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Mean-squared Displacements for Normal and Anomalous Diffusion of Grains

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Abstract. The problem of normal and anomalous space diffusion is formulated on the basis of the integral equations with various type of the probability transition functions for diffusion (PTD functions). For the cases of stationary and time-independent PTD functions the method of fractional differentiation is avoided to construct the correct probability distributions for arbitrary distances, what is important for applications to different stochastic problems. A new general integral equation for the particle distribution, which contains the time-dependent PTD function with one or, for more complicated physical situations, with two times, is formulated and discussed. On this basis fractional differentiation in time is also avoided and a wide class of time dependent PTD functions can be investigated. Calculations of the mean-squared displacements for the various cases are performed on the basis of formulated approach. The particular problems for the PTD functions, dependable from one and for two times, are solved.

1. Introduction

Diffusion is a fundamental transport process of matter and energy in various physical, chemical and biological systems [1,2]. The classical diffusion has been considered in the seminal papers of Smoluchowski, Einstein and Langevin (see the respective references in [3]), where the mass density of the grains diffusing in a background medium was related to the stochastic motion of these particles. Under usual conditions, the stochastic motion of grains leads to a second moment of the mass distribution that is linear in time. Such type of the diffusion processes play a crucial role in plasmas, including dusty plasma [4], nuclear physics [5], neutral systems in various phases [6] and in many other problems. At the same time in many systems the deviation from the linear in time dependence of the mean square displacement have been experimentally observed, in particular, under essentially non-equilibrium conditions or for some disordered systems. The average square separation of a pair of particles passively moving in a turbulent flow grows, according to Richardson’s law, with the third power of time [7]. For diffusion typical for glasses and related complex systems [8] the observed time dependence is slower than linear. These two types of anomalous diffusion obviously are characterized as superdiffusion and subdiffusion. For a description of these two diffusion regimes a number of effective models and methods have been suggested. The continuous time random walk (CTRW) model of Scher and Montroll [9], leading to strongly subdiffusion behavior, provides a basis for understanding
photoconductivity in strongly disordered and glassy semiconductors. The Levy-flight model [10], leading to superdiffusion, describes various phenomena as self-diffusion in micelle systems [11], reaction and transport in polymer systems [12] and is applicable even to the stochastic description of financial market indices [13]. For both cases the so-called fractional differential equations in coordinate and time spaces are applied as an effective approach [14].

In this paper we consider the diffusion of a macroscopic cloud of some grains or Brownian particles, which do not affect on medium in (for simplicity) homogeneous and isotropic case. We show that all known regimes of diffusion, as well as some new ones, can be effectively considered on the basis of the appropriate probability transition function for diffusion (PTD function) $W_D(r - r', t, t')$. Depending on from the specific structure of this function the usual diffusion, superdiffusion and subdiffusion processes can be described without fractional differentiation in space and in time. On this basis the more complicated (than power-type) probability transitions can be used for calculation of the grain density distribution function $f_g(r, t)$. The developed method can be applied for to a wide class of the other stochastic processes. As an example we can mention description of the observed non-Levy distributions for variations of the economical indexes and other economical parameters (e.g. [13]). The developed approach is also important for many other applications. At the end we introduce the generalized integral equation for two-time PTD function, which is able to describe wide classes of non-Markovian and Markovian diffusion processes.

Diffusion in velocity space and the relation between diffusion in coordinate and velocity space have been considered recently in [15] on the basis of the Fokker-Planck kinetic equation with a self-consistent description of friction and diffusion coefficients in velocity space on the basis of the probability transition function $W_V(V, t)$ for velocity space (PTV) [16]. However, this approach is not applicable for the calculation of some special types of spatial diffusion in coordinate space, when the PTD function has a specific form, in particular, possessing slowly decreasing tails in coordinate space, when the Fourier-components for such PTD functions are absent. As we already mentioned there are many important processes, where different types of anomalous diffusion exist [17]. Anomalous diffusion can be related not only with the coordinate, but also, in general, with time-dependent PTD function. Usually the problem of anomalous diffusion is studied on the basis of fractional differentiation [14]. This basis provides the universal description for the probability distribution of grains at large distances. At the same time, to describe all the distances we suggest to use the approach of the PTD function, when the fractional differentiation is not present at all. Naturally, the results for the power-type PTD functions are the same. However, for more complicated PTD functions, with non-power short-range behavior and long-range power tails, the PTD function approach gives the universal and simple description for all interesting cases. At the end we formulate a general integral equation for the distribution function of particles, which gives a most general description of normal and anomalous diffusion and can be useful for many applications. Naturally number of applications (when the specific models for structure of the PTD function permits to obtain the usual Fokker-Planck equation or to describe the processes of superdiffusion or subdiffusion) are well known and considered in the references to this paper (see, e.g., [6,14]). The new physical applications, especially for the case of two-time dependable PTD function, introduced below, will be considered in detail in a separate paper. Here we only demonstrate opportunity for the formal analytical solution in the particular case of two-time dependent PTD function, when the analytical solution is possible.

Necesary to underline that usually the specific form of PTD function is postulated on the basis of physical sense of the considering problem. Rigorous calculation of the PTD function on the basis of kinetic theory is possible for some class of the processes as, for example, for the case of the usual Fokker-Planck equation, when the PTD function can be related with the PTV function [15].

The main goal of this paper is to suggest a new general integral equation for diffusion, which
includes all known particular cases, as, e.g., the Fokker-Planck type equation for diffusion, the Scher-Montroll CTRW model and possibility of Levy-jumps. Our approach gives also opportunity to combine the respective processes as well as to obtain the results by simple way without fractional differentiation method. At the same time, as we believe, this approach gives a constructive way to consider the more general physical situations, which cannot be described by fractional differentiation method.

2. Master equation, probability transition and normal diffusion

Let us consider diffusion in coordinate space on the basis of the master equation, which describes the balance of grains coming in and out the point \( r \) at the moment \( t \). The structure of this equation is formally similar to the master equation in the momentum space. Of course, for coordinate space there is no conservation law, similar to that in momentum space:

\[
\frac{df_g(r,t)}{dt} = \int d r' \{ W(r,r')f_g(r',t) - W(r',r)f_g(r,t) \}.
\]

(1)

Here and below we use for the PTD function \( W_D \) the simplified notation \( W \). The probability transition \( W(r,r') \) describes the probability for a grain to transfer from the point \( r' \) to the point \( r \) per unit time. We can rewrite this equation in the coordinates \( u = r' - r \) and \( r \) as:

\[
\frac{df_g(r,t)}{dt} = \int du \{ W(u,r + u)f_g(r + u,t) - W(u,r)f_g(r,t) \}.
\]

(2)

Assuming that the characteristic displacements are small one may expand Eq. (2) and arrive at the Fokker-Planck form of the equation for the density distribution \( f_g(r,t) \)

\[
\frac{df_g(r,t)}{dt} = \frac{\partial}{\partial r_\alpha} \left[ A_\alpha(r)f_g(r,t) + \frac{\partial}{\partial r_\beta} \left( B_{\alpha\beta}(r)f_g(r,t) \right) \right].
\]

(3)

The coefficients \( A_\alpha \) and \( B_{\alpha\beta} \), describing the acting force and diffusion, respectively, can be written as the functionals of the probability function (PTD) in coordinate space \( W \) in the form:

\[
A_\alpha(r) = \int d^s u u_\alpha W(u,r)
\]

(4)

and

\[
B_{\alpha\beta}(r) = \frac{1}{2} \int d^s u u_\alpha u_\beta W(u,r),
\]

(5)

where \( s \) is the dimension of coordinate space. For the isotropic case the PTD function depends on \( r \) and the modulus of \( u \). For a homogeneous medium, when \( r \)-dependence of the PTD function is absent, the coefficients \( A_\alpha = 0 \) while the diffusion tensor is diagonal \( B_{\alpha\beta} = \delta_{\alpha\beta}B \). The function \( B \) can be written as the integral

\[
B = \frac{1}{2s} \int d^s u u^2 W(u).
\]

(6)

3. Generalized Levy-flights

This consideration cannot be applied to specific situations in which the integral in Eq. (6) is infinite. In that case we have to examine the general transport equation (1). We will now consider the problem for the homogeneous and isotropic case, when the PTD function depends only on \(|u|\). By Fourier-transformation we arrive at the following form of Eq. (1):

\[
\frac{df_g(k,t)}{dt} = \int d^s u [\exp(i k \cdot u) - 1] W(u)f_g(k,t) \equiv X(k)f_g(k,t),
\]

(7)
where $X(k) \equiv X(k)$ and $W(u) \equiv W(|u|)$. According to Eq. (7) for the case of enough fast decrease of PTD function for high distances the exponent in (7) can be expanded and for the function $X(k)$ the expression

$$X(k) = -\frac{k^2}{s} \int d^s u u^2 W(u),$$

(8)

is valid. This representation corresponds to Gaussian distribution for the function $f_g$ with the diffusion coefficient $D \equiv B$, determined by Eq. (6). Therefore the Gaussian distribution is the consequence of the fast decrease of the PTD-function and opportunity to expand for this case the general form Eq. (7) of the function $X(k)$ on small values of $k$. In general case the PTD function depends from two parameters: characteristic length $l$ and characteristic time $\tau$ for the typical mean path in the system under consideration. Instead one of these parameters it is possible, naturally, to use the effective diffusion coefficient $D^{eff}$ typical mean path in the system under consideration. Instead one of these parameters it is possible, naturally, to use the effective diffusion coefficient $D^{eff} \sim l^2/\tau$. For the Gaussian case, when the relation $kl \ll 1$ is fulfilled, the distribution $f_g(k,t)$ is the function of one parameter $D$, which is included in the composition $Dk^2t$. The specific peculiarities of the PTD function in Gaussian approximation manifest themselves only in the value of the diffusion coefficient $D$. In general case of arbitrary $kl$ the Gaussian distribution is violated and the distribution function $f_g$ depends from the parameters $kl$ and $t/\tau$ separately, or $kl^2$ and $(D^{eff} \cdot t)/l^2$.

Let us now assume a simple form of the PTD function with a power dependence of the distance: $W(u) = C/|u|^a$, where $C$ is a constant and $\alpha > 0$. Such a type singular dependence is typical for jump diffusion probability in heteropolymers in solution (see, e.g., [18], where the different applications of anomalous diffusion are considered on the basis of the fractional differentiation method). For the one-dimensional case we find:

$$X(k) \equiv -4 \int_0^\infty du \sin^2 \left(\frac{k u}{2}\right) W(u) = -2^{3-\alpha} C |k|^{\alpha-1} \int_0^\infty d\zeta \frac{\sin^2 \zeta}{\zeta^\alpha}. $$

(9)

For the values $1 < \alpha < 3$ this function is finite and equal to

$$X(k) = -\frac{2^{2-\alpha} \sqrt{\pi} C \Gamma((3 - \alpha)/2) |k|^{\alpha-1}}{\Gamma(\alpha/2)(\alpha - 1)}, $$

(10)

where $\Gamma$ is the Gamma-function. At the same time the integral (6) for a such type of PTD functions is infinite, because usual diffusion is absent. The considered procedure for the simplest cases of power dependence of the PTD function is equivalent to the equation with fractional space differentiation [14],[18]:

$$\frac{df_g(x,t)}{dt} = \bar{D} \Delta^{\mu/2} f_g(x,t), $$

(11)

where $\bar{D}$ is the diffusion coefficient for (in general) non Fokker-Planck type of processes. The linear operator $\Delta^{\mu/2}$ is a fractional Laplacian, whose action $\Delta^{\mu/2} f(x)$ on the function $f(x)$ in Fourier space is described by the relation $-(k^2)^{\mu/2} \tilde{f}(k) = -|k|\mu \tilde{f}(k)$, where $\tilde{f}(k)$ is the Fourier transformation of the function $f(x)$. In the case considered above $\mu \equiv (\alpha - 1)$, where $0 < \mu < 2$ and $\bar{D}$ is proportional to $C$. For more general PT functions, which (for arbitrary values $u$) are not proportional to the $\alpha$-th power of $u$, the method described above is also applicable, although the respective equation in fractional derivatives does not exist.

For the case of purely power dependence of PT the non-stationary solution for the density distribution describes the super-diffusive behavior (so-called Levy flights). The solution of Eq. (11) in Fourier space reads:

$$\tilde{f}_g(k,t) = \exp(-\bar{D}|k|^{\mu}t), $$

(12)
which in coordinate space corresponds to the so-called symmetric Levy stable distribution:

$$f_g(x, t) = \frac{1}{(Dt)^{1/\mu}} L \left[ \frac{x}{(Dt)^{1/\mu}}; \mu, 0 \right]. \quad (13)$$

For general case it follows from Eq. (7) that

$$f_g(k, t) = C_1 \exp[X(k)t], \quad (14)$$

with some constant $C_1$.

Let us now turn to the two-dimensional and three-dimensional cases for the PTD function equal to $W = C/|u|^\alpha$ with the constant $C > 0$. Then the functions $X(k)$ are respectively

$$X(k) \equiv 2\pi \int_0^\infty du \left( J_0(ku) - 1 \right) W(u), \quad (15)$$

where $J_0$ is the zero order Bessel function, and

$$X(k) \equiv 4\pi \int_0^\infty du u^2 \left( \frac{\sin(ku)}{ku} - 1 \right) W(u) = 4\pi Ck^{\alpha-3} \int_0^\infty d\zeta \frac{1}{\zeta^{\alpha-2}} \left( \frac{\sin \zeta}{\zeta} - 1 \right), \quad (16)$$

These integrals (negatively-valued for arbitrary $k$, as well as the integral (9) for the one-dimensional case) are finite for the values $2 < \alpha < 4$ and $3 < \alpha < 5$ respectively. If we return to the variable $\mu \equiv \alpha - s$ (s=3 for the 3-dimensional case), which is equal to the power of $k$ in the function $X(k)$, we find the same limitation as in the one-dimensional case $0 < \mu < 2$, which characterizes the fractional derivative power.

Of course, consideration on the basis of PTD function given above, permits to avoid the fractional differentiation method and to consider more general physical situations of the non-power probability transitions. Let us consider that for a simple example. Taking (for the one-dimensional case) the PT function $W(u)$ in the form

$$W(u) = C \frac{1 - \exp[-\sigma u^p]}{u^\alpha}, \quad (17)$$

with $p > 0$, we arrive at the function $X(k)$:

$$X(k) = -2^{3-\alpha} C |k|^{\alpha-1} \int_0^\infty d\zeta \frac{1 - \exp[-\sigma(2\zeta/|k|)^p]}{\zeta^\alpha} \sin^2 \zeta \equiv -2^{3-\alpha} C |k|^{\alpha-1} T(\sigma/|k|^p, \alpha). \quad (18)$$

It is easy to see that the function $T(\sigma/|k|^p, \alpha)$ is finite for $1 < \alpha < p + 3$, because for the small values of the distance for $p > 0$ divergence is suppressed also for some powers $\alpha > 3$. A simple calculation for $\alpha = 2$ and $p = 1$ leads to the following result, which cannot be found by the usual fractional differentiation method:

$$T(\sigma/|k|, 2) = \frac{\pi}{2} - \arctan(|k|/\sigma) + \frac{\sigma}{2|k|} \ln \left[ 1 + k^2/\sigma^2 \right]. \quad (19)$$

The asymptotic behavior of the function $X(k)$ for $k \to 0$ (or $\sigma \to \infty$) is similar, as follows from Eq. (19), to the case $W(u) = C/u^\alpha$. For the case under consideration $\alpha = 2$ the limit $X(k \to 0) \to -\pi Ck$. For large values of $k$ ($k \to \infty$ or $\sigma \to 0$) we find $X(k) \to (\sigma/k) \ln(k/\sigma)$. In general case, the universal behavior of the function $X(k)$ is provided by asymptotical properties of the PTD function for large distances for $1 < \alpha < p + 3$. 


4. Mean-squared displacements

For the normal Gaussian diffusion (see e.g. Eq. (3), (6)) for arbitrary $s$ the equality $X(k) = -Bk^2$ is valid. This leads to the Gaussian distribution in the coordinate space and to the usual time dependence of the mean-squared displacement $\langle r^2 \rangle \sim Bt$. For different types of anomalous diffusion this relation is violated and the dependence $\langle r^2(t) \rangle$ has to be calculated on the basis of a concrete anomalous distribution, which is determined by the respective equation for $f_g(r, t)$.

As follows from the analysis given above, the crucial function, determining the diffusion process, is the PTD function $W$. Let us find the relation between the mean-square displacement and the PTD function (or the function $X(k)$). It is easy to see that

$$< r^2(t) > = - \left\{ \frac{\partial^2 f(k, t)}{\partial k^2} + \left[ \frac{(s - 1)}{k} + \delta_{s, 1} \delta(k) \right] \frac{\partial f(k, t)}{\partial k} \right\}_{|k=0} ,$$

where $f(k)$ is the Fourier-component of the distribution $f_g(|r|, t)$ and differentiation is performed on $k$ equal to modulus of the vector $k$. According to Eq. (14) the function $\langle r^2(t) \rangle$ (with $C_1 = 1$) can be also rewritten in the form

$$< r^2(t) >= - \left\{ \frac{\partial^2 X(k)}{\partial k^2} t + \left[ \frac{(s - 1)}{k} + \delta_{s, 1} \delta(k) \right] \frac{\partial X(k)}{\partial k} t + \left( \frac{\partial X(k)}{\partial k} \right)^2 t^2 \right\}_{|k=0} \exp[X(k = 0) t].$$

As it follows from Eq. (21) for normal diffusion, when $X(k) = -sDk^2$ the standard result $\langle r^2(t) \rangle = 2sDt$ is reproduced. For the case of superdiffusion in the particular form of the Levy flights $X(k) \sim |k|^\mu$ (with $0 < \mu < 2$, where $\mu = \alpha - s$) the mean-squared displacement is infinite. These results follows immediately from determination of the mean-squared displacement and Eq. (20) and shows that for real physical problems in an infinite system always necessary to use PTD functions, which tend more fast to zero on large distances than $W(\rho) \sim 1/\rho^{\mu+s}$ with $\mu < 2$. Than the finite value of the mean-squared displacement can be provided. For the distribution function in coordinate space we can have, for example, the Gaussian-type shape for a large $r$ and Levy-type shape for a small $r$. It is some kind of, so-called, truncated Levy flights (see e.g. [19]). For a finite system the finite value of the mean-squared displacement is provided automatically as the consequence of some boundary conditions for the distribution function $f_g$.

As an example of the truncated Levy flight for $s = 1$ let us consider the PTD function in the form

$$W(\rho) = C \frac{1}{\rho^2 (\rho^2 + \lambda^2)},$$

which generate the function $X(k)$ in the form

$$X(k) = -\frac{\pi Ck}{\lambda^2} \left\{ 1 - \frac{1}{k\lambda} [1 - \exp(-k\lambda)] \right\} .$$

The function $X(k)$ is proportional to $k^2$ for small $k$ ($k\lambda << 1$), as for normal diffusion, and proportional to $k$ for large $k$ ($k\lambda >> 1$), as for Levy-flights. The mean-square displacement is similar to the usual diffusion process and equal to $\pi Ct/\lambda$. The distribution function in coordinate space for the case under consideration is the truncated Levy flight for the power $\alpha = 2$ in sense mentioned above.

5. Time-dependent PTD functions, subdiffusion and superdiffusion

It should be noted in the connection of the problem of a generalized description of diffusion that not only space dependence, but also time dependence of the PTD function can be very different from the case classical diffusion. This kind of problems relates to the class of stochastic transport,
which describes so-called subdiffusive behavior [9], which possess many other applications, e.g. photoconductivity in strongly disordered and glassy semiconductors or resonance radiative transfer in a plasma [20]. Here we only formulate the problem, leaving a detailed analysis for a separate paper. For this purpose we formulate a more general transport equation for the density distribution:

\[ f_g(r, t) = f_g(r, t = 0) + \int_0^t d\tau \int dr' \{ W(r, r', \tau, t - \tau) f_g(r', \tau) - W(r', r, \tau, t - \tau) f_g(r, \tau) \}. \] (24)

For the case of a stationary PTD function we return to Eq. (1). In absence of memory effects, but with the PT function being a function of the current time \( \tau \) we arrive at an equation that is more general than Eq. (1), which describes the density evolution with prescribed time-dependence of the PTD function:

\[ \frac{df_g(r, t)}{dt} = \int dr' \{ W(r, r', t)f_g(r', t) - W(r', r, t)f_g(r, t) \}. \] (25)

In particular in the case of slow space dependence of the function \( W \) we find an equation for diffusion similar to Eq. (3)-(5) with time-dependent coefficients \( A_\alpha(r, t) \) and \( B_{\alpha\beta}(r, t) \), which are calculated on the basis of the PTD function \( W(u, r, t) \).

If a system possesses memory, then, in the simplest case, when the function \( W \) can be expanded in the spirit of the Fokker-Planck approximation in coordinate space, we arrive at the following form of Eq. (24):

\[ f_g(r, t) = f_g(r, t = 0) + \int_0^t d\tau \frac{\partial}{\partial r_\alpha} \left[ A_\alpha(r, \tau, t - \tau) f_g(r, \tau) + \frac{\partial}{\partial r_\beta} (B_{\alpha\beta}(r, \tau, t - \tau) f_g(r, \tau)) \right]. \] (26)

If the function \( W \) possesses memory and depends only on the difference \( t - \tau \), but cannot be expanded in the coordinate space as in the case (26), we can use the Laplace-transformation in time to find:

\[ f_g(r, z) = \frac{f_g(r, t = 0)}{z} + \int dr' \{ W(r, r', z)f_g(r', z) - W(r', r, z)f_g(r, z) \}. \] (27)

The function \( W(r', r, z) \) is determined by the equality:

\[ W(r', r, z) = \int_0^\infty d\tau W(r', r, \tau) \exp(-z\tau) \] (28)

(The Laplace transformation can be applied also to simplify the Fokker-Planck type equation (26) if there is no the separate dependence PTD from \( \tau \).) For the space homogeneous case Eq. (27) can be Fourier-transformed and rewritten in the \( k, z \) variables:

\[ \tilde{f}_g(k, z) = \tilde{f}_g(k, t = 0) \frac{1}{z[1 - X(k, z)]}, \] (29)

where

\[ X(k, z) = \int d^s \rho [\exp(i k \cdot u) - 1] W(|u|, z). \] (30)

If the PT function is time-independent \( W(|u|, z) = W(|u|)/z \) and \( \tilde{f}_g(k, t = 0) = \text{constant} \), we return to the case of anomalous diffusion considered above in Eqs. (7)-(19). For a general multiplicative form of the PT function \( W(|u|, \tau) = W_1(|u|)W_2(\tau) \) (some examples of such
multiplicative functions with \( W_2(\tau) \neq 1 \) are considered in [14], the case \( W_2(\tau) = 1 \) describes as easy to see the generalized Levy flights) the function \( X(k, z) \) is equal:

\[
X(k, z) \equiv X_1(k)X_2(z),
\]

where

\[
X_1(k) = \int d^su \left[ \exp(i k \cdot u) - 1 \right] W_1(|u|)
\]

and

\[
X_2(z) = \int_0^\infty d\tau W_2(\tau) \exp(-z\tau).
\]

In result we arrive at the following form of the distribution \( \tilde{f}_g(k, z) \)

\[
\tilde{f}_g(k, z) = \frac{\tilde{f}_g(k, t = 0)}{z [1 - X_1(k)X_2(z)]},
\]

For a power dependence \( W_2 = C_2/\tau^\gamma \) with \( \gamma < 1 \), the integral \( X_2(z) \) is:

\[
X_2(z) = \frac{C_2}{z^{\gamma}} \Gamma(1 - \gamma).
\]

The distribution function (29) in \( k, z \) space in that case reads

\[
\tilde{f}_g(k, z) = \frac{\tilde{f}_g(k, t = 0)}{[z - z^{\gamma} C_2 \Gamma(1 - \gamma)X_1(k)]},
\]

Avoiding fractional differentiation on both coordinates and time [17],[21] we essentially extend and simplify the description of anomalous diffusion for more general than power-type PTD functions. For example, let us consider the PTD function with the simple time dependence:

\[
W_2(\tau) = C_2/\sqrt{\pi\tau} \exp(-k^2/4\tau),
\]

with \( k \geq 0 \). Then we immediately arrive at the expression for \( X_2(k, z) \):

\[
X_2(k, z) = C_2 \frac{1}{\sqrt{z}} \exp(-k\sqrt{z}).
\]

The respective distribution function is determined by Eq. (34).

Let us consider now the mean-square displacement for the general case of retardation, when the PTD function depends on time only via the argument \( t - \tau \). As it easy to see by Fourier- and Laplace-transformations of the distribution \( f_g(r, t) \) the mean-square displacement can be expressed in the form

\[
<r^2(t) > = - \int_{-\infty}^{+\infty} dz \frac{dz}{2\pi i} \left\{ \frac{\partial^2 f_g(k, z)}{\partial k^2} + \left( \frac{s-1}{k} + \delta_{s,1}\delta(k) \right) \frac{\partial f_g(k, z)}{\partial k} \right\}|_{k \to 0} \exp(zt).
\]

This equation can be rewritten as

\[
<r^2(t) > = - \int_{-\infty}^{+\infty} \frac{dz}{2\pi i} \frac{dz}{z} \exp(zt) \cdot \left\{ X''(k, z) + \left[ \frac{(s-1)}{k} + \delta_{s,1}\delta(k) \right] X'(k, z) \right\} \frac{1 - X(k, z) + 2X^2(k, z)}{[1 - X(k, z)]^3}|_{k \to 0}.
\]
For the particular case of the multiplicative form \( X(k, z) = -\tilde{D}k^2\psi(z) \) we arrive to the function \( \langle r^2(t) \rangle = \tilde{D} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{dz \exp(zt)}{\pi t} \psi(z). \) (41)

If, for example, \( \psi(z) = \Gamma(k)/z^k \) with \( k > 0 \) the mean-square displacement is equal to
\[
\langle r^2(t) \rangle = \tilde{D}t^k. \tag{42}
\]

For \( k < 1 \) and \( k > 1 \) Eq. (42) describes subdiffusion and superdiffusion respectively. The particular case of the power \( n = 3 \) describes superdiffusion related with the Richardson’s law.

### 6. Particular solution for two times

To consider the particular case of the general equation (24), in which both retardation and prescribed time dependence are present in the kern \( W \) we can choose the simplest form of the multiplicative PTD function with the exponential retardation:

\[
W(r, r', \tau, t - \tau) \equiv W_1(r, r') W_2(\tau, t - \tau) = W_1(r, r') \tilde{W}_2(\tau) \exp \left[ -\frac{(t - \tau)}{\tau_0} \right]. \tag{43}
\]

For a homogeneous case Eq. (24) after Fourier-transformation by the space variable \( u \) can be represented in the form:

\[
\tilde{f}_g(k, t) = \tilde{f}_g(k, t = 0) + X_1(k) \int_0^t dt' \tilde{W}_2(\tau) \exp \left[ -\frac{(t - \tau)}{\tau_0} \right] \tilde{f}_g(k, \tau), \tag{44}
\]

This equation can be written as the differential one:

\[
\frac{d\tilde{f}_g(k, t)}{dt} = X_1(k) \tilde{W}_2(t) \tilde{f}_g(k, t) - \tilde{f}_g(k, t = 0) - \frac{\tilde{f}_g(k, t)}{\tau_0}. \tag{45}
\]

If we introduce the notation \( U(k, t) \equiv [X_1(k) \tilde{W}_2(t) - \Omega_0] \), where \( \Omega_0 \equiv 1/\tau_0 \), the solution of Eq. (45) has a form:

\[
\tilde{f}_g(k, t) = \tilde{f}_g(k, t = 0) \left\{ 1 + \Omega_0 \int_0^t dt' \exp \left[ -\int_0^{t'} dt'' U(k, t'') \right] \right\} \exp \left[ \int_0^t d\zeta U(k, \zeta) \right]. \tag{46}
\]

The detailed consequences and physical applications of the dependencies \( W(|u|, \tau) \) introduced above as well as some more complicated dependence will be considered in a separate paper.

### 7. Conclusions

To summarize, the problems of anomalous diffusion are considered on the basis of the PTD function for a master-type equation in coordinate space. Consideration presented above permits to avoid the method of fractional space differentiation (coinciding with the results obtained by this method in the particular cases of power-type dependencies of the PTD function) and to extend the results for a wide class of PTD functions. General expressions for the mean-squared displacements are found for different diffusion regimes. An extension of this approach is used to formulate the general integral equation for the distribution function in coordinate space, applicable to description of very different types of time-dependent normal and anomalous diffusion. These results can be important for very different applications. Necessary to underline that microscopic calculation of the PTD functions for the various systems is unsolved and fascinating problem.
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