -component of the vector potential A describes the poloidal magnetic field

and B is the axial field. In terms of these scalar functions the set of EMHD equations takes the form (see, e.g., Refs. 24 and 28)

$$\frac{\partial}{\partial t} (A - d_e^2 \nabla^2 A) + [B, A - d_e^2 \nabla^2 A] = 0, \quad (8)$$

$$\frac{\partial}{\partial t} (B - d_e^2 \nabla^2 B) - d_e^2 [B, \nabla^2 B] + [A, \nabla^2 A] = 0, \quad (9)$$

where $[f, g] \equiv \mathbf{e}_z \cdot [\nabla f \times \nabla g]$ is the Poisson bracket. The second of these equations can be obtained by taking the z -component of Eq. (4), while the first one by subtracting Eq. (9) multiplied by $\mathbf{e}_z$ from Eq. (4), solving the resulting equation of the form $\text{curl} \mathbf{a} = 0$ and taking its z -component. In this model, the magnetic field has all three spatial components, but depends only on two coordinates. Therefore it is often called a 2½D model. This model can still describe helicon mode if a transverse background magnetic field associated with $A_{0x} = -B_{0y}x$ is present. It is generally assumed that only 3D models can describe systems with nonzero helicity [see, e.g., a statement made in Ref. 29, where the analysis of EMHD turbulence is based upon Eqs. (8) and (9)]. However, it can easily be checked that, generally speaking, the above 2½D model possesses a nonzero helicity

$$\mathbf{B} \cdot \text{curl} \mathbf{B} \equiv \nabla A \cdot \nabla B - B \nabla^2 A \neq 0. \quad (10)$$

Thus, the 2½D model, being simple compared to a 3D one, still allows the description of helical motions of the electron fluid.

B. Formulation of the problem and basic assumptions

The physical problem that we consider in this section is the following. Let us assume that there is a homogeneous plasma turbulence with spatial scale l which is maintained at a stationary level by an external source and with prescribed correlation properties to be discussed below. Our main purpose is to study the effect of this turbulence on the evolution of a spontaneous infinitesimal magnetic field perturbation with typical spatial scale L which is large compared to the corresponding scale l of the turbulence, $L \gg l$. In other words, we are going to study the stability of small-scale turbulence with respect to a large-scale perturbation. The assumption of this separation of space scales introduces a small parameter l/L into the problem and allows to apply a two-scale expansion method. We represent both A and B as sums of regular, large-scale mean fields $\langle A \rangle, \langle B \rangle$, which are considered to be infinitesimal, and small-scale, random fields $\tilde{A}, \tilde{B}$:

$$A = \langle A \rangle + \tilde{A}, \quad B = \langle B \rangle + \tilde{B}, \quad \langle \tilde{A} \rangle = \langle \tilde{B} \rangle. \quad (11)$$

Hereafter $\langle \dots \rangle$ implies the ensemble averaging which is equivalent to time averaging with the appropriate ergodic assumption. In the absence of the large-scale field the random components are reduced to A' and B' , which correspond to the background turbulence (the zeroth order of the two-scale expansion procedure), so that

$$\tilde{A} = A' + \delta A, \quad \tilde{B} = B' + \delta B, \quad (12)$$

where δA and δB are infinitesimal and represent the inhomogeneous part of the turbulent field due to its nonlinear interaction with the mean field. The background turbulence is considered to be homogeneous.

C. Equations for the mean and random fields

Our main goal is to study the evolution of the mean fields. The equations describing the interaction of small-scale turbulence with the mean-field follow from Eqs. (8) and (9) after substituting Eqs. (11) and (12), and separating the mean and random parts. The mean-field equations are

$$\frac{\partial}{\partial t} \{ \langle A \rangle - d_e^2 \nabla_s^2 \langle A \rangle \} + \langle [B^t, \delta A - d_e^2 \nabla^2 \delta A] \rangle + \langle [\delta B, A^t - d_e^2 \nabla^2 A^t] \rangle = 0, \quad (13)$$

$$\frac{\partial}{\partial t} \{ \langle B \rangle - d_e^2 \nabla_s^2 \langle B \rangle \} + \langle [A^t, \nabla^2 \delta A] + [\delta A, \nabla^2 A^t] \rangle - d_e^2 \langle [B^t, \nabla^2 \delta B] + [\delta B, \nabla^2 B^t] \rangle = 0. \quad (14)$$

Due to the assumption that the background turbulence is homogeneous, the effect of small-scale turbulence in the mean-field equations is proportional to the turbulent responses to the large-scale perturbations (δA and δB).

The turbulent responses satisfy the equations,

$$\frac{\partial}{\partial t} (\delta A - d_e^2 \nabla^2 \delta A) + [B^t, \langle A \rangle - d_e^2 \nabla^2 \langle A \rangle] + [\langle B \rangle, A^t - d_e^2 \nabla^2 A^t] = 0, \quad (15)$$

$$\frac{\partial}{\partial t} (\delta B - d_e^2 \nabla^2 \delta B) + [A^t, \nabla^2 \langle A \rangle] + [\langle A \rangle, \nabla^2 A^t] - d_e^2 \langle [B^t, \nabla^2 \langle B \rangle] + [\langle B \rangle, \nabla^2 B^t] \rangle = 0. \quad (16)$$

Here, we have limited ourselves to the quasi-linear approximation and, thus, have neglected quadratic terms like $\langle \delta \mathbf{B} \mathbf{B}^t \rangle - \delta \mathbf{B} \mathbf{B}^t$. Such an approach is justified if the turbulence is controlled by the correlation properties of the source rather than by nonlinear interactions between turbulent pulsations, i.e., when the following conditions are satisfied:

$$\langle (B^t)^2 \rangle^{1/2} / l^2 \ll \frac{1}{\tau}, \quad \langle (A^t)^2 \rangle^{1/2} / l^3 \ll \frac{1}{\tau}, \quad (17)$$

where τ is the correlation time of the turbulent pulsations.

D. The mean-field equations

To obtain a closed set of equations for the evolution of the mean fields, Eqs. (15) and (16) should be solved for the inhomogeneous turbulent fields δA and δB in terms of A^t , B^t and the mean fields $\langle A \rangle, \langle B \rangle$. The easiest way to do this is to apply the Fourier representation of the fields

$$(A^t, \delta A) = \int d\mathbf{k} (A_{\mathbf{k}}^t, \delta A_{\mathbf{k}})(t) e^{i\mathbf{k}\mathbf{x}}, \quad (18)$$

$$\langle A \rangle = \int d\mathbf{q} \langle A \rangle_{\mathbf{q}}(t) e^{i\mathbf{q}\mathbf{x}}, \text{ etc.}$$

Due to the above assumption of space-scale separation $q \ll k$. The Fourier transformed Eqs. (15) and (16) are

$$(1 + k^2 d_e^2) \frac{\partial}{\partial t} \delta A_{\mathbf{k}}(t) + \int d\mathbf{q} (\mathbf{k} \times \mathbf{q})_z \{ [1 + (\mathbf{k} - \mathbf{q})^2 d_e^2] \times A_{\mathbf{k}-\mathbf{q}}^t(t) \langle B \rangle_{\mathbf{q}}(t) - (1 + q^2 d_e^2) B_{\mathbf{k}-\mathbf{q}}^t(t) \langle A \rangle_{\mathbf{q}}(t) \} = 0, \quad (19)$$

$$(1 + k^2 d_e^2) \frac{\partial}{\partial t} \delta B_{\mathbf{k}}(t) + \int d\mathbf{q} (\mathbf{k} \times \mathbf{q})_z (k^2 - 2\mathbf{k}\mathbf{q}) \times \{ d_e^2 B_{\mathbf{k}-\mathbf{q}}^t(t) \langle B \rangle_{\mathbf{q}}(t) - A_{\mathbf{k}-\mathbf{q}}^t(t) \langle A \rangle_{\mathbf{q}}(t) \} = 0. \quad (20)$$

In addition to space-scale separation we assume that the time scales of the turbulence and of the large-scale perturbation are separated too. We will show shortly that such an assumption is always justified. Then, we find from Eqs. (19) and (20)

$$\delta A_{\mathbf{k}}(t) = \int d\mathbf{q} \frac{(\mathbf{k} \times \mathbf{q})_z}{1 + k^2 d_e^2} \times \left\{ (1 + q^2 d_e^2) \langle A \rangle_{\mathbf{q}}(t) \int_{-\infty}^t B_{\mathbf{k}-\mathbf{q}}^t(t') dt' - [1 + (\mathbf{k} - \mathbf{q})^2 d_e^2] \langle B \rangle_{\mathbf{q}}(t) \int_{-\infty}^t A_{\mathbf{k}-\mathbf{q}}^t(t') dt' \right\}, \quad (21)$$

$$\delta B_{\mathbf{k}}(t) = \int d\mathbf{q} \frac{(\mathbf{k} \times \mathbf{q})_z (k^2 - 2\mathbf{k}\mathbf{q})}{1 + k^2 d_e^2} \times \left\{ \langle A \rangle_{\mathbf{q}}(t) \int_{-\infty}^t A_{\mathbf{k}-\mathbf{q}}^t(t') dt' - d_e^2 \langle B \rangle_{\mathbf{q}}(t) \int_{-\infty}^t B_{\mathbf{k}-\mathbf{q}}^t(t') dt' \right\} = 0. \quad (22)$$

It has been assumed that the turbulence is stationary and homogeneous. These two conditions mean that the two-point, two-time correlators of the turbulence depend only on $\mathbf{x} - \mathbf{x}'$, $t - t'$. In addition, we adopt the reasonable assumption that turbulence is isotropic in the $x-y$ plane. Then, the correlators can depend only on $|\mathbf{x} - \mathbf{x}'|$. In the most general case we will describe the turbulence by three spectral correlation functions of second order

$$\langle (B_{\mathbf{k}}^t(t) B_{\mathbf{k}'}^t(t')), \langle A_{\mathbf{k}}^t(t) A_{\mathbf{k}'}^t(t')), \langle A_{\mathbf{k}}^t(t) B_{\mathbf{k}'}^t(t')) \rangle = I^{(B,A,h)}(k) \delta(\mathbf{k} + \mathbf{k}') \Phi(t - t'). \quad (23)$$

A nonzero value of $\int k^2 I^h(k) d\mathbf{k}$ implies that the turbulence is helical and possesses nonzero (cross-)helicity $\langle \mathbf{v} \cdot \mathbf{B} \rangle = -\langle \mathbf{B}^t \cdot \text{curl } \mathbf{B}^t \rangle \neq 0$. Further, if $\int d\mathbf{k} k^2 (k^2 I^A(k) - I^B(k)) \neq 0$, the turbulence is anisotropic in the sense that $\langle (\nabla B_z^t)^2 - (\nabla B_x^t)^2 - (\nabla B_y^t)^2 \rangle \neq 0$. It is this latter effect that results into the turbulent resistivity found in our previous paper.³⁰

In agreement with the assumption that the characteristic time T of the evolution of large-scale perturbations is large

compared to the correlation time of turbulence, $T \gg \tau$, the turbulence may be taken to be delta-correlated in time to a first approximation

$$\Phi(t-t') = \delta\left(\frac{t-t'}{\tau}\right). \quad (24)$$

Below we will show that this time separation is consistent with the assumption of space-scale separation.

Substituting Eqs. (21) and (22) into Eqs. (13) and (14) we arrive at the following equations for the Fourier components of the mean fields (the details of the calculation of the averages of the Poisson brackets are given in Appendix A):

$$\begin{aligned} (1+q^2 d_e^2) \frac{\partial}{\partial t} \langle A \rangle_{\mathbf{q}} + \int_0^\infty dk \pi \tau k^3 q^2 \left\{ \left[\frac{1}{2} (1+q^2 d_e^2) I^B(k) \right. \right. \\ \left. \left. - \kappa(\epsilon) \frac{(1+k^2 d_e^2)(k^2-q^2)}{1+k^2 d_e^2 + q^2 d_e^2} I^A(k) \right] \langle A \rangle_{\mathbf{q}} \right. \\ \left. - \kappa(\epsilon) \frac{1+k^2 d_e^2}{1+k^2 d_e^2 + q^2 d_e^2} \right. \\ \left. \times [1+2q^2 d_e^2 - kq d_e^2 \epsilon \kappa(\epsilon)] I^h(k) \langle B \rangle_{\mathbf{q}} \right\} = 0, \end{aligned} \quad (25)$$

$$\begin{aligned} (1+q^2 d_e^2) \frac{\partial}{\partial t} \langle B \rangle_{\mathbf{q}} + \int_0^\infty dk \frac{\pi \tau k^3 q^2}{1+k^2 d_e^2 + q^2 d_e^2} \kappa(\epsilon) \\ \times [q^2 - kq \epsilon \kappa(\epsilon)] \\ \times \{ [(1+k^2 d_e^2) I^A(k) - d_e^4 (k^2 - q^2) I^B(k)] \langle B \rangle_{\mathbf{q}} \\ - I^h(k) (1 - k^2 d_e^2 + 2q^2 d_e^2) \langle A \rangle_{\mathbf{q}} \} = 0, \end{aligned} \quad (26)$$

where

$$\kappa(\epsilon) = \frac{1 - \sqrt{1 - \epsilon^2}}{\epsilon^2}, \quad \epsilon \equiv \frac{2kq d_e^2}{1+k^2 d_e^2 + q^2 d_e^2} \ll 1. \quad (27)$$

Note that the evolution equations (25) and (26) become decoupled in the limit of vanishing helicity, $I^h \rightarrow 0$. In the next subsections we will consider several limiting cases of these averaged equations.

1. The case $kd_e \ll 1$

First we will consider $2\frac{1}{2}$ D EMHD turbulence with spatial scales larger than the collisionless electron skin depth, $l > d_e$. According to the discussion given in the beginning of this section, the large-scale perturbation in this limiting case has the scale-length which is close to the limits of applicability of EMHD. At the same time, this limit is easier to analyze, while it still possesses most features that are pertinent to the full model. Such an approach allows to reveal in a more clear way the main physical processes.³⁰

When electron inertia effects are negligible ($kd_e \ll 1$), we obtain from Eqs. (25) and (26), after back transformation to real space, the following leading order equations

$$\frac{\partial}{\partial t} \langle A \rangle - \eta \nabla^2 \langle A \rangle + \mu \nabla^4 \langle A \rangle + \alpha \nabla^2 \langle B \rangle = 0, \quad (28)$$

$$\frac{\partial}{\partial t} \langle B \rangle + \mu \nabla^4 \langle B \rangle - \alpha \nabla^2 \langle A \rangle = 0, \quad (29)$$

where

$$\begin{aligned} \eta &= \frac{\pi \tau}{2} \int_0^\infty dk k^3 [I^B(k) - k^2 I^A(k)] \\ &\equiv (\tau/4) \langle (\nabla B_z^t)^2 - (\nabla B_x^t)^2 - (\nabla B_y^t)^2 \rangle, \\ \mu &= \frac{\pi \tau}{2} \int_0^\infty dk k^3 I^A(k) \equiv (\tau/2) \langle (\mathbf{B}_\perp^t)^2 \rangle, \\ \alpha &= \frac{\pi \tau}{2} \int_0^\infty dk k^3 I^h(k) \equiv (\tau/8) \langle \mathbf{B}^t \cdot \text{curl } \mathbf{B}^t \rangle. \end{aligned} \quad (30)$$

The coefficients (30) correspond to turbulent resistivity (or magnetic viscosity), turbulent viscosity, and to the α -effect, respectively. The turbulent viscosity is always positive and leads to damping of the mean field on the time scale L^4/μ . The turbulent resistivity can have either sign depending on the values of the mean squares of the gradients of the axial and poloidal components of the turbulent magnetic field. In particular, when $\langle (\nabla B_z^t)^2 - (\nabla B_x^t)^2 - (\nabla B_y^t)^2 \rangle < 0$, i.e., when the effect from the poloidal components of the magnetic field prevails over the effect from the axial component, resistivity becomes negative and tends to amplify the spontaneous mean-field perturbation.

Recalling that the magnetic field in $2\frac{1}{2}$ D EMHD model is described by Eq. (7), we can rewrite the set of equations (28) and (29) in the form of a single vector equation

$$\frac{\partial \langle \mathbf{B} \rangle}{\partial t} = \eta \nabla^2 \langle \mathbf{B}_\perp \rangle - \alpha \nabla^2 \text{curl} \langle \mathbf{B} \rangle - \mu \nabla^4 \langle \mathbf{B} \rangle, \quad (31)$$

where $\langle \mathbf{B}_\perp \rangle$ is the mean poloidal magnetic field. Except for the contribution of turbulent resistivity, Eq. (31) has the same form as the result (2) given in Ref. 9 for 3D isotropic turbulence. A similar problem has been considered recently in Ref. 29, where the authors were mistaken in stating that their $2\frac{1}{2}$ D model cannot possess finite helicity. As a consequence their treatment is restricted to mirror-symmetric turbulence. They did find turbulent resistivity, but overlooked that it can become negative and restricted their discussion to the suppression of turbulent diffusivity as a result of small-scale turbulence in the poloidal components of the magnetic field.

An external magnetic field B_{0y} corresponding to $A_0 = -B_{0y}x$ was taken into account in Ref. 29. This allows to investigate finite frequency helicon (whistler) modes. It has been found that for a turbulent state that consists only of a collection of helicons [$B(\mathbf{k}) = \pm kA(\mathbf{k})$] the turbulent diffusivity vanishes. The question of whistlerization of the turbulent spectrum has been investigated numerically in Ref. 29, and a tendency towards whistlerization and equipartition of energy between poloidal and axial components of the magnetic field has been observed. In another recent paper³¹ a detailed numerical simulation is presented of the decaying 2D EMHD turbulence in the regime where the spatial exci-

tation scales are longer than the electron skin depth. The study shows that the short-wavelength part of the spectrum tends to be more whistlerized and that the spectrum is quantitatively more whistlerized with increasing strength of the external magnetic field. Also, anisotropy of the spectrum in the presence of a uniform external field along a particular direction is found. These numerical results are at variance with the results of two other papers^{28,32} on 2D simulations without external magnetic field. The study of Ref. 28 on EMHD turbulence is based upon a numerically observed scaling of the power spectrum and categorically rules out the influence of helicon modes on the energy transfer rate. The numerical simulation of decaying EMHD turbulence in Ref. 32 shows a fast evolution towards equipartition of energy when the system is initialized in such a state that the energy is concentrated in the poloidal components of the magnetic field, but no equipartition has actually been observed. No such evolution is found in the opposite case, when initially the energy is concentrated in the axial component of the magnetic field: Rather a slow increase of the ratio of energy in the axial component to that in the poloidal components has been found. Thus the question whether helicons play a role in the evolution of EMHD turbulence remains open and requires further comprehensive studies.

2. The case $kd_e \gg 1$

Here, we consider a generalization of the equations considered in the previous subsection to the case of smaller-scale EMHD turbulence where the effects of electron inertia are important. Taking into account that $\epsilon \ll 1$, expanding the expressions under the integrals in Eqs. (25) and (26) into a power series over k/q , and transforming the resulting equations back to real space, we arrive at the following set of mean-field equations

$$(1 - d_e^2 \nabla^2) \frac{\partial}{\partial t} \langle A \rangle = \eta \nabla^2 \langle A \rangle - \mu_1 \nabla^4 \langle A \rangle - \alpha_1 (1 - d_e^2 \nabla^2) \nabla^2 \langle B \rangle, \quad (32)$$

$$(1 - d_e^2 \nabla^2) \frac{\partial}{\partial t} \langle B \rangle = \alpha_2 \nabla^4 \langle A \rangle - \mu_2 \nabla^4 \langle B \rangle. \quad (33)$$

The turbulent coefficients are given by the first one of Eqs. (30) and by

$$\begin{aligned} \mu_1 &= \frac{\pi \tau}{2} \int_0^\infty dk k^3 \left[d_e^2 I^B(k) + \left(1 + \frac{k^2 d_e^2}{(1 + k^2 d_e^2)^2} \right) I^A(k) \right], \\ \mu_2 &= \frac{\pi \tau}{2} \int_0^\infty dk \frac{k^3}{(1 + k^2 d_e^2)^2} \\ &\quad \times [(1 + k^2 d_e^2) I^A(k) - k^2 d_e^4 I^B(k)], \\ \alpha_1 &= \frac{\pi \tau}{2} \int_0^\infty dk k^3 I^h(k), \\ \alpha_2 &= \frac{\pi \tau}{2} \int_0^\infty dk \frac{k^3 (1 - k^2 d_e^2)}{(1 + k^2 d_e^2)^2} I^h(k). \end{aligned} \quad (34)$$

Electron inertia does not change the resistivity η , which arises from the anisotropy of the turbulence. On the other hand, turbulent viscosity and helicity effect become different in the equations for the evolution of $\langle A \rangle$ and of $\langle B \rangle$. The turbulent viscosity effect on the poloidal component of the magnetic field Eq. (32) is increased due to electron inertia. Inertia can qualitatively modify the turbulent viscosity coefficient μ_2 , which can become negative when the turbulence spectrum is anisotropic and the axial component of the turbulent magnetic field prevails over its poloidal components. This condition is opposite to the one for negative resistivity. The influence of inertia is most clearly seen in the evolution of the axial component of large-scale magnetic field $\langle B \rangle$. If the spectral correlator $I^h(k)$ has a peak at $k > d_e^{-1}$, electron inertia results in a decrease and even a change of sign of the helicity effect α_2 in Eq. (34). In the case when the turbulence is concentrated at $k \gg d_e^{-1}$ its effect on the poloidal components of the mean field is much stronger than on the axial component $\alpha_1 \approx -(k^2 d_e^2) \alpha_2 \gg |\alpha_2|$ and $\mu_1 \approx (k^2 d_e^2) |\mu_2| \gg |\mu_2|$.

E. Instabilities of the mean-field perturbations

The turbulence leads to unstable mean-field perturbations. We consider mean-field perturbations of the form

$$(\langle A \rangle, \langle B \rangle) = (A^0, B^0) \exp(i \mathbf{q} \mathbf{x} - i \omega t), \quad (35)$$

where ω and $\mathbf{q}$ are the frequency and the wave number of the perturbation, and discuss the two limiting cases $kd_e \rightarrow 0$ and $kd_e \gg 1$.

1. The case $kd_e \ll 1$

The instabilities described by Eqs. (28) and (29) have been analyzed in Ref. 30. The dispersion relation follows from the above set of equations

$$(\gamma + \eta q^2 + \mu q^4)(\gamma + \mu q^4) - |\alpha|^2 q^6 = 0, \quad \gamma = -i\omega. \quad (36)$$

In the limiting case of essentially anisotropic turbulence with negligible helicity, $\alpha^2 \ll |\eta| \mu$, Eqs. (28) and (29) decouple, and the poloidal components of the large-scale magnetic field are unstable if turbulent resistivity is negative $\eta < 0$. The most unstable mode is characterized by the growth rate and wave number:

$$\gamma_{\max} = \frac{\eta^2}{4\mu}, \quad q_{\max} = \left(\frac{|\eta|}{2\mu} \right)^{1/2}. \quad (37)$$

The assumption of separation of length and time scales requires that the wavelength of the most unstable mode should be large compared to the scale length of the turbulence, i.e., $q_{\max} l \ll 1$, and its growth rate should be relatively slow, $\gamma_{\max} \tau \ll 1$. Eliminating η and expressing $\gamma_{\max}$ in terms of $q_{\max}$, this condition takes the form

$$\gamma_{\max} \tau = \mu q_{\max}^4 \tau \ll 1. \quad (38)$$

According to Eq. (30) $\mu \sim \tau \langle (\mathbf{B}'_\perp)^2 \rangle$. On the basis of the inequality (17), which justifies the application of the quasilinear approximation, we have $\mu \ll l^4 / \tau$. The substitution of this estimate into Eq. (38) shows that the inequality is satisfied automatically, i.e., any large-scale perturbation is slow

compared to the correlation time of turbulence. This justifies the approximation that the turbulence is δ -correlated in time.

In the opposite limit of helical, but isotropic turbulence, $\alpha^2 \gg |\eta|\mu$, helicity results in unstable large-scale perturbations. The most unstable mode is characterized by

$$\gamma_{\max} = \frac{\alpha}{4} \left(\frac{3|\alpha|}{4\mu} \right)^3, \quad q_{\max} = \frac{3|\alpha|}{4\mu}. \quad (39)$$

Again, like in the previous case, one finds that large-scale perturbations $q_{\max} l \ll 1$ are automatically slow with respect to the correlation time of the turbulence.

2. The case $kd_e \gg 1$

The dispersion relation for large-scale perturbations follows from Eqs. (32) and (33)

$$\bar{\gamma}^2 + [\eta q^2 + (\mu_1 + \mu_2)q^4] \bar{\gamma} - \alpha_1 \alpha_2 q^6 (1 + q^2 d_e^2) + \mu_2 q^6 (\eta + \mu_1 q^2) = 0, \quad (40)$$

where $\bar{\gamma} = -i\omega(1 + q^2 d_e^2)$. Its roots are given by

$$\bar{\gamma} = -\frac{1}{2} [\eta q^2 + (\mu_1 + \mu_2)q^4] \pm \left\{ \frac{1}{4} [\eta q^2 + (\mu_1 - \mu_2)q^4]^2 + \alpha_1 \alpha_2 q^6 (1 + q^2 d_e^2) \right\}^{1/2}. \quad (41)$$

In the limit of essentially anisotropic, but nonhelical ($I^h = 0$) turbulence, Eqs. (32) and (33) decouple, and the poloidal components of the large-scale magnetic field are unstable if the turbulent resistivity is negative $\eta < 0$. The axial component of the large-scale magnetic field either grows due to negative turbulent viscosity ($\mu_2 < 0$) or damps if turbulent viscosity is positive ($\mu_2 > 0$). However, the growth (damping) rate of such an instability is small,

$$\gamma = -\mu_2 q^4. \quad (42)$$

One of the components of the large-scale magnetic field will grow and the another one will be damped. In the opposite limit of helical, but isotropic ($\eta = 0$) turbulence, helicity results in the instability of the mean field only if

$$\alpha_1 \alpha_2 > 0. \quad (43)$$

In particular, if the cross-correlator responsible for helicity $I^h(k)$ is sufficiently smooth and helicity is concentrated at $kd_e \gg 1$, there is no instability due to helicity; helicity results in oscillating modes $\text{Re } \omega \neq 0$, which are damped due to turbulent viscosity. Again it can easily be checked by analogy with the case $l > d_e$ that the large-scale perturbation is automatically slow compared to the correlation time of turbulence and the approximation of δ -correlated turbulence is justified (see Ref. 33).

IV. GENERATION OF LARGE-SCALE HELICONS BY SMALL-SCALE HELICON TURBULENCE

In this section we will generalize our discussion to the 3D case. Let us assume that plasma is imbedded in a stationary and homogeneous background magnetic field $\mathbf{B}_0 = \mathbf{e}_z$. Then the equation for the self-consistent magnetic field of EMHD motions takes the form

$$\frac{\partial}{\partial t} (\mathbf{B} - d_e^2 \nabla^2 \mathbf{B}) + \frac{\partial}{\partial z} \text{curl } \mathbf{B} = \text{curl} [(\mathbf{B} - d_e^2 \nabla^2 \mathbf{B}) \times \text{curl } \mathbf{B}], \quad (44)$$

where the magnetic field is normalized to the background field. As has already been mentioned in Sec. II, this equation has solutions in the form of helicons with eigenfrequencies $\omega_{\mathbf{k}}$ described by the normalized dispersion relation

$$\omega_{\mathbf{k}}^2 = \frac{k^2 k_z^2}{(1 + k^2 d_e^2)^2}. \quad (45)$$

The spatio-temporal structure of the helicon is determined by the expression

$$\begin{aligned} \mathbf{B} &= C_{\mathbf{k}} \mathbf{f}_{\mathbf{k}} \exp(i\mathbf{k}\mathbf{x} - i\omega_{\mathbf{k}} t), \\ \mathbf{f}_{\mathbf{k}} &= \mathbf{k} \times (\mathbf{e}_x \times \mathbf{k}) - \frac{i\omega_{\mathbf{k}}(1 + k^2 d_e^2)}{k_z} (\mathbf{e}_x \times \mathbf{k}), \end{aligned} \quad (46)$$

where $C_{\mathbf{k}}$ corresponds to its amplitude and $\mathbf{f}_{\mathbf{k}}$ defines its spatial structure, $\mathbf{e}_x$ is the unit vector along the x axis. Note that, as follows from Eq. (46),

$$\text{curl } \mathbf{B} = \frac{\omega_{\mathbf{k}}}{k_z} (1 + k^2 d_e^2) \mathbf{B}, \quad (47)$$

so that a helicon is a nonlinear solution of the EMHD equations. It belongs to the class of force-free solutions and possesses a nonzero helicity.

We assume that the helicon turbulence has the typical scale-length l and that a large scale helicon with characteristic length $L \gg l$ has spontaneously arisen. We will study the evolution of this larger-scale helicon due to its interaction with the small-scale turbulence following the approach suggested in Ref. 17 for studying the adiabatic interaction of waves with different space and time scales.

We present the self-consistent magnetic field $\mathbf{B}$ in a form similar to Eq. (11):

$$\mathbf{B} = \mathbf{H} + \mathbf{b}, \quad (48)$$

where $\mathbf{H}$ corresponds to the large-scale helicon and depends on slow time and space scales, and $\mathbf{b}$ describes the small-scale helicon turbulence ($\langle \mathbf{b} \rangle = 0$). Due to its interaction with the large-scale helicon, $\mathbf{b}$ depends on both fast and slow time and space variables. The treatment is restricted to the initial stage when $\mathbf{H}$ is still infinitesimal.

Upon substituting Eq. (48) into (44) and averaging over the small scales, assuming that $l = O(d_e)$, we arrive at the following equation:

$$\begin{aligned} \frac{\partial \mathbf{H}}{\partial t} + \frac{\partial}{\partial z} \text{curl } \mathbf{H} &= \langle \text{curl} [(\mathbf{b} - d_e^2 \nabla^2 \mathbf{b}) \times \text{curl } \mathbf{b}] \rangle \\ &= -\text{curl } \nabla \cdot \langle \mathbf{b} \mathbf{b} - d_e^2 (\text{curl } \mathbf{b})(\text{curl } \mathbf{b}) \rangle. \end{aligned} \quad (49)$$

The helicon branch of plasma oscillations is strongly dispersive [see Eq. (45)]. Therefore, it seems to be reasonable that typically the helicon turbulence can be considered as a weak turbulence in the sense that the nonlinear frequency shift (due to wave-wave interaction) is small compared to the helicon frequency. In the range where the spatial scales are

longer than the electron skin depth, isotropic³⁴ and anisotropic^{35,36} spectra of the weak helicon turbulence have been found. Here, in contrast with Refs. 34–36, we actually study stability of the helicon turbulence spectra with respect to large-scale perturbations. We follow a quasi-linear approach and neglect the interaction between the small-scale helicons in the evolution of $\mathbf{b}$ and take into account only their interaction with the large-scale helicon, which results in the modulation of the small-scale helicons on the slow time and space scales. Then, the equation for $\mathbf{b}$ takes the form

$$\frac{\partial}{\partial t}(\mathbf{b} - d_e^2 \nabla^2 \mathbf{b}) + \frac{\partial}{\partial z} \text{curl } \mathbf{b} = \text{curl}\{(\mathbf{b} - d_e^2 \nabla^2 \mathbf{b}) \times \text{curl } \mathbf{H} + \mathbf{H} \times \text{curl } \mathbf{b}\}. \quad (50)$$

It is seen from Eq. (49) that, in order to obtain a closed set of equations describing the interaction of the large-scale helicon with the small-scale helicon turbulence, it is necessary to derive the equation for the correlation function of the turbulence. To that end it is convenient to Fourier transform Eq. (50) and, in the spirit of Ref. 17, obtain an analogue of the wave-kinetic equation, i.e., an equation for the turbulent spectral function.

A. The equation for the spectral function of helicon turbulence

The application of the spatial Fourier transform to Eq. (50) yields

$$(1 + k^2 d_e^2) \frac{\partial \mathbf{b}_\mathbf{k}}{\partial t} - k_z \mathbf{k} \times \mathbf{b}_\mathbf{k} = i \mathbf{k} \times \mathbf{F}_\mathbf{k}, \quad (51)$$

where

$$\mathbf{F}_\mathbf{k} = i \int d\mathbf{q} \{ [1 + (\mathbf{k} - \mathbf{q})^2 d_e^2] \mathbf{b}_{\mathbf{k}-\mathbf{q}} \times (\mathbf{q} \times \mathbf{H}_\mathbf{q}) + \mathbf{H}_\mathbf{q} \times [(\mathbf{k} - \mathbf{q}) \times \mathbf{b}_{\mathbf{k}-\mathbf{q}}] \}. \quad (52)$$

In order to derive the required equation which will allow to describe the correlation properties of the helicon turbulence modulated by the large-scale helicon, we introduce the scalar function $\psi_\mathbf{k}$ defined by

$$\psi_\mathbf{k} = \frac{1}{k_y^2 + k_z^2} \mathbf{f}_\mathbf{k}^* \cdot \mathbf{b}_\mathbf{k}, \quad \mathbf{f}_\mathbf{k}^* \equiv \mathbf{f}_{-\mathbf{k}}. \quad (53)$$

If the interaction between small-scale helicons and the large-scale one is weak and, to leading order, only results in the slow modulation of their amplitudes, we may neglect the RHS of Eq. (51). Then, we arrive at the following expression [compare with Eq. (46)]:

$$\mathbf{b}_\mathbf{k} = \frac{\psi_\mathbf{k}}{2k^2} \mathbf{f}_\mathbf{k}. \quad (54)$$

Substituting this relation into the RHS of Eq. (52), which is justified by the assumption of weak interaction between the helicons of the different space and time scales, and multiplying Eq. (51) by $\mathbf{f}_\mathbf{k}^*/(k_y^2 + k_z^2)$, one finally arrives at an equation that contains only $\psi_\mathbf{k}$ and has a canonical left-hand side

$$\begin{aligned} \frac{\partial \psi_\mathbf{k}}{\partial t} + i \omega_\mathbf{k} \psi_\mathbf{k} = & \frac{\omega_\mathbf{k}}{2k_z(k_y^2 + k_z^2)} \int d\mathbf{q} \frac{1 + (\mathbf{k} - \mathbf{q})^2 d_e^2}{(\mathbf{k} - \mathbf{q})^2} \psi_{\mathbf{k}-\mathbf{q}} \\ & \times (\mathbf{f}_\mathbf{k}^* \times \mathbf{f}_{\mathbf{k}-\mathbf{q}}) \cdot \left(i \mathbf{q} \times \mathbf{H}_\mathbf{q} - \frac{\omega_{\mathbf{k}-\mathbf{q}}}{k_z - q_z} \mathbf{H}_\mathbf{q} \right). \end{aligned} \quad (55)$$

We describe the small-scale turbulence by the spectral function (the Wigner function) $I_\mathbf{k}(\mathbf{x}, t)$ defined by

$$I_\mathbf{k}(\mathbf{x}, t) = \int d\mathbf{q} \langle \psi_\mathbf{k} \psi_{-\mathbf{k}+\mathbf{q}} \rangle \exp(i\mathbf{q}\mathbf{x}). \quad (56)$$

The slow time and space dependence of the spectral function corresponds to the modulation of the amplitudes of the small-scale helicons due to their interaction with the large-scale helicon. Hence, we may take $q \ll k$.

To derive the evolution equation for $I_\mathbf{k}(\mathbf{x}, t)$ we multiply Eq. (55) by $\psi_{-\mathbf{k}+\mathbf{q}}$ and add a similar equation obtained by interchanging $\mathbf{k}$ and $-\mathbf{k} + \mathbf{q}$. Upon averaging the resulting equation over small scales, and applying the operator $\int d\mathbf{q} \exp(i\mathbf{q}\mathbf{x})$, we obtain

$$\begin{aligned} \frac{\partial I_\mathbf{k}(\mathbf{x}, t)}{\partial t} + i \int d\mathbf{q} (\omega_\mathbf{k} + \omega_{-\mathbf{k}+\mathbf{q}}) \langle \psi_\mathbf{k} \psi_{-\mathbf{k}+\mathbf{q}} \rangle \exp(i\mathbf{q}\mathbf{x}) \\ = S_{1\mathbf{k}} + S_{2\mathbf{k}}, \end{aligned} \quad (57)$$

where

$$\begin{aligned} S_{1\mathbf{k}} = i \int d\mathbf{q} d\mathbf{p} \left[\frac{\omega_\mathbf{k}}{2k_z(k_y^2 + k_z^2)} \frac{1 + (\mathbf{k} - \mathbf{p})^2 d_e^2}{(\mathbf{k} - \mathbf{p})^2} \langle \psi_{-\mathbf{k}+\mathbf{q}} \psi_{\mathbf{k}-\mathbf{p}} \rangle \right. \\ \left. \times (\mathbf{f}_\mathbf{k}^* \times \mathbf{f}_{\mathbf{k}-\mathbf{p}}) + \mathbf{k} \rightarrow \mathbf{q} - \mathbf{k} \right] \cdot \mathbf{p} \times \mathbf{H}_\mathbf{p} \exp(i\mathbf{q}\mathbf{x}), \end{aligned} \quad (58)$$

$$\begin{aligned} S_{2\mathbf{k}} = \int d\mathbf{q} d\mathbf{p} \exp(i\mathbf{q}\mathbf{x}) \mathbf{H}_\mathbf{p} \\ \cdot \left[\frac{\omega_\mathbf{k}}{2k_z(k_y^2 + k_z^2)} \frac{\omega_{\mathbf{k}-\mathbf{p}}}{k_z - p_z} \frac{1 + (\mathbf{k} - \mathbf{p})^2 d_e^2}{(\mathbf{k} - \mathbf{p})^2} \right. \\ \left. \times \langle \psi_{-\mathbf{k}+\mathbf{q}} \psi_{\mathbf{k}-\mathbf{p}} \rangle \mathbf{f}_{\mathbf{k}-\mathbf{p}} \times \mathbf{f}_\mathbf{k}^* + \mathbf{k} \rightarrow \mathbf{q} - \mathbf{k} \right]. \end{aligned} \quad (59)$$

According to Eq. (45) the eigenfrequencies of the helicons are real, $\text{Im } \omega_\mathbf{k} = 0$, and therefore, $\omega_{-\mathbf{k}} = -\omega_\mathbf{k}$. Then, taking into account a space-scale separation and expanding $\omega_{-\mathbf{k}+\mathbf{q}}$ on the LHS of Eq. (57) in a series over the small parameter q/k , we obtain

$$\begin{aligned} i \int d\mathbf{q} (\omega_\mathbf{k} + \omega_{-\mathbf{k}+\mathbf{q}}) \langle \psi_\mathbf{k} \psi_{-\mathbf{k}+\mathbf{q}} \rangle \exp(i\mathbf{q}\mathbf{x}) \\ \approx i \int d\mathbf{q} d\mathbf{p} \cdot \frac{\partial \omega_\mathbf{k}}{\partial \mathbf{k}} \langle \psi_\mathbf{k} \psi_{-\mathbf{k}+\mathbf{q}} \rangle \exp(i\mathbf{q}\mathbf{x}) \\ = \frac{\partial \omega_\mathbf{k}}{\partial \mathbf{k}} \cdot \nabla \int d\mathbf{q} \langle \psi_\mathbf{k} \psi_{-\mathbf{k}+\mathbf{q}} \rangle \exp(i\mathbf{q}\mathbf{x}) = \frac{\partial \omega_\mathbf{k}}{\partial \mathbf{k}} \cdot \nabla I_\mathbf{k}(\mathbf{x}, t). \end{aligned} \quad (60)$$

To express the quantities averaged over the ensemble in Eqs. (58) and (59) in terms of the spectral function we use the inverse of Eq. (56)

$$\langle \psi_{\mathbf{k}-\mathbf{p}} \psi_{-\mathbf{k}+\mathbf{q}} \rangle = \frac{1}{(2\pi)^3} \int d\mathbf{x}' I_{\mathbf{k}-\mathbf{p}}(\mathbf{x}', t) \exp(-i(\mathbf{q}-\mathbf{p})\mathbf{x}'). \quad (61)$$

Expanding the integrands in Eqs. (58) and (59) to powers of the small parameters p/k and q/k and restricting the calculation to linear terms, we finally arrive at the following expressions (the details of the calculation are given in Appendix B):

$$S_{1\mathbf{k}} \approx 0, \quad (62)$$

$$S_{2\mathbf{k}} \approx \nabla \left[\frac{\omega_{\mathbf{k}}}{k_z} (\mathbf{k} \cdot \mathbf{H}) \right] \cdot \frac{\partial}{\partial \mathbf{k}} \left[\frac{1+k^2 d_e^2}{k^2} (k_y^2 + k_z^2) I_{\mathbf{k}}(\mathbf{x}, t) \right] \\ \times \frac{k^2}{(1+k^2 d_e^2)(k_y^2 + k_z^2)} - \nabla I_{\mathbf{k}}(\mathbf{x}, t) \cdot \frac{\partial}{\partial \mathbf{k}} \left[\frac{\omega_{\mathbf{k}}}{k_z} (\mathbf{k} \cdot \mathbf{H}) \right]. \quad (63)$$

Substituting Eqs. (60)–(63) into Eq. (57), we finally obtain

$$\frac{\partial N_{\mathbf{k}}}{\partial t} + \frac{\partial}{\partial \mathbf{k}} \left[\omega_{\mathbf{k}} + \frac{\omega_{\mathbf{k}}}{k_z} (\mathbf{k} \cdot \mathbf{H}) \right] \cdot \nabla N_{\mathbf{k}} \\ - \nabla \cdot \left[\omega_{\mathbf{k}} + \frac{\omega_{\mathbf{k}}}{k_z} (\mathbf{k} \cdot \mathbf{H}) \right] \frac{\partial N_{\mathbf{k}}}{\partial \mathbf{k}} = 0, \quad (64)$$

where

$$N_{\mathbf{k}} = \frac{1+k^2 d_e^2}{k^2} (k_y^2 + k_z^2) I_{\mathbf{k}}. \quad (65)$$

Equation (64) is in conservation form which means that $\int N_{\mathbf{k}} d\mathbf{k} d\mathbf{x} = \text{const}$. Therefore, $N_{\mathbf{k}}$ can be interpreted as the generalized action invariant of the system under consideration. It can also be interpreted as an analogue of the distribution function of helicon quasi-particles. At the same time, one can check, with an accuracy up to a multiplicative factor, that $N_{\mathbf{k}}$ is equal to the spectral energy density of the small-scale helicon turbulence $(1+k^2 d_e^2) \langle \mathbf{b}_{\mathbf{k}} \mathbf{b}_{-\mathbf{k}} \rangle$.

B. The mean-field equation

To obtain a closed set of equations it is necessary to express the RHS of Eq. (49) in terms of $N_{\mathbf{k}}$. Since this RHS does not vanish to leading order one may take $\mathbf{b}$ in the form of the set of helicons with slightly modulated amplitudes and substitute

$$\mathbf{b} = \int \frac{d\mathbf{k}}{2k^2} \psi_{\mathbf{k}} \mathbf{f}_{\mathbf{k}} \exp(i\mathbf{k}\mathbf{x}) \quad (66)$$

and

$$\text{curl } \mathbf{b} = \int \frac{d\mathbf{k}}{2k^2} \frac{\omega_{\mathbf{k}}}{k_z} (1+k^2 d_e^2) \psi_{\mathbf{k}} \mathbf{f}_{\mathbf{k}} \exp(i\mathbf{k}\mathbf{x}). \quad (67)$$

Then, one obtains (the details of the calculation are given in Appendix C)

$$\langle b_l b_m - d_e^2 (\text{curl } \mathbf{b})_l (\text{curl } \mathbf{b})_m \rangle = \int d\mathbf{k} \frac{I_{\mathbf{k}}}{4k^4} f_{l\mathbf{k}} f_{m\mathbf{k}}^* (1 - k^2 d_e^2). \quad (68)$$

The product $f_{l\mathbf{k}} f_{m\mathbf{k}}^*$ is calculated from definition (46) and the Chandrasekhar identities³⁷

$$(\epsilon_{jpl} k_i - \epsilon_{ipl} k_j) e_p k_l \equiv k^2 \epsilon_{ijr} e_r - (\mathbf{e} \cdot \mathbf{k}) \epsilon_{ijr} k_r, \quad (69)$$

$$(\epsilon_{jpl} e_i - \epsilon_{ipl} e_j) e_p k_l \equiv (\mathbf{e} \cdot \mathbf{k}) \epsilon_{ijr} e_r - \epsilon_{ijr} k_r, \quad (70)$$

where $\mathbf{e}$ is an arbitrary vector. As a result, we arrive, after rather lengthy but straightforward calculations, at

$$f_{l\mathbf{k}} f_{m\mathbf{k}}^* = (k_y^2 + k_z^2) \left[k^2 \left(\delta_{lm} - \frac{k_l k_m}{k^2} \right) - i \frac{\omega_{\mathbf{k}}}{k_z} (1 + k^2 d_e^2) \epsilon_{lmr} k_r \right]. \quad (71)$$

The first term here is mirror-symmetric and corresponds to the isotropic part of the correlation tensor of turbulence, while the second is related to the helicity of turbulence. Finally, the substitution of Eqs. (68) and (71) into Eq. (49) yields the mean-field equation

$$\frac{\partial \mathbf{H}}{\partial t} + \frac{\partial}{\partial z} \text{curl } \mathbf{H} = \text{curl} \int d\mathbf{k} \frac{1 - k^2 d_e^2}{4k^2 (1 + k^2 d_e^2)} \\ \times \left[\mathbf{k} (\mathbf{k} \cdot \nabla N_{\mathbf{k}}) + i \frac{\omega_{\mathbf{k}}}{k_z} (1 + k^2 d_e^2) \nabla N_{\mathbf{k}} \times \mathbf{k} \right]. \quad (72)$$

C. Instability of large-scale helicon

The evolution of the large-scale helicon due to its interaction with the small-scale helicon turbulence is described by Eqs. (64) and (72). We represent $N_{\mathbf{k}} = N_{\mathbf{k}}^0 + \delta N_{\mathbf{k}}$, where $N_{\mathbf{k}}^0$ is the equilibrium part and $\delta N_{\mathbf{k}}$ its perturbation by the large-scale helicon, and assume that the perturbations are proportional to $\exp(-i\Omega t + i\mathbf{q} \cdot \mathbf{x})$. Then from Eq. (64) we find the correction to the generalized action invariant due to the small-amplitude long-wavelength perturbations:

$$\delta N_{\mathbf{k}} = - \frac{\omega_{\mathbf{k}}}{k_z} (\mathbf{k} \cdot \mathbf{H}) \frac{\mathbf{q} \cdot \partial N_{\mathbf{k}}^0 / \partial \mathbf{k}}{\Omega - \mathbf{q} \cdot \partial \omega_{\mathbf{k}} / \partial \mathbf{k}}. \quad (73)$$

Substituting this result into Eq. (72), we obtain

$$-i\Omega \mathbf{H}_{\mathbf{q}} - q_z \mathbf{q} \times \mathbf{H}_{\mathbf{q}} \\ = \mathbf{q} \times \int d\mathbf{k} \frac{1 - k^2 d_e^2}{4k^2 (1 + k^2 d_e^2)} \frac{\omega_{\mathbf{k}}}{k_z} (\mathbf{k} \cdot \mathbf{H}_{\mathbf{q}}) \frac{\mathbf{q} \cdot \partial N_{\mathbf{k}}^0 / \partial \mathbf{k}}{\Omega - \mathbf{q} \cdot \partial \omega_{\mathbf{k}} / \partial \mathbf{k}} \\ \times \left[\mathbf{k} (\mathbf{k} \cdot \mathbf{q}) + i \frac{\omega_{\mathbf{k}}}{k_z} (1 + k^2 d_e^2) \mathbf{q} \times \mathbf{k} \right]. \quad (74)$$

This equation completely defines the dispersion relation. Due to its rather complicated vector form it is not easy to carry out a stability analysis. We restrict ourselves to two cases.

First, we assume that the leading order correction to the frequency of the large-scale helicon, due to the interaction with the turbulence, is small because of the small amplitude of helicon turbulence. Then, the solution of Eq. (74) can be represented in the form

$$\mathbf{H}_q = \mathbf{H}_q^{(0)} + \mathbf{H}_q^{(1)} + \dots, \quad \Omega = \Omega^{(0)} + \delta\Omega^{(1)} + \dots \quad (75)$$

To zero order we have

$$-i\Omega\mathbf{H}_q^{(0)} - q_z\mathbf{q} \times \mathbf{H}_q^{(0)} = 0. \quad (76)$$

This equation describes the large-scale helicon ($q^2 d_e^2 \ll 1$), whose eigenfrequency and eigenfunction are

$$\Omega^{(0)} = \omega_q, \quad \omega_q^2 = q^2 q_z^2, \quad \mathbf{H}_q^{(0)} = A_0 \mathbf{f}_q, \quad (77)$$

where $\mathbf{f}_q$ is defined by Eq. (46) in the limit $q d_e \rightarrow 0$. To next order we have

$$\begin{aligned} & -i\omega_q \mathbf{H}_q^{(1)} - q_z \mathbf{q} \times \mathbf{H}_q^{(1)} - i\delta\Omega^{(1)} \mathbf{H}_q^{(0)} \\ & = \mathbf{q} \times \int d\mathbf{k} \frac{1 - k^2 d_e^2}{4k^2(1 + k^2 d_e^2)} \frac{\omega_k}{k_z} (\mathbf{k} \cdot \mathbf{H}_q^{(0)}) \frac{\mathbf{q} \cdot \partial N_k^0 / \partial \mathbf{k}}{\omega_q - \mathbf{q} \cdot \partial \omega_k / \partial \mathbf{k}} \\ & \quad \times \left[\mathbf{k}(\mathbf{k} \cdot \mathbf{q}) + i \frac{\omega_k}{k_z} (1 + k^2 d_e^2) \mathbf{q} \times \mathbf{k} \right]. \end{aligned} \quad (78)$$

Since the Fourier component $\mathbf{H}_q^{(1)}$ is perpendicular to $\mathbf{q}$, it can always be expanded over the vectors $\mathbf{e}_x \times \mathbf{q}$ and $\mathbf{q} \times (\mathbf{e}_x \times \mathbf{q})$, i.e., it can be represented in the form

$$\mathbf{H}_q^{(1)} = A_1^{(+)} \mathbf{f}_q + A_1^{(-)} \mathbf{f}_q^*. \quad (79)$$

Then, substituting Eqs. (77) and (79) into Eq. (78) we arrive at the equation

$$\begin{aligned} & -2i\omega_q A_1^{(-)} \mathbf{f}_q^* - i\delta\Omega^{(1)} A_0 \mathbf{f}_q \\ & = A_0 \mathbf{q} \times \int d\mathbf{k} \frac{1 - k^2 d_e^2}{4k^2(1 + k^2 d_e^2)} \frac{\omega_k}{k_z} (\mathbf{k} \cdot \mathbf{f}_q) \frac{\mathbf{q} \cdot \partial N_k^0 / \partial \mathbf{k}}{\omega_q - \mathbf{q} \cdot \partial \omega_k / \partial \mathbf{k}} \\ & \quad \times \left[\mathbf{k}(\mathbf{k} \cdot \mathbf{q}) + i \frac{\omega_k}{k_z} (1 + k^2 d_e^2) \mathbf{q} \times \mathbf{k} \right]. \end{aligned} \quad (80)$$

Use the identities $\mathbf{f}_q \cdot \mathbf{f}_q = \mathbf{f}_q^* \cdot \mathbf{f}_q^* = 0$ to eliminate $A_1^{(-)}$. Scalar multiplication of Eq. (80) with $\mathbf{f}_q^*$, and using the identity $\mathbf{q} \times \mathbf{f}_q^* = i(\omega_q / q_z) \mathbf{f}_q^*$, yields the dispersion relation

$$\begin{aligned} \delta\Omega^{(1)} & = \frac{1}{\mathbf{f}_q \cdot \mathbf{f}_q^*} \int d\mathbf{k} \frac{1 - k^2 d_e^2}{4k^2(1 + k^2 d_e^2)} (\mathbf{k} \cdot \mathbf{f}_q) (\mathbf{k} \cdot \mathbf{f}_q^*) \\ & \quad \times \frac{\mathbf{q} \cdot \partial N_k^0 / \partial \mathbf{k}}{\omega_q - \mathbf{q} \cdot \partial \omega_k / \partial \mathbf{k}} \left[(\mathbf{k} \cdot \mathbf{q}) \frac{\omega_k}{k_z} \frac{\omega_q}{q_z} + \frac{k^2 q^2}{1 + k^2 d_e^2} \right]. \end{aligned} \quad (81)$$

When the spectral width of the helicon turbulence $\Delta\omega_k$ exceeds the nonlinear growth rate (decrement), the resonant instability can take place. For such a resonant instability the phase velocity of the long-wavelength helicon approaches the group velocity of the package of small-scale helicons so that

$$\frac{1}{\omega_q - \mathbf{q} \cdot \partial \omega_k / \partial \mathbf{k}} = \mathcal{P} \frac{1}{\omega_q - \mathbf{q} \cdot \partial \omega_k / \partial \mathbf{k}} - i\pi \delta \left(\omega_q - \mathbf{q} \cdot \frac{\partial \omega_k}{\partial \mathbf{k}} \right). \quad (82)$$

The growth rate of such an instability follows from Eqs. (81) and (82)

$$\begin{aligned} \Gamma_q & \equiv \text{Im } \delta\Omega^{(1)} \\ & = -\pi \frac{1}{\mathbf{f}_q \cdot \mathbf{f}_q^*} \int d\mathbf{k} \frac{1 - k^2 d_e^2}{4k^2(1 + k^2 d_e^2)} (\mathbf{k} \cdot \mathbf{f}_q) (\mathbf{k} \cdot \mathbf{f}_q^*) \left(\mathbf{q} \cdot \frac{\partial N_k^0}{\partial \mathbf{k}} \right) \\ & \quad \times \left[(\mathbf{k} \cdot \mathbf{q}) \frac{\omega_k}{k_z} \frac{\omega_q}{q_z} + \frac{k^2 q^2}{1 + k^2 d_e^2} \right] \delta \left(\omega_q - \mathbf{q} \cdot \frac{\partial \omega_k}{\partial \mathbf{k}} \right). \end{aligned} \quad (83)$$

Since $\mathbf{f}_q \cdot \mathbf{f}_q^* > 0$ and $(\mathbf{k} \cdot \mathbf{f}_q)(\mathbf{k} \cdot \mathbf{f}_q^*) \geq 0$, and because also the term within the curly brackets is also non-negative on the basis of the helicon dispersion relations [see Eqs. (46) and (77)], we find that the instability condition is

$$(1 - k^2 d_e^2) \left(\mathbf{q} \cdot \frac{\partial N_k^0}{\partial \mathbf{k}} \right) \bigg|_{\omega_q = \mathbf{q} \cdot \partial \omega_k / \partial \mathbf{k}} < 0. \quad (84)$$

The mechanism of this instability is similar to Landau damping on particles: The long-wavelength helicon is driven by its resonant interaction with quasi-particles characterized by velocity $\partial \omega_k / \partial \mathbf{k}$ and a distribution function N_k .

The second problem we consider is similar to the one studied in the framework of the $2\frac{1}{2}$ D model. We assume that the correlation properties of the helicon turbulence are controlled by an external source, and replace the resonant denominator $\Omega - \mathbf{q} \cdot \partial \omega_k / \partial \mathbf{k}$ on the RHS of Eq. (73) by its broadened counterpart $\Omega - \mathbf{q} \cdot \partial \omega_k / \partial \mathbf{k} + i\Delta\omega_k$, where $\Delta\omega_k$ is nonlinear broadening. For a white noise source, which in fact corresponds to the formulation of the problem in Ref. 9 and in Sec. III, we should simply replace $\Omega - \mathbf{q} \cdot \partial \omega_k / \partial \mathbf{k}$ by $i\Delta\omega_k$ on the RHS of Eq. (73). Then, instead of Eq. (73) we have

$$\delta N_k = \frac{1}{\Delta\omega_k} \frac{\omega_k}{k_z} \nabla \cdot \left[(\mathbf{k} \cdot \mathbf{H}) \frac{\partial N_k^0}{\partial \mathbf{k}} \right]. \quad (85)$$

After substitution of Eq. (85) into Eq. (72) it follows

$$\begin{aligned} \frac{\partial \mathbf{H}}{\partial t} + \frac{\partial}{\partial z} \text{curl } \mathbf{H} & = \text{curl} \int d\mathbf{k} \frac{1 - k^2 d_e^2}{4k^2(1 + k^2 d_e^2)} \frac{\omega_k}{k_z} \frac{1}{\Delta\omega_k} \\ & \quad \times \left\{ \mathbf{k}(\mathbf{k} \cdot \nabla) \left[\nabla(\mathbf{k} \cdot \mathbf{H}) \cdot \frac{\partial N_k^0}{\partial \mathbf{k}} \right] \right. \\ & \quad \left. + i \frac{\omega_k}{k_z} (1 + k^2 d_e^2) \nabla \left[\nabla(\mathbf{k} \cdot \mathbf{H}) \cdot \frac{\partial N_k^0}{\partial \mathbf{k}} \right] \times \mathbf{k} \right\}. \end{aligned} \quad (86)$$

Assuming that both N_k and $\Delta\omega_k$ are isotropic and depend only on k , so that $\partial N_k / \partial \mathbf{k} = (\mathbf{k}/k) \partial N_k / \partial k$, and taking into account that ω_k / k_z also depends only on k , according to the dispersion relation (46), we can integrate the RHS of Eq. (86) over the angle in $\mathbf{k}$ -space. The second term inside the curly brackets is odd with respect to $\mathbf{k}$ and vanishes when averaged over the angle. As a result we arrive at the equation

$$\frac{\partial \mathbf{H}}{\partial t} + \frac{\partial}{\partial z} \text{curl } \mathbf{H} = \alpha \nabla^2 \text{curl } \mathbf{H}, \quad (87)$$

where α is defined by

$$\alpha = \frac{\pi}{15} \int_0^\infty dk \frac{k^3(1-k^2 d_e^2)}{1+k^2 d_e^2} \frac{\omega_{\mathbf{k}}}{k_z} \frac{1}{\Delta \omega_{\mathbf{k}}} \frac{\partial N_{\mathbf{k}}}{\partial k} \quad (88)$$

and

$$\begin{aligned} & \int f(k) k_i k_j k_l k_m d\mathbf{k} \\ &= \frac{4\pi}{15} \int_0^\infty dk k^6 f(k) (\delta_{ij} \delta_{lm} + \delta_{il} \delta_{jm} + \delta_{im} \delta_{jl}), \end{aligned} \quad (89)$$

has been used. Note that α differs from zero only if there is an additional asymmetry between helicons with positive and negative phase velocities $\omega_{\mathbf{k}}/k_z$ and possessing the positive and negative values of helicity, correspondingly, in the sense, that they have different values of $(1/\Delta \omega_{\mathbf{k}}) \partial N_{\mathbf{k}} / \partial k$ (e.g., due to the properties of the turbulence source). A similar conclusion but in a different form has been drawn earlier in Ref. 9. The dependence of α on $k d_e$, namely, a change of its sign when k crosses the value $1/d_e$ is similar to the result found in Ref. 38, where in the framework of the dissipative EMHD stability of the helicon maintained by the external force with respect to the large-scale perturbations has been investigated. Equation (87) has a form which is similar to Eq. (2), except for the turbulent viscosity term in the latter. The dispersion relation for the large-scale helicon follows from Eq. (87)

$$\Omega^2 = q^2 (q_z - i \alpha q^2)^2, \quad (90)$$

which means that independently of the sign of α , one of two helicons with the opposite signs of the phase velocity along z axis is unstable. When the coefficient α becomes small (of order l/L) due to a partial compensation related to the above mentioned effect, the turbulent viscosity, which is the next order term in expansion over a small parameter l/L , starts to play a significant role and can result in damping of the large-scale helicon.

V. CONCLUSION

In this paper we have considered the problem of the generation of large-scale perturbations by small-scale turbulence within the framework of electron magnetohydrodynamics. We have investigated two EMHD models: A $2\frac{1}{2}$ D model in which the magnetic field has all three spatial components, but depends only on two coordinates, and a fully 3D model with an imposed stationary and homogeneous magnetic field.

It is shown that $2\frac{1}{2}$ D EMHD turbulence has properties that usually are considered to be pertinent to both 2D and 3D turbulence. On the one hand, similarly to 3D EMHD turbulence,⁹ the $2\frac{1}{2}$ D turbulence with nonzero helicity (actually cross-helicity) results into an α -effect, like in MHD. On the other hand, if the spectrum of the turbulence is anisotropic, then $2\frac{1}{2}$ D turbulence results into dissipation (resistivity, viscosity) of the mean field, like purely 2D turbulence,

which can be *negative*. Both mechanisms, the α -effect and negative dissipation lead to spontaneous amplification of the mean field.

We have shown that EMHD turbulence with scale length $l > c/\omega_{pe}$ leads to two mechanisms that can destabilize mean fields, the α -like effect and turbulent resistivity. Turbulent viscosity is always positive in this case. The turbulent resistivity is different from zero only in the absence of equipartition between the energies in the axial and poloidal components of the turbulent magnetic field. The effect of negative resistivity is largest in the 2D case where the turbulent magnetic field is purely poloidal. The characteristic scale lengths and growth rates of the instabilities caused by both negative resistivity and the α effect have been analyzed. These quantities are completely defined by the correlation properties of the turbulence.

The influence of inertia on the generation of mean fields is also analyzed in the $2\frac{1}{2}$ D model by considering turbulence with scale length $l \leq c/\omega_{pe}$. It is shown that the coefficient of resistivity does not depend on electron inertia. At the same time, the effect of inertia results in an asymmetry of the α -effect and of viscosity in the equations describing the evolution of the axial and poloidal components of the large-scale magnetic field. The α -effect in the equation for the axial component of the perturbed magnetic field decreases and even a change of sign may occur.

The influence of inertia on viscosity is particularly important in the equation for the axial component of the large-scale magnetic field. The viscosity coefficient becomes negative when the axial component of the turbulent magnetic field prevails over its poloidal components. This condition is opposite to the one for resistivity to takes its maximal negative value, i.e., in the case of purely 2D EMHD turbulence.

We have studied the modification of the stability conditions of the mean field perturbations compared to the case $l > c/\omega_{pe}$. The most important difference is the absence of instability due to the helicity of the turbulent field when the spectral density of helicity $I^h(k)$ is a monotonous function and concentrated in the scale lengths below d_e .

A significant difference of 3D EMHD model with a background magnetic field from $2\frac{1}{2}$ D model is the existence of waves with nonzero frequency called helicons (or whistlers). If the background magnetic field is sufficiently strong, the helicon turbulence can be described by the wave-kinetic equation within the standard weak turbulence approach. We have studied the interaction of the small-scale helicon turbulence with the large-scale helicon based upon the adiabatic approximation technique proposed in Ref. 17. A closed set of equations for the evolution of both the magnetic field of the large-scale helicon and the generalized action invariant of the small-scale turbulence is obtained [see Eqs. (64) and (72)]. We have used this set of equations to study the linear stability of the large-scale helicon. Two cases have been considered. In the case when the spectral width of the helicon turbulence $\Delta \omega_{\mathbf{k}}$ exceeds the nonlinear growth rate (decrement) the resonant instability takes place. The mechanism of this instability is similar to Landau damping on particles. When condition Eq. (84) is fulfilled the long-

wavelength helicon is driven by its resonant interaction with quasi-particles, that correspond to the small-scale helicons. In the other case, we have considered the problem which is similar to that studied within the framework of the $2\frac{1}{2}D$ model. Assuming that the correlation properties of the helicon turbulence are controlled by the external source, we have replaced the resonant denominator in the expression for the perturbation of the generalized action by its broadened counterpart, and in the case of an isotropic white noise source, we have derived the equation for the large-scale helicon evolution similar to Ref. 9, in which the small-scale turbulence has resulted in the α -like effect. We have found that in order that the coefficient α does not vanish, the additional condition of asymmetry between the helicons with positive and negative values of the phase velocity along the background field is required.

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APPENDIX A: AVERAGING THE POISSON BRACKETS

To derive the averages of the Poisson brackets we use Eq. (18) and take into account the correlation properties (23). For example,

$$\begin{aligned}
 \langle [A^t, \nabla^2 \delta A] + [\delta A, \nabla^2 A^t] \rangle &= \int \int d\mathbf{k}' d\mathbf{k} (\mathbf{k} \times \mathbf{k}')_z (k'^2 - k^2) \langle A_{\mathbf{k}}^t \delta A_{\mathbf{k}'} \rangle e^{i(\mathbf{k}' + \mathbf{k})\mathbf{x}} \\
 &= \int \int \int d\mathbf{k}' d\mathbf{k} d\mathbf{q} \frac{(\mathbf{k}' \times \mathbf{q})_z}{1 + k'^2 d_e^2} (\mathbf{k} \times \mathbf{k}')_z (k'^2 - k^2) e^{i(\mathbf{k}' + \mathbf{k})\mathbf{x}} \\
 &\quad \times \left[(1 + q^2 d_e^2) \langle A \rangle_{\mathbf{q}} \int_{-\infty}^t \langle A_{\mathbf{k}}^t(t) B_{\mathbf{k}' - \mathbf{q}}^t(t') \rangle dt' - (1 + (\mathbf{k}' - \mathbf{q})^2 d_e^2) \langle B \rangle_{\mathbf{q}} \int_{-\infty}^t \langle A_{\mathbf{k}'}^t(t) A_{\mathbf{k} - \mathbf{q}}^t(t') \rangle dt' \right] \\
 &= \int \int \int d\mathbf{k}' d\mathbf{k} d\mathbf{q} \frac{\tau (\mathbf{k}' \times \mathbf{q})_z}{2} \frac{(\mathbf{k} \times \mathbf{k}')_z}{1 + k^2 d_e^2} (k'^2 - k^2) e^{i(\mathbf{k}' + \mathbf{k})\mathbf{x}} \delta(\mathbf{k} + \mathbf{k}' - \mathbf{q}) \\
 &\quad \times [(1 + q^2 d_e^2) I^h(k) \langle A \rangle_{\mathbf{q}} - (1 + (\mathbf{k} - \mathbf{q})^2 d_e^2) I^A(k') \langle B \rangle_{\mathbf{q}}] \\
 &= \int \int d\mathbf{k} d\mathbf{q} \frac{\tau (\mathbf{k} \times \mathbf{q})_z^2 (2\mathbf{k}\mathbf{q} - q^2)}{1 + (\mathbf{k} - \mathbf{q})^2 d_e^2} e^{i\mathbf{q}\mathbf{x}} [(1 + q^2 d_e^2) I^h(k) \langle A \rangle_{\mathbf{q}} - (1 + k^2 d_e^2) I^A(k) \langle B \rangle_{\mathbf{q}}]. \quad (A1)
 \end{aligned}$$

The averages of the other Poisson brackets are calculated in a similar way,

$$\begin{aligned}
 \langle [B^t, \nabla^2 \delta B] + [\delta B, \nabla^2 B^t] \rangle \\
 = \int \int d\mathbf{k} d\mathbf{q} \frac{\tau (\mathbf{k} \times \mathbf{q})_z^2 (2\mathbf{k}\mathbf{q} - q^2)(k^2 - q^2)}{1 + (\mathbf{k} - \mathbf{q})^2 d_e^2} \\
 \times \{ I^h(k) \langle A \rangle_{\mathbf{q}} - d_e^2 I^B(k) \langle B \rangle_{\mathbf{q}} \} e^{i\mathbf{q}\mathbf{x}}, \quad (A2)
 \end{aligned}$$

$$\begin{aligned}
 \langle [B^t, \delta A - d_e^2 \nabla^2 \delta A] + [\delta B, A^t - d_e^2 \nabla^2 A^t] \rangle \\
 = \int \int d\mathbf{k} d\mathbf{q} \frac{\tau}{2} (\mathbf{k} \times \mathbf{q})_z^2 \left[(1 + q^2 d_e^2) I^B(k) \right. \\
 \left. - \frac{(1 + k^2 d_e^2)(k^2 - q^2)}{1 + (\mathbf{k} - \mathbf{q})^2 d_e^2} I^A(k) \right] \langle A \rangle_{\mathbf{q}} \\
 \left. - \frac{1 + 2q^2 d_e^2 - 2\mathbf{k}\mathbf{q} d_e^2}{1 + (\mathbf{k} - \mathbf{q})^2 d_e^2} (1 + k^2 d_e^2) I^h(k) \langle B \rangle_{\mathbf{q}} \right\} e^{i\mathbf{q}\mathbf{x}}. \quad (A3)
 \end{aligned}$$

The integration over the angle in $\mathbf{k}$ -space can be performed using the formula:³⁹

$$\int_0^\pi \frac{\cos n\theta d\theta}{1 + a \cos \theta} = \frac{\pi}{\sqrt{1 - a^2}} \left(\frac{\sqrt{1 - a^2} - 1}{a} \right)^n, \quad a^2 < 1. \quad (A4)$$

Applying this formula to Eqs. (A1)–(A3) we obtain

$$\begin{aligned}
 \langle [A^t, \nabla^2 \delta A] + [\delta A, \nabla^2 A^t] \rangle \\
 = \int d\mathbf{q} \int_0^\infty d\mathbf{k} \frac{\pi \tau k^3 q^2 dk}{1 + k^2 d_e^2 + q^2 d_e^2} \kappa(\epsilon) e^{i\mathbf{q}\mathbf{x}} [k q \epsilon \kappa(\epsilon) - q^2] \\
 \times [(1 + q^2 d_e^2) I^h(k) \langle A \rangle_{\mathbf{q}} - (1 + k^2 d_e^2) I^A(k) \langle B \rangle_{\mathbf{q}}], \quad (A5)
 \end{aligned}$$

$$\begin{aligned}
& \langle [B^t, \nabla^2 \delta B] + [\delta B, \nabla^2 B^t] \rangle \\
&= \int d\mathbf{q} \int_0^\infty dk \frac{\pi \tau k^3 q^2 (k^2 - q^2)}{1 + k^2 d_e^2 + q^2 d_e^2} \\
&\quad \times \kappa(\epsilon) e^{i\mathbf{q}\mathbf{x}} [kq \epsilon \kappa(\epsilon) - q^2] \\
&\quad \times \{I^h(k) \langle A \rangle_{\mathbf{q}} - d_e^2 I^B(k) \langle B \rangle_{\mathbf{q}}\}, \quad (\text{A6})
\end{aligned}$$

$$\begin{aligned}
& \langle [B^t, \delta A - d_e^2 \nabla^2 \delta A] + [\delta B, A^t - d_e^2 \nabla^2 A^t] \rangle \\
&= \int d\mathbf{q} \int_0^\infty dk \pi \tau k^3 q^2 e^{i\mathbf{q}\mathbf{x}} \left\{ \left[\frac{1}{2} (1 + q^2 d_e^2) I^B(k) \right. \right. \\
&\quad \left. \left. - \frac{(1 + k^2 d_e^2)(k^2 - q^2)}{1 + k^2 d_e^2 + q^2 d_e^2} \kappa(\epsilon) I^A(k) \right] \langle A \rangle_{\mathbf{q}} \right. \\
&\quad \left. - \frac{1 + k^2 d_e^2}{1 + k^2 d_e^2 + q^2 d_e^2} \kappa(\epsilon) \right. \\
&\quad \left. \times [1 + 2q^2 d_e^2 - kq \epsilon \kappa(\epsilon) d_e^2] I^h(k) \langle B \rangle_{\mathbf{q}} \right\}, \quad (\text{A7})
\end{aligned}$$

where ϵ and $\kappa(\epsilon)$ are defined by Eq. (27). Substituting Eq. (A7) in Eq. (13), and Eqs. (A5), (A6) in Eq. (14) and applying the Fourier transformation we arrive at Eqs. (25) and (26) for the Fourier components of the mean fields.

APPENDIX B: CALCULATION OF $S_{1\mathbf{k}}$ AND $S_{2\mathbf{k}}$

Using Eq. (61) we represent Eq. (58) in the form

$$\begin{aligned}
S_{1\mathbf{k}} &= \frac{i}{(2\pi)^3} \int \int \int d\mathbf{x}' d\mathbf{q} d\mathbf{p} \exp[i\mathbf{q}\mathbf{x} - i(\mathbf{q} - \mathbf{p})\mathbf{x}'] \\
&\quad \times \left\{ \frac{\omega_{\mathbf{k}}}{2k_z(k_y^2 + k_z^2)} \frac{1 + (\mathbf{k} - \mathbf{p})^2 d_e^2}{(\mathbf{k} - \mathbf{p})^2} I_{\mathbf{k} - \mathbf{p}}(\mathbf{x}') \mathbf{f}_{\mathbf{k} - \mathbf{p}}^* \times \mathbf{f}_{\mathbf{k} - \mathbf{p}} \right. \\
&\quad \left. + \frac{\omega_{\mathbf{q} - \mathbf{k}}}{2(q_z - k_z)[(q_y - k_y)^2 + (q_z - k_z)^2]} \right. \\
&\quad \left. \times \frac{1 + (\mathbf{q} - \mathbf{k} - \mathbf{p})^2 d_e^2}{(\mathbf{q} - \mathbf{k} - \mathbf{p})^2} I_{\mathbf{k}}(\mathbf{x}') \mathbf{f}_{\mathbf{q} - \mathbf{k}}^* \times \mathbf{f}_{\mathbf{q} - \mathbf{k} - \mathbf{p}} \right\} \cdot (\mathbf{p} \times \mathbf{H}_{\mathbf{p}}). \quad (\text{B1})
\end{aligned}$$

In the calculation of $S_{1\mathbf{k}}$ and $S_{2\mathbf{k}}$ we restrict ourselves to the terms of lowest order with respect to the ratio of the small-scale length l to the large-scale one L , namely, we take into account only the terms of order l/L and neglect terms of higher orders. Therefore, we retain only linear terms with respect to p and q . As a result Eq. (B1) is reduced to

$$\begin{aligned}
S_{1\mathbf{k}} &\approx \frac{i}{(2\pi)^3} \int \int \int d\mathbf{x}' d\mathbf{q} d\mathbf{p} \exp[i\mathbf{q}\mathbf{x} - i(\mathbf{q} - \mathbf{p})\mathbf{x}'] \\
&\quad \times \frac{1 + k^2 d_e^2}{k^2(k_y^2 + k_z^2)} I_{\mathbf{k}}(\mathbf{x}') \left\{ \frac{\omega_{\mathbf{k}}}{2k_z} \mathbf{f}_{\mathbf{k}}^* \times \mathbf{f}_{\mathbf{k}} - \frac{\omega_{-\mathbf{k}}}{2k_z} \mathbf{f}_{-\mathbf{k}}^* \times \mathbf{f}_{-\mathbf{k}} \right\} \\
&\quad \cdot (\mathbf{p} \times \mathbf{H}_{\mathbf{p}}) \\
&= i \int d\mathbf{p} \exp(i\mathbf{p}\mathbf{x}) \frac{\omega_{\mathbf{k}}(1 + k^2 d_e^2)}{2k_z k^2(k_y^2 + k_z^2)} I_{\mathbf{k}}(\mathbf{x}) \\
&\quad \times (\mathbf{f}_{\mathbf{k}}^* \times \mathbf{f}_{\mathbf{k}} + \mathbf{f}_{\mathbf{k}} \times \mathbf{f}_{\mathbf{k}}^*) \cdot (\mathbf{p} \times \mathbf{H}_{\mathbf{p}}) = 0. \quad (\text{B2})
\end{aligned}$$

The calculation of $S_{2\mathbf{k}}$ is a similar but much more complicated task which requires rather lengthy calculations. First, similarly to Eq. (B1) we can rewrite the expression for $S_{2\mathbf{k}}$ in the form

$$\begin{aligned}
S_{2\mathbf{k}} &= \frac{1}{(2\pi)^3} \int \int \int d\mathbf{x}' d\mathbf{q} d\mathbf{p} \exp[i\mathbf{q}\mathbf{x} - i(\mathbf{q} - \mathbf{p})\mathbf{x}'] \\
&\quad \times \left\{ \frac{\omega_{\mathbf{k}}}{2k_z(k_y^2 + k_z^2)} \frac{\omega_{\mathbf{k} - \mathbf{p}}}{k_z - p_z} \frac{1 + (\mathbf{k} - \mathbf{p})^2 d_e^2}{(\mathbf{k} - \mathbf{p})^2} I_{\mathbf{k} - \mathbf{p}}(\mathbf{x}') \right. \\
&\quad \times \mathbf{f}_{\mathbf{k} - \mathbf{p}} \times \mathbf{f}_{\mathbf{k}}^* + \frac{\omega_{\mathbf{q} - \mathbf{k}}}{2(q_z - k_z)[(q_y - k_y)^2 + (q_z - k_z)^2]} \\
&\quad \times \frac{\omega_{\mathbf{q} - \mathbf{k} - \mathbf{p}}}{q_z - k_z - p_z} \frac{1 + (\mathbf{q} - \mathbf{k} - \mathbf{p})^2 d_e^2}{(\mathbf{q} - \mathbf{k} - \mathbf{p})^2} \\
&\quad \left. \times I_{\mathbf{k}}(\mathbf{x}') \mathbf{f}_{\mathbf{q} - \mathbf{k} - \mathbf{p}} \times \mathbf{f}_{\mathbf{q} - \mathbf{k}}^* \right\} \cdot \mathbf{H}_{\mathbf{p}}. \quad (\text{B3})
\end{aligned}$$

We represent $S_{2\mathbf{k}}$ in the form of a series $S_{2\mathbf{k}} \approx S_{2\mathbf{k}}^{(0)} + S_{2\mathbf{k}}^{(1)}$ with respect to a small parameters p/k and q/k . Then, we obtain

$$\begin{aligned}
S_{2\mathbf{k}}^{(0)} &= \frac{1}{(2\pi)^3} \int \int \int d\mathbf{x}' d\mathbf{q} d\mathbf{p} \exp[i\mathbf{q}\mathbf{x} - i(\mathbf{q} - \mathbf{p})\mathbf{x}'] \\
&\quad \times \frac{1 + k^2 d_e^2}{2k_z^2 k^2(k_y^2 + k_z^2)} I_{\mathbf{k}}(\mathbf{x}') \mathbf{H}_{\mathbf{p}} \cdot (\omega_{\mathbf{k}}^2 \mathbf{f}_{\mathbf{k}} \times \mathbf{f}_{\mathbf{k}}^* + \omega_{-\mathbf{k}}^2 \mathbf{f}_{-\mathbf{k}} \times \mathbf{f}_{-\mathbf{k}}^*) \\
&= \int d\mathbf{p} \exp(i\mathbf{p}\mathbf{x}) \frac{\omega_{\mathbf{k}}^2(1 + k^2 d_e^2)}{2k_z^2 k^2(k_y^2 + k_z^2)} I_{\mathbf{k}}(\mathbf{x}) \\
&\quad \times \mathbf{H}_{\mathbf{p}} \cdot (\mathbf{f}_{\mathbf{k}} \times \mathbf{f}_{\mathbf{k}}^* + \mathbf{f}_{\mathbf{k}}^* \times \mathbf{f}_{\mathbf{k}}) = 0. \quad (\text{B4})
\end{aligned}$$

To simplify further calculations it is useful to take into account that the expansion of any arbitrary function $A_{\mathbf{k} - \mathbf{q}}$ can be represented in the form

$$A_{\mathbf{k} - \mathbf{q}} \equiv A_{\mathbf{k} + (\mathbf{p} - \mathbf{q}) - \mathbf{p}} \approx A_{\mathbf{k}} + \left[(\mathbf{p} - \mathbf{q}) \cdot \frac{\partial}{\partial \mathbf{k}} \right] A_{\mathbf{k}} - \left(\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{k}} \right) A_{\mathbf{k}}. \quad (\text{B5})$$

Then $S_{2\mathbf{k}}^{(1)}$ takes the form

$$\begin{aligned}
S_{2\mathbf{k}}^{(1)} = & \frac{1}{(2\pi)^3} \int \int \int d\mathbf{x}' d\mathbf{q} d\mathbf{p} \exp[i\mathbf{q}\mathbf{x} - i(\mathbf{q}-\mathbf{p})\mathbf{x}'] \mathbf{H}_\mathbf{p} \cdot \left\{ -\frac{\omega_{\mathbf{k}}}{2k_z(k_y^2+k_z^2)} \mathbf{f}_{\mathbf{k}} \times \mathbf{f}_{\mathbf{k}}^* \left(\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{k}} \right) \left[\frac{\omega_{\mathbf{k}}(1+k^2 d_e^2)}{k_z k^2} I_{\mathbf{k}}(\mathbf{x}') \right] \right. \\
& - \frac{\omega_{\mathbf{k}}^2(1+k^2 d_e^2)}{2k_z^2 k^2 (k_y^2+k_z^2)} I_{\mathbf{k}}(\mathbf{x}') \left[\left(\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{k}} \right) \mathbf{f}_{\mathbf{k}} \times \mathbf{f}_{\mathbf{k}}^* \right] - \frac{\omega_{\mathbf{k}}(1+k^2 d_e^2)}{k_z k^2} I_{\mathbf{k}}(\mathbf{x}') \mathbf{f}_{\mathbf{k}}^* \times \mathbf{f}_{\mathbf{k}} \left(\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{k}} \right) \left[\frac{\omega_{\mathbf{k}}}{2k_z(k_y^2+k_z^2)} \right] \\
& \left. - \frac{\omega_{\mathbf{k}}^2(1+k^2 d_e^2)}{2k_z^2 k^2 (k_y^2+k_z^2)} I_{\mathbf{k}}(\mathbf{x}') \left[\mathbf{f}_{\mathbf{k}}^* \times \left(\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{k}} \right) \mathbf{f}_{\mathbf{k}} \right] + I_{\mathbf{k}}(\mathbf{x}') \left[(\mathbf{p}-\mathbf{q}) \cdot \frac{\partial}{\partial \mathbf{k}} \right] \left(\frac{\omega_{\mathbf{k}}^2(1+k^2 d_e^2)}{2k_z^2 k^2 (k_y^2+k_z^2)} \mathbf{f}_{\mathbf{k}}^* \times \mathbf{f}_{\mathbf{k}} \right) \right\}. \quad (\text{B6})
\end{aligned}$$

The terms in Eq. (B6) that are proportional to $[(\mathbf{q} \cdot \partial/\partial \mathbf{k}) \mathbf{f}_{\mathbf{k}} \times \mathbf{f}_{\mathbf{k}}^*]$ cancel each other. We represent the remaining part of the integral in the form $S_{2\mathbf{k}}^{(1)} = S_{2\mathbf{k}}^{(1a)} + S_{2\mathbf{k}}^{(1b)}$, where

$$\begin{aligned}
S_{2\mathbf{k}}^{(1a)} = & -\frac{1}{(2\pi)^3} \int \int \int d\mathbf{x}' d\mathbf{q} d\mathbf{p} \exp[i\mathbf{q}\mathbf{x} - i(\mathbf{q}-\mathbf{p})\mathbf{x}'] \mathbf{H}_\mathbf{p} \\
& \cdot \mathbf{f}_{\mathbf{k}} \times \mathbf{f}_{\mathbf{k}}^* \left\{ \frac{\omega_{\mathbf{k}}}{2k_z(k_y^2+k_z^2)} \left(\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{k}} \right) \left[\frac{\omega_{\mathbf{k}}(1+k^2 d_e^2)}{k_z k^2} I_{\mathbf{k}}(\mathbf{x}') \right] \right. \\
& \left. - \frac{\omega_{\mathbf{k}}(1+k^2 d_e^2)}{k_z k^2} I_{\mathbf{k}}(\mathbf{x}') \left(\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{k}} \right) \left[\frac{\omega_{\mathbf{k}}}{2k_z(k_y^2+k_z^2)} \right] \right\}, \quad (\text{B7})
\end{aligned}$$

$$\begin{aligned}
S_{2\mathbf{k}}^{(1b)} = & \frac{1}{(2\pi)^3} \int \int \int d\mathbf{x}' d\mathbf{q} d\mathbf{p} \exp[i\mathbf{q}\mathbf{x} - i(\mathbf{q}-\mathbf{p})\mathbf{x}'] \\
& \times I_{\mathbf{k}}(\mathbf{x}') \mathbf{H}_\mathbf{p} \cdot \left[(\mathbf{p}-\mathbf{q}) \cdot \frac{\partial}{\partial \mathbf{k}} \right] \left[\frac{\omega_{\mathbf{k}}^2(1+k^2 d_e^2)}{2k_z^2 k^2 (k_y^2+k_z^2)} \mathbf{f}_{\mathbf{k}}^* \times \mathbf{f}_{\mathbf{k}} \right]. \quad (\text{B8})
\end{aligned}$$

Equation (B7) is easily integrated over $\mathbf{q}$ and $\mathbf{x}'$,

$$\begin{aligned}
S_{2\mathbf{k}}^{(1a)} = & -\int d\mathbf{p} \exp(i\mathbf{p}\mathbf{x}) \mathbf{H}_\mathbf{p} \cdot \mathbf{f}_{\mathbf{k}} \times \mathbf{f}_{\mathbf{k}}^* \frac{\omega_{\mathbf{k}}^2}{2k_z^2(k_y^2+k_z^2)} \left(\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{k}} \right) \\
& \times \left[\frac{(1+k^2 d_e^2)(k_y^2+k_z^2)}{k^2} I_{\mathbf{k}}(\mathbf{x}) \right] \\
= & i \frac{\omega_{\mathbf{k}}^2}{2k_z^2(k_y^2+k_z^2)} \nabla(\mathbf{H} \cdot \mathbf{f}_{\mathbf{k}} \times \mathbf{f}_{\mathbf{k}}^*) \\
& \cdot \frac{\partial}{\partial \mathbf{k}} \left[\frac{(1+k^2 d_e^2)(k_y^2+k_z^2)}{k^2} I_{\mathbf{k}}(\mathbf{x}) \right]. \quad (\text{B9})
\end{aligned}$$

The integral in Eq. (B8) can be calculated using the identity

$$i(\mathbf{p}-\mathbf{q}) \exp[i(\mathbf{p}-\mathbf{q})\mathbf{x}'] = \frac{\partial}{\partial \mathbf{x}'} \exp[i(\mathbf{p}-\mathbf{q})\mathbf{x}'],$$

and applying the integration by parts over $\mathbf{x}'$. As a result we obtain

$$S_{2\mathbf{k}}^{(1b)} = i \nabla I_{\mathbf{k}}(\mathbf{x}) \cdot \frac{\partial}{\partial \mathbf{k}} \left[\frac{\omega_{\mathbf{k}}^2(1+k^2 d_e^2)}{2k_z^2 k^2 (k_y^2+k_z^2)} \mathbf{H} \cdot \mathbf{f}_{\mathbf{k}}^* \times \mathbf{f}_{\mathbf{k}} \right]. \quad (\text{B10})$$

Finally, taking into account that

$$\mathbf{f}_{\mathbf{k}}^* \times \mathbf{f}_{\mathbf{k}} \equiv \frac{2i\omega_{\mathbf{k}}}{k_z} (1+k^2 d_e^2)(k_y^2+k_z^2) \mathbf{k}, \quad (\text{B11})$$

and adding Eqs. (B9) and (B10) we arrive at Eq. (63).

APPENDIX C: CALCULATION OF THE AVERAGE IN EQ. (49)

To calculate the average on the RHS of Eq. (49) in terms of $N_{\mathbf{k}}$ we take $\mathbf{b}$ and $\text{curl } \mathbf{b}$ in the forms defined by Eqs. (66) and (67). Their substitution results into

$$\begin{aligned}
& \langle b_l b_m - d_e^2 (\text{curl } \mathbf{b})_l (\text{curl } \mathbf{b})_m \rangle \\
& \simeq \int \int d\mathbf{k} d\mathbf{k}' \frac{\langle \psi_{\mathbf{k}} \psi_{\mathbf{k}'} \rangle}{4k^2 k'^2} f_{l\mathbf{k}} f_{m\mathbf{k}'} \\
& \times \left[1 - d_e^2 \frac{\omega_{\mathbf{k}} \omega_{\mathbf{k}'}}{k_z k'_z} (1+k^2 d_e^2)(1+k'^2 d_e^2) \right] \\
& \times \exp[i(\mathbf{k}+\mathbf{k}')\mathbf{x}] \\
& = \int \int d\mathbf{K} d\boldsymbol{\kappa} \frac{\langle \psi_{\mathbf{K}+\boldsymbol{\kappa}/2} \psi_{-\mathbf{K}+\boldsymbol{\kappa}/2} \rangle}{4(\mathbf{K}+\boldsymbol{\kappa}/2)^2 (-\mathbf{K}+\boldsymbol{\kappa}/2)^2} \\
& \times f_{l, \mathbf{K}+\boldsymbol{\kappa}/2} f_{m, -\mathbf{K}+\boldsymbol{\kappa}/2} \exp(i\boldsymbol{\kappa}\mathbf{x}) \\
& \times \left\{ 1 - d_e^2 \frac{\omega_{\mathbf{K}+\boldsymbol{\kappa}/2} \omega_{-\mathbf{K}+\boldsymbol{\kappa}/2}}{(K_z+\kappa_z/2)(-K_z+\kappa_z/2)} \right. \\
& \times [1 + (\mathbf{K}+\boldsymbol{\kappa}/2)^2 d_e^2] [1 + (-\mathbf{K}+\boldsymbol{\kappa}/2)^2 d_e^2] \left. \right\} \\
& = \frac{1}{(2\pi)^3} \int \int \int d\mathbf{K} d\boldsymbol{\kappa} d\mathbf{x}' \frac{I_{\mathbf{K}+\boldsymbol{\kappa}/2} \exp[i\boldsymbol{\kappa}(\mathbf{x}-\mathbf{x}')]}{4(\mathbf{K}+\boldsymbol{\kappa}/2)^2 (-\mathbf{K}+\boldsymbol{\kappa}/2)^2} \\
& \times f_{l, \mathbf{K}+\boldsymbol{\kappa}/2} f_{m, -\mathbf{K}+\boldsymbol{\kappa}/2} \\
& \times \left\{ 1 - d_e^2 \frac{\omega_{\mathbf{K}+\boldsymbol{\kappa}/2} \omega_{-\mathbf{K}+\boldsymbol{\kappa}/2}}{(K_z+\kappa_z/2)(-K_z+\kappa_z/2)} \right. \\
& \times [1 + (\mathbf{K}+\boldsymbol{\kappa}/2)^2 d_e^2] [1 + (-\mathbf{K}+\boldsymbol{\kappa}/2)^2 d_e^2] \left. \right\}. \quad (\text{C1})
\end{aligned}$$

Here $\mathbf{K} \equiv (\mathbf{k}-\mathbf{k}')/2$ corresponds to small scales, and $\boldsymbol{\kappa} \equiv \mathbf{k} + \mathbf{k}'$ to large scales, so that $\boldsymbol{\kappa} \ll K$. Therefore, Eq. (C1) can be significantly simplified by neglecting terms of order $\boldsymbol{\kappa}/K$ in the integrand. Carrying out the integration over $\boldsymbol{\kappa}$ and $\mathbf{x}'$, recalling that $\omega_{-\mathbf{k}} = -\omega_{\mathbf{k}}$, and taking into account the dispersion relation (45), one obtains Eq. (68).

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