Time scales and FSI in oscillatory liquid-filled pipe flow

Arris S Tijsseling  
Dept. of Mathematics and Computer Science  
Eindhoven University of Technology  
P.O. Box 513, 5600 MB Eindhoven  
The Netherlands

Alan E Vardy  
Civil Engineering Division  
University of Dundee  
Dundee DD1 4HN  
United Kingdom

ABSTRACT

The validity of various methods of modelling unsteady flows – namely quasi-steady, rigid-column (with and without friction) and water-hammer (with and without fluid-structure interaction (FSI)) is examined at various time scales of excitation. One aim is to find the time scales for which 1D-FSI is of importance in forced oscillating pipe flow. Illustrative examples are given for starting and oscillating laminar flows in a single pipe.

NOMENCLATURE

Scalars

\(c\)  fluid wave speed, m/s
\(c_s\)  solid wave speed, m/s
\(D\)  inner diameter of pipe, m
\(d\)  dispersion coefficient, m\(^3\)/s
\(E\)  Young modulus of pipe wall material, Pa
\(e\)  pipe wall thickness, m
\(f\)  frequency, Hz
\(FSI\)  fluid-structure interaction
\(I_m\)  modified Bessel function of the first kind of order \(m\)
\(Im\)  imaginary part of complex number
\(J_{0,1}\)  Bessel function of the first kind of order 0, 1
\(K\)  fluid bulk modulus, Pa
\(L\)  pipe length, m
\(l\)  wave length, m
\(P\)  fluid pressure, Pa
\(r\)  radial coordinate, m
\(T\)  period of harmonic oscillation, s
\(t\)  time, s
\(V\)  cross-sectional average of axial fluid velocity, m/s
\(v\)  axial fluid velocity, m/s
\(z\)  axial coordinate, m
\(\alpha\)  added fluid mass coefficient
\(\beta\)  inertia coefficient
\(\gamma\)  phase angle
\(\Delta P\)  (linear) pressure rise within time \(\Delta t\), Pa
\(\Delta t\)  duration of ramp excitation, s
\(\lambda\)  Darcy-Weisbach friction coefficient
\(\lambda_0\)  \(n\)th zero of \(J_0\)
\(\mu\)  Poisson ratio
\(\nu\)  kinematic viscosity, m\(^2\)/s
\(\rho\)  fluid mass density, kg/m\(^3\)
\(\rho_s\)  solid mass density, kg/m\(^3\)
\(\omega\)  angular frequency, rad/s

Subscripts

\(qs\)  quasi steady
\(r\)  reversal
\(rc,l\)  rigid column, laminar
\(rc:t\)  rigid column, turbulent
\(s\)  structure, solid, tube
\(sh\)  “steel hammer”
\(su\)  start up
\(wh\)  water hammer
\(0\)  initial (steady) state
\(\infty\)  final steady (oscillatory) state
1 INTRODUCTION

The validity of a mathematical model describing a dynamic physical system strongly depends on the time scales of the system in relation to the time scale of the excitation. Four one-dimensional mathematical models describing unsteady flow in circular pipes are considered herein: quasi-steady flow, rigid-column motion, water hammer, and water hammer with FSI. The performance of each model at different time scales of excitation is investigated in dimensional test examples. One important goal is to find the characteristic time scales for water hammer with FSI; this is where pipe motion starts to influence the unsteady flow. The present study is an extension to work presented at the previous conference in the Pressure Surges series.

The earlier paper (Ref. 1) gives all relevant equations governing one-dimensional pipe flow. Representative time scales are defined and discussed, and flows are classified according to these time scales. The FSI mechanisms in each type of flow are explained and the equations needed in a pipe stress analysis are given. The dimensional examples included both slow and fast acceleration from rest of turbulent flow and the instantaneous starting of laminar flow. The identification of suitable non-dimensional parameters, equivalent to those in Refs. (2) and (3), was a future goal, but these can be obtained from the ratios of different time scales. Analytical rigid-column solutions were given in appendices.

The present paper should be considered in combination with Ref. (1). It summarizes the time scales defined in Ref. (1) and it studies the same test problem. The examples include starting laminar flow (revisited) and forced oscillating laminar flow. Appendices A, B and C list analytical solutions. Appendix A for turbulent flow is given here for completeness: it is an extension and improvement of Ref. (1). Appendices B and C give closed-form solutions for laminar rigid-column 1D and 2D flows respectively. Appendix D looks at rigid-column transients.

The Nomenclature defines the symbols; these are not declared in the text.

2 TIME SCALES

Representative time scales for laminar flow have been defined and explained in Ref. (1). They are listed in Table 1, where the conventional acoustic wave speeds are

\[ c = \sqrt{\frac{K}{\rho}} \sqrt{1 + \frac{D}{e \rho \rho_s}} \quad \text{and} \quad c_s = \sqrt{\frac{E}{\rho_s}}. \] (1)

The apparent inertia factor \( \beta \) used in Table 1 is needed for laminar flow (Refs. 3 and 4). For rigid-column analyses allowing for liquid inertia, we get different time scales depending upon whether the resistance is (a) absent, (b) proportional to the length of the liquid column (e.g. wall friction), (c) independent of the length of the liquid column (e.g. a valve) or (d) a combination of these. Herein we consider situation (b) with time scale \( \beta \frac{D^2}{32 \nu} \). This important time scale has many different names in the literature; Sarpkaya (Ref. 5) wrote a nice historical review on it. The start-up time is the minimum time required to accelerate from zero to the actual final velocity and – although it is conceptually different – it is equal to the [situation (b)] rigid-column time scale for \( \beta = 1 \).

Representative time scales and frequencies for turbulent flow are listed in Table 2, noting that the factor \( \beta \) has not been included here because it is assumed to be close to 1 (for turbulent flow). In Table 2, the following terms are used for simulations including water hammer. See Ref. (6) for an extensive review of water hammer with FSI.

Water hammer denotes calculations with no FSI;
Table 1  Representative time scales in laminar pipe flow
[Numerical values are given for the particular case of water ($V_\infty=2$ m/s, $c=1000$ m/s, $\beta = 1$) in a steel pipe ($L=20$ m, $D=0.0008$ m, $c_s=5000$ m/s)].

| Time scales | $t_{su} = \frac{\rho |V_\infty| L}{\Delta P} = \frac{\beta D^2}{32 \bar{V}} = t_{rc,l} = 0.02$ s |
|-------------|---------------------------------------------------------------|
| frictionless start-up, rigid column | $t_{2D} = \frac{D^2}{4 \bar{V}} = 0.16$ s |
| water hammer | $0.04 \ s = \frac{2L \sqrt{\beta}}{c} \leq t_{wh} \leq \frac{4L \sqrt{\beta}}{c} = 0.08 \ s$ |
| steel hammer | $0.008 \ s = \frac{2L}{c_s} \leq t_{sh} \leq \frac{4L}{c_s} = 0.016 \ s$ |

Table 2  Representative time scales and frequencies in turbulent pipe flow
[Numerical values are given for the particular case of water ($V_\infty=2$ m/s, $c=1000$ m/s) in a steel pipe ($L=200$ m, $D=0.8$ m, $e=8$ mm, $c_s=5000$ m/s), see Ref. (1)].

<table>
<thead>
<tr>
<th>Time scales</th>
<th>Frequencies</th>
</tr>
</thead>
<tbody>
<tr>
<td>start-up, rigid column</td>
<td>$t_{su} = \frac{\rho</td>
</tr>
<tr>
<td>water hammer</td>
<td>$0.4 \ s = \frac{2L}{c} \leq t_{wh} \leq \frac{4L}{c} = 0.8 \ s$</td>
</tr>
<tr>
<td>steel hammer</td>
<td>$0.08 \ s = \frac{2L}{c_s} \leq t_{sh} \leq \frac{4L}{c_s} = 0.16 \ s$</td>
</tr>
<tr>
<td>ringing</td>
<td>$f_{ring}(\alpha) = \frac{c_s}{\pi D} \sqrt{1 + \alpha \frac{D}{2e} \frac{\rho}{\rho_s}} = 2150$ Hz for $\alpha = 0$ and $1050$ Hz for $\alpha = 1/2$</td>
</tr>
<tr>
<td>dispersive FSI</td>
<td>In present example: $d = 4$ m$^3$/s.</td>
</tr>
<tr>
<td>ovaling</td>
<td>$f_{FSI(t)} \approx 0.36 \frac{c}{\sqrt{dt}}$</td>
</tr>
<tr>
<td>ovaling</td>
<td>$f_{FSI(1s)} = 130$ Hz</td>
</tr>
<tr>
<td>ovaling</td>
<td>$f_{oval} = f_{ring}(0) \cdot 2\sqrt{\frac{e}{D}} \sqrt{\frac{5 + \frac{D}{e} \frac{\rho}{\rho_s}}{18} = 18$ Hz</td>
</tr>
</tbody>
</table>

Steel hammer denotes water hammer plus axial FSI, but with no allowance for radial inertia in either the pipe wall or the liquid. It is an idealised form of FSI where coupled axial-stress waves and pressure waves travel without dispersion;

Ringing denotes steel hammer plus radial inertia of the pipe and the liquid, but without coupling between radial and axial FSI effects. The pipe vibrates at the relatively high ring-frequency of a radially unrestrained hoop. Added liquid mass is accounted for by the coefficient $\alpha$ which may have values between 1/4 and 1/2, depending on the assumed distribution of the radial liquid velocity;
Dispersive FSI denotes ringing plus coupling between radial and axial FSI effects. The coupling causes smearing of wave fronts and causes trailing oscillations (see Ref. 7). None of these analyses include the small influence of Ovaling frequencies that are much lower than ringing frequencies and that have been confirmed in the pipe wall experimentally by the authors (although the relevant measurements have not yet been published).

3 TEST PROBLEM

To illustrate the relative importance of friction, inertia, elasticity and FSI [Poisson and junction coupling, Ref. (6)] at different time scales, and to gain insight into associated phenomena, calculations have been carried out for accelerating and oscillating laminar flows in a straight tube. The steel tube is 20 m long, has a diameter of 0.8 mm, and is filled with water of mass density 1000 kg/m$^3$, bulk modulus 2.1 GPa and kinematic viscosity $10^{-6}$ m$^2$/s. The tube wall has mass density 7900 kg/m$^3$, modulus of elasticity 210 GPa and Poisson ratio 0.3. The conventional wave speeds of pressure (in the liquid) and axial stress (in the wall) have round values $c = 1000$ m/s and $c_s = 5000$ m/s, respectively. At one end ($z = 0$) the system is excited by either a linearly increasing pressure (from zero) or a harmonically oscillating pressure (Fig. 1), whereas at the other end ($z = L$) the pressure remains zero. All pressures are relative to a sufficiently high and constant static pressure. For starting flow, the pressure rise (at $z = 0$) is $\Delta P = 20$ bar within time $\Delta t$, after which the pressure remains constant. For oscillating flow, the pressure amplitude is $\Delta P = 2$ bar and the cycle period is $T$. The ramp interval $\Delta t$ and the oscillation period $T$ are important time scales of excitation, which should be compared with the system time scales listed in Table 1. Laminar flow is quite rare in civil engineering practice although there are some examples (e.g. Ref. 8). It is investigated here because of the availability of analytical 1D and 2D solutions (see Appendices B and C) which give direct insight into the system behaviour and because friction is of low importance at small time scales (transients) anyway.

![Fig 1](image)

**Fig 1** Ramp (continuous line) and oscillatory (dashed line) pressure excitation (relative to arbitrary static pressure) at upstream end of pipe.

4 RESULTS OF CALCULATIONS

The governing equations underlying the calculations are given in Ref. (1). The basic equation, Newton’s second law [Eqs. (9A, 9B) in Ref. (1)], is stated here without gravity, but with a laminar-flow friction-term and an apparent inertia factor $\beta$ that is introduced in Refs. (3) and (4):
\[ \beta \frac{\partial V}{\partial t} = -\frac{1}{\rho} \frac{\partial P}{\partial z} - \frac{32\bar{v}}{D^2} V, \]  

(2)

where \( \beta \frac{\partial V}{\partial t} \) should not be interpreted directly as a modified inertia term. Rather, \((\beta - 1) \frac{\partial V}{\partial t}\) is an approximate term that arises from velocity re-distribution in the cross section when the mean velocity changes. Schönfeld (Ref. 4) and others have shown that the greatest value of \( \beta - 1 \) occurs at lowest frequencies and that \( \beta - 1 \) tends to zero at the highest frequencies. The \( \beta \)-inertia should not be confused with \( \beta \)-momentum (although Schönfeld shows that they are equal in the limiting case of low-frequency laminar flow).

All quasi-steady and rigid-column solutions in the following discussion have been obtained analytically [see Appendix B, see also Ref. (9)] whereas all water-hammer and FSI solutions have been computed numerically (with the method of characteristics). In each figure, continuous lines represent solutions obtained with the most advanced model, dotted lines are solutions of the least advanced model, and dashed and dash-dot lines are in between.

4.1 Starting laminar flow (revisited)

In the first example, a pressure \textit{step} (ramp with \( \Delta t = 0 \)) of \( \Delta P = 20 \) bar is applied to accelerate the liquid column from rest to a final steady-state velocity of \( V_\infty = 2 \) m/s. The flow is assumed to remain laminar with paraboloid (Hagen-Poiseuille) velocity profiles assumed in steady, quasi-steady and 1D rigid-column behaviour. Figure 2 compares three 1D solutions (broken lines) with the 2D solution (continuous line) that includes the effect of non-paraboloid velocity profiles (see Appendix C). The time scale of the event (transient, homogeneous solution) is in the order of 0.1 second, which is in between \( t_{\text{rcl}} \) and \( t_{2D} \) (see Table 1), the time scale of the final steady state (constant, particular solution) is infinity, and the time scale of the excitation is zero. The 1D model predicts a too fast response to the pressure step when the factor \( \beta = 1 \) in Eq. (2), but a much better response when \( \beta = 4/3 \). The frictionless solution (dotted line) reaches \( V = 2 \) m/s at time \( t = 0.02 \) s, which is the same as \( t_{su} \). For fast excitation, with \( \Delta t = 0 \) as the extreme, elasticity effects are always important. Figure 3 compares the effect of water hammer when a pressure step is applied suddenly at the end \( z = 0 \) (continuous line) and when a pressure \textit{gradient} is (hypothetically) applied suddenly throughout \( 0 \leq z \leq L \) (dashed line) as in the rigid-column 1D and 2D solutions. In contrast to a sudden pressure at a pipe end, a sudden constant gradient does not generate significant water hammer.

![Fig 2](image-url)  

**Fig 2** Starting laminar flow driven by 20 bar instantaneous pressure rise: \textit{continuous line} = 2D analytical solution, cross-sectional average of velocity, Eq. (C2); \textit{lower dashed line} = rigid column (1D, \( \beta =4/3 \)); \textit{upper dashed line} = rigid column (1D, \( \beta =1 \)); \textit{dotted line} = frictionless rigid column (1D, \( \beta =1, \bar{v} =0 \)).
4.2 Oscillating laminar flow

In the second example, a harmonic pressure oscillation of period $T$ and amplitude $\Delta P = 2$ bar is applied at $z = 0$ (see Fig. 1) to accelerate and decelerate the liquid column from rest (or from any other initial state with $|V_0| \leq 0.2$ m/s and $P = 0$) to a steady oscillatory state with velocity amplitude $V_\infty$. For quasi-steady flow $V_\infty = 0.2$ m/s, for rigid-column oscillation $V_\infty < 0.2$ m/s, and for water hammer it is possible that $V_\infty > 0.2$ m/s. The time scale $t_{rc,l}$ characterises the rigid-column transient (App. D) and it defines the amplitude (Eq. B4) of the steady oscillation in which the only visible time scale is $T$. For the single tube with prescribed pressures at both ends $t_{wh} = 2 L \sqrt{\beta} / c$.

Figure 4 shows a rigid-column oscillation with a period $T = 1$ s, so that $T/t_{rc,l} = 50$. The oscillation is sufficiently slow for inertia not to be important. Friction dominates. The quasi-steady solution (dotted line) closely matches the rigid-column solutions (continuous and dashed lines). Note that $t_{rc,l} \ll T$ (see Table 1).
Figure 5 shows an oscillation with a shorter period $T = 0.1$ s, so that $T/t_{rc,1} = 5$ and $T/t_{wh} \approx 2.5$. In this example, the best approximate solution is the rigid-column analysis with friction (dashed line), showing that both friction and inertia have significant influence. Note that $t_{rc,1} < t_{wh} < T$. Also note that $\beta$ is taken as $4/3$ in the frictionless rigid-column analysis even though a truly inviscid wall-slip analysis would imply $\beta = 1$. This has been done to facilitate comparison with the rigid-column analysis with friction. The depicted water-hammer solution (continuous line, $\beta = 4/3$) is the velocity history at the midpoint at $z = L/2$; it is not constant between $t = 0$ and $t = L\sqrt{\beta}/(2c) \approx 0.01$ s, because of the non-equilibrium initial condition $V_0 \neq 0, P_0 = 0$. In the water-hammer solution, the axial distribution of pressure and velocity is not linear, but spatially oscillating. This explains the difference from the rigid-column solution, where the pressure gradient and velocity are spatially constant (but oscillating in time).

Figure 6a shows an oscillation with an even shorter period $T = 0.01$ s, so that $T/t_{wh} = 0.25$. In this oscillation, friction, inertia and elasticity are all found to be important. The frictionless water-hammer solution (dashed line, $\beta = 1.15$) develops a beat, which is suppressed by friction (continuous line), as shown in Fig. 6b. Very high amplitudes may occur when $2L\sqrt{\beta}/c$ is close to multiples of $T$ – because of resonance. The rigid-column solutions with (dash-dot line) and without (dotted line) friction are nearly the same, but neither is valid. Here $T < t_{wh}$.

---

Fig 5 Oscillating laminar flow driven by harmonic pressure of amplitude 2 bar and period 0.1 s and starting at $V_0 = -0.088$ m/s (Eq. D1) (1D, $\beta=4/3$); velocity at midpoint: continuous line = elastic column; dashed line = rigid column; dash-dot line = frictionless rigid column; dotted line = quasi-steady (massless rigid column).
Fig 6 Oscillating laminar flow driven by harmonic pressure of amplitude 2 bar and period 0.01 s and starting at \( V_0 = -0.014 \) m/s (Eq. D1) (1D, \( \beta = 1.15 \)); velocity at the midpoint: (a) continuous line = elastic column; dashed line = frictionless elastic column; dash-dot line = rigid column; dotted line = frictionless rigid column; (b) continuous line = elastic column; dashed line = frictionless elastic column.

Figure 7 shows oscillations with a yet shorter period \( T = 0.001 \) s, so that \( T/t_{wh} = 0.025 \) and \( T/t_{sh} = 0.0625 \). “Steady”-oscillatory amplitude behaviour is studied for water hammer with quasi-steady friction (Fig. 7a), water hammer with friction and Poisson coupling (Fig. 7b), and water hammer with friction, Poisson and junction coupling (Fig. 7c), all with \( \beta = 1 \). Note that \( T < t_{sh} < t_{wh} \) (see Table 1). Classical water-hammer theory predicts that the midpoint velocity oscillation grows in about 5 time intervals \( L \sqrt{\beta} / c = 0.1 \) s to its final amplitude of 0.34 m/s (Fig. 7a, left), which is higher than the 0.19 m/s in Fig. 6b (continuous line). The midpoint pressure amplitude stabilises at about \( 10^5 \) Pa (Fig. 7a, right), with a nodal point \( P = 0 \) at \( z = L \), this is half of the driving pressure amplitude at \( z = 0 \). Figure 7b shows the influence of Poisson coupling. The tube is structurally fixed at the excited end and stress-free at the other end so that junction coupling is absent. After \( t = 0.1 \) s the amplitude of the velocity oscillation gradually increases and has magnitude 0.42 m/s at \( t = 0.2 \) s (Fig. 7b, left). A slow beat develops (Ref. 10). The midpoint pressure amplitude stabilizes at about \( 10^5 \) Pa (Fig. 7b, right). Figure 7c shows results for a tube with a free-to-move closed end at \( z = L \), where junction coupling takes place. The closed end undergoes an oscillating axial motion, thereby acting on the liquid as a displacement piston. It is not a nodal point for the pressure anymore and as a result the midpoint pressure amplitude is more than 50% higher (Fig. 7c, right). On the other hand, the midpoint velocity amplitude is much lower at 0.23 m/s (Fig. 7c, left). Junction coupling is, as expected, a much stronger effect than Poisson coupling.

The FSI examples are rather unrealistic – because the wall thickness of the 0.8 mm diameter steel tube is 0.0073 mm to obtain the round value \( c = 1000 \) m/s – and the midpoint time histories are not more than an illustration. The tube accommodates 20 wave lengths \( l = cT = 1 \) m with 40 possible nodal (zero) points. The shown results are sensitive to the midpoint’s position relative to its nearest (moving) nodal point. The distribution of, and interplay between, kinetic energy (inertia) and potential energy (elasticity), in both liquid and tube, should be considered. Furthermore, a Fourier analysis of the calculated signals, or a full frequency-domain analysis, is the appropriate way to make effects like changed natural frequencies visible (Ref. 11). These aspects are beyond the scope of the present paper.
(7a) No FSI: no coupling and quasi-steady friction.

(7b) FSI: Poisson and friction coupling.

(7c) FSI: junction, Poisson and friction coupling.

4 CONCLUSIONS

Four illustrating examples of oscillating laminar flow driven at four different periods nicely show the decreasing influence of friction and the increasing influence of elasticity of liquid and pipe as the time scale of excitation reduces (Figs. 4-7). The smallest time scale considered is small enough for FSI Poisson coupling to be of some importance in forced oscillation, but less so than in free vibration (Ref. 1). FSI junction coupling is the stronger effect that cannot be ignored. The obtained results have been verified against the time scales defined in Table 1.

ACKNOWLEDGEMENT

The authors gratefully acknowledge financial assistance from the UK EPSRC Grant EP/CO15479/1.
REFERENCES


APPENDIX A Analytical 1D solutions in rigid-column theory for turbulent flow

The solutions are assumed to describe the transfer from one steady state to another: \( V(0) = V_0 \geq 0 \) defines the initial steady state, and \( \pm \Delta P \) defines the final steady state.

Positive constants used in the analytical solutions are:

\[ C_1 = \frac{\Delta P}{\rho L} > 0, \quad C_2 = \frac{\lambda}{2D} > 0. \]

The representative time scale is

\[ t_{\text{rc,t}} = \frac{1}{\sqrt{C_1C_2}} \frac{2D}{(\Lambda|V_0|)} = t_{\text{rc,t}} \quad (\text{see Table 2}). \]

Non-reversing 1D turbulent flow, step \( \Delta P \).

Solution of the equation:

\[ \frac{\partial V}{\partial t} = C_1 - C_2 V^2, \quad V(0) = V_0 \geq 0, \]

\[ V(t) = \frac{V_0}{1 + C_2 V_0 t}, \quad \text{with} \quad V_\infty = \frac{1}{C_1/C_2} > 0. \]

For \( V_0 = 0 \) this solution corresponds to the solution given by Wylie and Streeter (Ref. 12, Eq. 5-10, which misses the factor \( 1/(2V_\infty) \)). Note that \( V_\infty > V_0 \) if \( C_1 > C_2 V_0^2 \) or \( \Delta P_\infty = \Delta P > \Lambda (L/D) (\rho V_0^2 / 2) = \Delta P_0 \).

Stopping 1D turbulent flow, step \( \Delta P = 0 \).

Solution of the equation:

\[ \frac{\partial V}{\partial t} = -C_1 - C_2 V^2, \quad V(0) = V_0 \geq 0, \]

\[ V(t) = \frac{V_0}{1 - C_2 V_0 t}, \quad \text{with} \quad V_\infty = V_0. \]

Reversing 1D turbulent flow, step \( -\Delta P \). Solution given by Wylie and Streeter (Ref. 12, Eq. 5-6).

Solution of the equation:

\[ \frac{\partial V}{\partial t} = -C_1 - C_2 V^2, \quad V(0) = V_0 \geq 0, \]

\[ \frac{V(t)}{V_\infty} = \tan \left( \arctan \left( \frac{V_0}{V_\infty} \right) + \sqrt{C_1C_2} t \right) = \frac{V_\infty \sinh \left( \sqrt{C_1C_2} t \right) + V_0 \cosh \left( \sqrt{C_1C_2} t \right)}{V_\infty \cosh \left( \sqrt{C_1C_2} t \right) - V_0 \sinh \left( \sqrt{C_1C_2} t \right)}, \quad t \leq t_t, \]

\[ (A2a) \]
with $V_{\infty} = -\sqrt{C_1 / C_2} < 0$ and which is valid until flow reversal occurs at

t_r = -\arctan \left( V_0 / V_{\infty} \right) / \sqrt{C_1 C_2} .

After flow reversal the solution is Eq. (A1) with $V(t_r) = 0$:

$$V(t) = V_{\infty} \tanh \left( \sqrt{C_1 C_2} (t-t_r) \right), \quad t \geq t_r ,$$  \hspace{1cm} (A2b)

and with negative $V_{\infty} = -\sqrt{C_1 / C_2} < 0$.

Starting 1D turbulent flow, ramp $\Delta P / \Delta t$.

Solution of the equation: \[
\frac{\partial V}{\partial t} = \begin{cases} 
C_1 \frac{t}{\Delta t} - C_2 V^2, & 0 < t \leq \Delta t \\
C_1 - C_2 V^2, & V(\Delta t) = V(\Delta t), \quad t \geq \Delta t 
\end{cases}
\]

\[
\frac{V(t)}{V_{\infty}} = \begin{cases} 
\frac{I_2}{2} \left( \frac{2}{3} \sqrt{C_1 C_2} t \frac{t}{\Delta t} \right) \tanh \left( \frac{t}{\Delta t} \right), & 0 < t \leq \Delta t \\
\frac{I_{-1}}{3} \left( \frac{2}{3} \sqrt{C_1 C_2} t \frac{t}{\Delta t} \right) \tanh \left( \arctan \left( V(\Delta t) / V_{\infty} \right) + \sqrt{C_1 C_2} (t-\Delta t) \right), & t > \Delta t
\end{cases}
\]

with $V_{\infty} = \sqrt{C_1 / C_2} > 0$ and where $I_m$ is a modified Bessel function of the first kind of order $m$.

Non-reversing oscillating 1D laminar flow with driving pressure amplitude $\Delta P$ and circular frequency $\omega$. Solution of the equation: \[
\frac{\partial V}{\partial t} = C_1 \sin(\omega t) - C_2 V^2, \quad V(0) = V_0 \geq 0 ,
\]

not yet found by the authors, showing the difficulty of nonlinear terms.

Quasi-steady oscillating 1D laminar flow with driving pressure amplitude $\Delta P$ and circular frequency $\omega$. Solution of the equation:

\[
0 = C_1 \sin(\omega t + \gamma) - C_2 V |V|, \quad V(0) = V_0 \geq 0 ,
\]

\[
V_{qs}(t) = \begin{cases} 
\frac{C_1}{C_2} \sqrt{\sin(\omega t + \gamma)} \quad & \text{if } 0 \leq \omega t + \gamma < \pi (\text{mod.} 2\pi) \\
- \frac{C_1}{C_2} \sqrt{\sin(\omega t + \gamma)} \quad & \text{if } \pi \leq \omega t + \gamma < 2\pi (\text{mod.} 2\pi)
\end{cases}
\]

where $\gamma = \arcsin \left( \frac{C_2 V_0^2}{C_1} \right)$.

**APPENDIX B** Analytical 1D solutions in rigid-column theory for laminar flow

The solutions are assumed to describe the transfer from one steady state to another:

$V(0) = V_0 \geq 0$ defines the initial steady state, and $\pm \Delta P$ defines the final steady state. Positive constants used in the analytical solutions are:

$C_1 = \frac{\Delta P}{\rho L} > 0$, \quad $C_3 = \frac{32 \bar{v}}{D^2} > 0$. 

The inertia factor in Eq. (2) is included after dividing both $C_1$ and $C_3$ by $\beta$. The representative time scale is $1/C_3 = t_{re,1}$ (see Table 1).

Non-reversing 1D laminar flow, step $\Delta P$.
Solution of the equation: \[ \frac{\partial V}{\partial t} = C_1 - C_3 V, \quad V(0) = V_0 \geq 0, \]
\[ V(t) = V_\infty + (V_0 - V_\infty) e^{-C_1 t}, \] (B1)
with $V_\infty = C_1 / C_3 > 0$.

Stopping 1D laminar flow, step $\Delta P = 0$.
Solution of the equation: \[ \frac{\partial V}{\partial t} = -C_3 V, \quad V(0) = V_0 \geq 0, \]
\[ V(t) = V_0 e^{-C_3 t}, \] (B1a)
with $V_\infty = 0$.

Reversing 1D laminar flow, step $-\Delta P$.
Solution of the equation: \[ \frac{\partial V}{\partial t} = -C_1 - C_3 V, \quad V(0) = V_0 \geq 0, \]
\[ V(t) = V_\infty + (V_0 - V_\infty) e^{-C_1 t}, \] (B2)
with $V_\infty = -C_1 / C_3 < 0$ and which is also valid after flow reversal occurred at $t_{r} = \frac{1}{C_3} \ln \left(1 - V_0/V_\infty\right)$.

Except for the sign of $V_\infty$, the solutions (B2) and (B1) and (B1a) are the same.

Starting 1D laminar flow, ramp $\Delta P / \Delta t$.
Solution of the equation: \[ \frac{\partial V}{\partial t} = \begin{cases} C_1 \frac{t}{\Delta t} - C_3 V, & V(0) = 0, \quad 0 < t \leq \Delta t \\ C_1 - C_3 V, & V(\Delta t) = V(\Delta t), \quad t \geq \Delta t \end{cases}, \]
\[ V(t) = \begin{cases} \left(\frac{t}{\Delta t} - \frac{1}{C_3 \Delta t} (1 - e^{-C_1 t})\right) & \text{if } t \leq \Delta t \\ 1 + (V(\Delta t) V_\infty - 1) e^{-C_1 (t - \Delta t)} & \text{if } t > \Delta t \end{cases}, \] (B3)
with $V_\infty = C_1 / C_3 > 0$.

Oscillating 1D laminar flow with driving pressure amplitude $\Delta P$ and circular frequency $\omega$. Solution of the equation: \[ \frac{\partial V}{\partial t} = C_1 \sin(\omega t) - C_3 V, \quad V(0) = V_0, \]
\[ V(t) = V_0 e^{-C_3 t} + \frac{C_1 C_3}{\omega^2 + C_3^2} \sin(\omega t) - \frac{C_1 \omega}{\omega^2 + C_3^2} \left(\cos(\omega t) - e^{-C_3 t}\right) \]
\[ = \left(V_0 + \frac{C_1 \omega}{C_3} \frac{1}{1 + (\omega / C_3)^2}\right) e^{-C_3 t} - \frac{C_1}{C_3} \frac{1}{\sqrt{1 + (\omega / C_3)^2}} \cos \left(\omega t + \arctan\left(\frac{C_3}{\omega}\right)\right), \]
which is also valid for $C_3 = 0$.  

\[ V_{qs}(t) = \frac{C_1}{C_3} \sin \left( \omega t + \arcsin \left( \frac{C_3}{C_1} V_0 \right) \right). \]  

\[ \lambda_n = -0.635 + 3.131 n. \]

APPENDIX C  Analytical 2D solutions in rigid-column theory for laminar flow

The constants $C_1$ and $C_3$ are defined in Appendix B, and the first 21 values of $\lambda_n$ ($n^{th}$ zero of $J_0$) used herein are:

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$\lambda_8$</th>
<th>$\lambda_{15}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.40482555769577</td>
<td>24.3524715307493</td>
<td>46.3411883716618</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>$\lambda_9$</td>
<td>$\lambda_{16}$</td>
</tr>
<tr>
<td>5.52007811028631</td>
<td>27.4934791320403</td>
<td>49.4826098973978</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>$\lambda_{10}$</td>
<td>$\lambda_{17}$</td>
</tr>
<tr>
<td>8.65372791291101</td>
<td>30.6346064684320</td>
<td>52.6240518411150</td>
</tr>
<tr>
<td>$\lambda_4$</td>
<td>$\lambda_{11}$</td>
<td>$\lambda_{18}$</td>
</tr>
<tr>
<td>11.7915344390143</td>
<td>33.7758202135736</td>
<td>55.7655107550200</td>
</tr>
<tr>
<td>$\lambda_5$</td>
<td>$\lambda_{12}$</td>
<td>$\lambda_{19}$</td>
</tr>
<tr>
<td>14.9309177084878</td>
<td>36.9170983536640</td>
<td>58.9069839260809</td>
</tr>
<tr>
<td>$\lambda_6$</td>
<td>$\lambda_{13}$</td>
<td>$\lambda_{20}$</td>
</tr>
<tr>
<td>18.0710639679109</td>
<td>40.0584257646282</td>
<td>62.0484691902272</td>
</tr>
<tr>
<td>$\lambda_7$</td>
<td>$\lambda_{14}$</td>
<td>$\lambda_{21}$</td>
</tr>
<tr>
<td>21.2116366298793</td>
<td>43.1997917131767</td>
<td>65.1899644802069</td>
</tr>
</tbody>
</table>

For these 21 values, approximately: $\lambda_n = -0.635 + 3.131 n$.

Starting 2D laminar flow, step $\Delta P$. Solution given by Boussinesq (Ref. 13), Szymanski (Ref. 14, Eq. 33) and Schönfeld (Ref. 4, Eq. 31), see White (Ref. 15). Solution of the equation:

\[ \frac{\partial v}{\partial t} = C_1 + \nu \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right), \quad v(r, 0) = 0, \quad \nu \left( \frac{D}{2} \right) = 0, \quad |v(0, t)| < \infty, \]

\[ v(r, t) = 2V_\infty \left\{ 1 - \left( \frac{2r}{D} \right)^2 \right\} - 8 \sum_{n=1}^{\infty} \frac{\lambda_n^2}{\lambda_n^3 J_1(\lambda_n)} \text{e}^{-\frac{1}{8} \lambda_n^2 C_1 t}. \]  

\[ \text{(C1)} \]

and

\[ V(t) = V_\infty \left( 1 - 32 \sum_{n=1}^{\infty} \frac{\text{e}^{-\frac{1}{8} \lambda_n^2 C_1 t}}{\lambda_n^4} \right), \]  

\[ \text{(C2)} \]

with $V_\infty = C_1 / C_3 > 0$.

Steady oscillating 2D laminar flow, amplitude of driving pressure is $\Delta P$. Solution given by Crandall (Ref. 16), Grace (Ref. 17), Sexl (Ref. 18) and Schönfeld (Ref. 4, Eq. 14), see White (Ref. 15). Solution of the equation:

\[ \frac{\partial v}{\partial t} = C_1 \sin(\omega t) + \nu \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right), \quad \nu \left( \frac{D}{2} \right) = 0, \quad |v(0, t)| < \infty, \]
\[ v(r,t) = \text{Im} \left\{ \frac{C_1}{i\omega} e^{i\omega t} \left[ 1 - \frac{J_0 \left( \frac{2r}{D \sqrt{C_3}} \right)}{-8i\omega \sqrt{C_3}} \right] \right\} \]  
\[ V(t) = \text{Im} \left\{ \frac{C_1}{i\omega} e^{i\omega t} \left[ 1 - \frac{2J_1 \left( -8i\omega \sqrt{C_3} \right)}{\sqrt{-8i\omega \sqrt{C_3}}} \right] \right\} = C_1 \text{Im} \left\{ \frac{1}{i} e^{i\omega t} \left[ a(\omega) + ib(\omega) \right] \right\} \]

APPENDIX D  Rigid-column transient

Rigid-column transients, shown in Fig. 8, occur in the 1D model because the initial velocity \( V_{0,1D} \) in general does not match the steady oscillatory velocity at \( t = 0 \). Only if

\[ V_{0,1D}(\omega) = -\frac{C_1 \omega \beta(\omega)}{(\omega \beta(\omega))^2 + C_3^2}, \]  

where \( \omega = 2\pi / T, C_1 = \Delta P / (\rho L) = 10 \text{ m/s}^2 \) and \( C_3 = 32 \tilde{v} / D^2 = 1 / T_{e,1} = 50 \text{ Hz} \), see Eq. (B4), is the flow steady oscillatory from \( t = 0 \) on. Equation (D1) describes the phase difference between driving pressure-gradient and trailing velocity. Although rigid-column transients make systems numerically stiff, they are not the focus of the present study and in Section 4.2 the initial velocities are chosen according to Eq. (D1) (with \( P_0(z) = 0 \) in the water-hammer calculations). The time scale of the event in Fig. 8 (transient, homogeneous solution) is of the order of 0.1 second, which is in between \( t_{e,1} \) and \( t_{2D} \) (see Table 1).

![Fig 8](image-url) Oscillating laminar rigid-column (1D, \( \beta=4/3 \)) driven by harmonic pressure of amplitude 2 bar and period 1 s, for different initial velocities: \( V_0 = -0.0326 \text{ m/s} \) (Eq. D1) (continuous line); \( V_0 = 0.05 \text{ m/s} \) (dashed line); \( V_0 = 0.1 \text{ m/s} \) (dash-dot line); \( V_0 = 0.2 \text{ m/s} \) (dotted line).

The initial velocity \( V_{0,2D} \) in the 2D solution is \( V(0) \) given by Eq. (C4) as

\[ V_{0,2D}(\omega) = -\frac{C_1}{\omega} a(\omega). \]  

(D2)
Figure 9 shows $V_{0,1D}$ and $V_{0,2D}$ as a function of the dimensionless frequency $\alpha = \sqrt{8/\omega C_3}$. The independently obtained initial velocities $V_{0,1D}$ (Eq. D1 combined with either Eq. E3 or Eq. E4) and $V_{0,2D}$ (Eq. D2) are nearly the same.

**Figure 9** Initial mean velocity $V(0) = V_0(\omega)$ as a function of dimensionless frequency $\alpha$. 
*Continuous line*: $V_{0,2D}$, Eq. (D2); *dotted line*: $V_{0,1D}$, Eq. (D1) with $\beta$ from Eq. (E3); *dashed line*: $V_{0,1D}$, Eq. (D1) with $\beta$ from Eq. (E4). The dotted and dashed lines almost coincide in this graph.

**APPENDIX E  Inertia factor $\beta$**

The apparent inertia factor $\beta$ in Eq. (2) accounts for non-paraboloid velocity profiles in the pipe cross-section. The velocity profile determines the resistance (the shear stress at the wall) experienced by the liquid column. The factor $\beta$ in oscillatory flow depends on the dimensionless time $Tl_{c,1}$. Figure 10 shows steady rigid-column oscillation of period $T = 0.01$ s. The continuous line is the 2D analytical solution, Eq. (C4) (steady oscillatory, particular solution). The three other lines are 1D solutions, Eq. (B4), for three different values of $\beta$. The amplitude of each 1D solution is

$$V_{\infty,1D}(\omega) = \frac{C_1}{C_3} \frac{1}{\sqrt{1 + (\omega \beta(\omega)/C_3)^2}}.$$  \hfill (E1)

Figure 10 shows that, for this particular oscillation, the best amplitude fit (with the 2D solution) is obtained for $\beta \approx 1.15$. The circles in Fig. 11 show the corresponding best-fit values obtained in a similar manner for other values of $T = 2\pi/\omega$, varying $\beta$ in steps of 0.05. For slow oscillations, $\beta \approx 1.35$, and for fast oscillations, $\beta \approx 1$. For slow oscillations, the velocity profiles remain close to a paraboloid and, for fast oscillations, they remain close to uniform (except in the Stokes layer near to the wall). Schönfeld (Ref. 4) has demonstrated that the theoretical limit ratio $\beta_{\text{slow}}/\beta_{\text{fast}}$ is equal to the steady-state momentum-correction factor (4/3) [Boussinesq (Ref. 13) or Coriolis (Ref?) coefficient]. See also Ref. 3 for $\beta_{\text{slow}} = 4/3$.

A theoretical expression for $\beta(\omega)$ is obtained by setting the amplitude of the 1D solution (Eq. E1) equal to the amplitude of the 2D solution (Eq. C4),

$$V_{\infty,2D}(\omega) = \frac{C_1}{\omega} \sqrt{a^2(\omega) + b^2(\omega)}.$$  \hfill (E2)

where $a(\omega)$ and $b(\omega)$ are the real and imaginary parts of the expression between square brackets in Eq. (C4), respectively. Solving $V_{\infty,1D} = V_{\infty,2D}$ for $\beta$ yields
\[ \beta^2(\omega) = \frac{1}{a^2(\omega) + b^2(\omega)} \left( \frac{C_3}{\omega} \right)^2 \left\{ \left[ \frac{V_{\infty,1D}(0)}{V_{\infty,2D}(\omega)} \right]^2 - \left[ \frac{V_{\infty,2D}(\omega)}{V_{\infty,2D}(\omega)} \right]^2 \right\}. \]  

(E3)

This expression, plotted as \( \beta(T) \) in Fig. 11 (dashed line), has limit values \( \beta^2(0) = 17/9 \) and \( \beta^2(\infty) = 1 \). The low-frequency limit differs from the expected \( (4/3)^2 = 16/9 \). The reason is that Eq. (E3) is based on matching amplitudes (or cycle averages of \( V^2 \)) instead of minimising cycle averages of \( (V_{1D} - V_{2D})^2 \). The latter approach gives:

\[ \beta^2(\omega) = \left( \frac{a(\omega)}{a^2(\omega) + b^2(\omega)} \right)^2 \left( \frac{C_3}{\omega} \right)^2 \left\{ \left[ \frac{V_{\infty,1D}(0)}{V_{\infty,2D}(\omega)} \right]^2 \right\}. \]  

(E4)

This expression, plotted as \( \beta(T) \) in Fig. 11 (continuous line), has limit values \( \beta^2(0) = 16/9 \) and \( \beta^2(\infty) = 1 \). Figure 12 shows the factor \( \beta \) given by the Eqs. (E3) and (E4) as a function of the dimensionless frequency \( \alpha = \sqrt{8\omega/C_3} \). This figure is in agreement with Fig. 1 in Ref. (19). Note that for low frequencies \( T > 0.1 \text{ s} \) the influence of 2D effects (unsteadiness, apparent inertia) is so small that the exact value of \( \beta \) [4/3 or \( \sqrt{17}/3 \)] is unimportant.

Figure 13 shows the amplitudes of oscillation given by the Eqs. (E1) and (E2) as a function of the dimensionless frequency \( \alpha \) for different values of \( \beta \). For all \( \beta \) the 1D and 2D amplitudes agree well in absolute sense (Figs. 13a and 13b), but less so in relative sense (Fig. 13c).

In summary, the inertia correction factor \( \beta \) is 4/3, or \( \sqrt{17}/3 \), for slow oscillations, where velocity profiles remain close to paraboloid, and 1 for fast oscillations, where the velocity profiles become close to uniform. The in-between values shown in Figs. 11 and 12 are given by the newly derived Eq. (E4), or alternatively Eq. (E3), which is a useful expression for analysis in the frequency domain.

**Fig 10** Oscillating laminar flow driven by harmonic pressure of amplitude 2 bar and period 0.01 s and starting at \( V_0 \) according to Eq. (D1): *continuous line* = 2D analytical solution, cross-sectional average of velocity, Eq. (C4); *dashed line* = rigid column (1D, \( \beta = 4/3 \)); *dash-dot line* = rigid column (1D, \( \beta = 1.15 \)); *dotted line* = rigid column (1D, \( \beta = 1 \)).
Fig 11  Inertia factor $\beta$ as function of period of oscillation $T$. *Dashed line*: theoretical curve, Eq. (E3); *continuous line*: theoretical curve, Eq. (E4); *circles*: visual checks, like in Fig. 10.

Fig 12  Inertia factor $\beta$ as function of dimensionless frequency $\alpha$. *Dashed line*: theoretical curve, Eq. (E3); *continuous line*: theoretical curve, Eq. (E4).

Fig 13a  Velocity amplitude $V_\infty(\omega)$ as function of dimensionless frequency $\alpha$. *Continuous line*: $V_{\infty,2D}$, Eq. (E2); *dotted line*: $V_{\infty,1D}$, Eq. (E1) with $\beta=4/3$; *dashed line*: $V_{\infty,1D}$, Eq. (D1) with $\beta=1$. The continuous and dotted lines almost coincide in this graph.
Fig 13b  Velocity amplitude $V_{\infty}(\omega)$ as function of dimensionless frequency $\alpha$. 

Continuous line: $V_{\infty,2D}$, Eq. (E2); dotted line: $V_{\infty,1D}$, Eq. (E1) with $\beta$ from Eq. (E3); 
dashed line: $V_{\infty,1D}$, Eq. (D1) with $\beta$ from Eq. (E4). All lines almost coincide in this graph.

Fig 13c  Relative velocity amplitude $V_{\infty,1D} / V_{\infty,2D}$ as function of dimensionless frequency $\alpha$. Horizontal dash-dot line: $V_{\infty,1D}$, Eq. (E1) with $\beta$ from Eq. (E3); continuous line: $V_{\infty,1D}$, Eq. (D1) with $\beta$ from Eq. (E4); dotted line: $V_{\infty,1D}$, Eq. (E1) with $\beta=4/3$; 
dashed line: $V_{\infty,1D}$, Eq. (D1) with $\beta=1$. 