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UPPER BOUND ON THE EXPECTED SIZE OF THE INTRINSIC BALL

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Abstract
We give a short proof of Theorem 1.2(i) from [5]. We show that the expected size of the intrinsic ball of radius \( r \) is at most \( C r \) if the susceptibility exponent \( \gamma \) is at most 1. In particular, this result follows if the so-called triangle condition holds.

Let \( G = (V, E) \) be an infinite connected graph. We consider independent bond percolation on \( G \). For \( p \in [0, 1] \), each edge of \( G \) is open with probability \( p \) and closed with probability \( 1 - p \) independently for distinct edges. The resulting product measure is denoted by \( P_p \). For two vertices \( x, y \in V \) and an integer \( n \), we write \( x \leftrightarrow y \) if there is an open path from \( x \) to \( y \), and we write \( x \leftrightarrow^\leq n y \) if there is an open path of at most \( n \) edges from \( x \) to \( y \). Let \( C(x) \) be the set of all \( y \in V \) such that \( x \leftrightarrow y \). For \( x \in V \), the intrinsic ball of radius \( n \) at \( x \) is the set \( B_I(x, n) \) of all \( y \in V \) such that \( x \leftrightarrow^\leq n y \). Let \( p_c = \inf\{ p : P_p(|C(x)| = \infty) > 0 \} \) be the critical percolation probability. Note that \( p_c \) does not depend on a particular choice of \( x \in V \), since \( G \) is a connected graph. For general background on Bernoulli percolation we refer the reader to [2].

In this note we give a short proof of Theorem 1.2(i) from [5]. Our proof is robust and does not require particular structure of the graph.

Theorem 1. Let \( x \in V \). If there exists a finite constant \( C_1 \) such that \( \mathbb{E}_p|C(x)| \leq C_1(p_c - p)^{-1} \) for all \( p < p_c \), then there exists a finite constant \( C_2 \) such that for all \( n \),

\[
\mathbb{E}_{p_c}|B_I(x, n)| \leq C_2 n.
\]

Before we proceed with the proof of this theorem, we discuss examples of graphs for which the assumption of Theorem 1 is known to hold. It is believed that as \( p \nearrow p_c \), the expected size of \( C(x) \) diverges like \( (p_c - p)^{-\gamma} \). The assumption of Theorem 1 can be interpreted as the mean-field bound \( \gamma \leq 1 \). It is well known that for vertex-transitive graphs this bound is satisfied if the triangle condition holds at \( p_c \) [1]: For \( x \in V \),

\[
\sum_{y, z \in V} \mathbb{P}_{p_c}(x \leftrightarrow y) \mathbb{P}_{p_c}(y \leftrightarrow z) \mathbb{P}_{p_c}(z \leftrightarrow x) < \infty.
\]

\(^{1}\)RESEARCH PARTIALLY SUPPORTED BY EXCELLENCE FUND GRANT OF TU/E OF REMCO VAN DER HOFSTAD.
This condition holds on certain Euclidean lattices $\mathbb{Z}^d$ including the nearest-neighbor lattice $\mathbb{Z}^d$ with $d \geq 19$ and sufficiently spread-out lattices with $d > 6$. It also holds for a rather general class of non-amenable transitive graphs $[6,8,9,10]$. It has been shown in [7] that for vertex-transitive graphs, the triangle condition is equivalent to the so-called open triangle condition. The latter is often used instead of the triangle condition in studying the mean-field criticality.

Proof of Theorem 1. Let $p < p_c$. We consider the following coupling of percolation with parameter $p$ and with parameter $p_c$. First delete edges independently with probability $1 - p_c$, then every present edge is deleted independently with probability $1 - (p/p_c)$. This construction implies that for $x, y \in V, p < p_c$, and an integer $n$,

$$P_p(x \leftrightarrow^n y) \geq \left(\frac{p}{p_c}\right)^n P_{p_c}(x \leftrightarrow^n y).$$

Summing over $y \in V$ and using the inequality $P_p(x \leftrightarrow^n y) \leq P_p(x \leftrightarrow y)$, we obtain

$$E_p\left|B_I(x,n)\right| \leq \left(\frac{p_c}{p}\right)^n E_p\left|C(x)\right|.$$

The result follows by taking $p = p_c(1 - \frac{1}{2n})$. □

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References


