Entropy and power spectrum of asymmetrically DC-constrained binary sequences

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possible rational capacities is given by \(1/4, 1/2, 3/4\). A rate
3/4 dc- and Nyquist-free code demonstrates the ability to code
at the highest rational capacity available.

References
1986.
[7] A. Galoppoulos, C. Heegard, and P. Siegel, "The power spectrum of
finite-state machines generating certain asymmetrically dc-constrained
digital magnetic recording. The eigenstructure is used to calculate the
entropy is given by

\[ H = \sum_{k=1}^{n} \alpha_k \text{log}_2 \alpha_k, \]

where \( p = \frac{1}{4}, \ldots, 1 \). These matrices arise as follows. Suppose we
have a binary sequence \( s_n = \pm 1 \), a rational number \( a = q/p \) \( \in [0,1] \) with \( \gcd(q,p) = 1 \), and a \( b > 0 \) such that the constraint

\[ t_n = \sum_{k=1}^{n} r_k - n a \in [-b, b], \quad n \in \mathbb{Z} \]  

(1.2)
is satisfied. We associate with \( t_n \) the finite-state machine with
states \( l/p, \ |l| \leq pb \). From state \( t_n = l/p \) state \( t_{n+1} = \frac{l \pm p - q}{p} \) can be reached, provided that \( |l \pm p - q| \leq pb \),
according as \( s_{n+1} = \pm 1 \). This gives rise to the skeleton matrix

\[
\begin{pmatrix}
0 & \cdots & 1 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
0 & \cdots & 0 & 1 & \cdots & 0 & 1 & \cdots & 0 \\
1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
0 & \cdots & 0 & 1 & \cdots & 0 & 1 & \cdots & 0 \\
\end{pmatrix}_{M+1} \cdot (1.3)
\]

where \( M = 2 \lfloor pb \rfloor \), of which (1.1) is the special case with \( q = p - 1 \).
From the largest eigenvalue \( \lambda_0 \) of \( A \) the maximum Shannon
entropy of the finite-state machine is found as \( \log_2 \lambda_0 \) and the transition matrix \( B \) for which the machine yields the maximum
entropy is given by

\[ B = \left( B_i \right)_{i=1}^{M+1}, \quad \lambda_0 = \lambda^M \left( \frac{P_i}{P_i} A_{ij} \right)_{i=0}^{M-1}, \]

(1.4)

where \( p = [p_0, \ldots, p_M]^T \) is the right eigenvector of \( A \) corresponding to \( \lambda_0 \). It will turn out that the left eigenvector of \( B \) corresponding to the eigenvalue 1 is a multiple of

\[ q = [p_0, p_1, \ldots, p_M]^T. \]

(1.5)

This vector \( q \) is the vector of stationary probabilities. Finally,
the power spectrum of the process \( t_n \) generated by the finite
state machine with transition matrix \( B \) is proportional to

\[ \mathcal{S}_n(\theta) = \sum_{k=-N}^N R(k)e^{-2\pi i k\theta}, \]

(1.6)

where

\[ R(k) = u^{*} F B^* u, \quad k = 0, 1, \ldots, \]

(1.7)

with \( F = \text{diag}(q_0, q_1, \ldots, q_M) \) and \( u = 1/p(-m, \ldots, m, 1, \ldots, m)^T \) \( m = \lfloor pb \rfloor \). The power spectrum \( S_n(\theta) \) of the process
\( s_n = t_n - t_{n-1} + \alpha \) (the original binary data) is then given (since \( E\delta_0 = 0 \)) by

\[ S_n(\theta) = \alpha^2 \mathcal{S}_n(\theta) + 4 \mathcal{S}_n(\theta) \sin^2 \pi \theta. \]

(1.8)

For the basic results concerning finite-state machines and their
spectra we refer to [1], [4].

The main results of this correspondence are Theorem 1 and
Theorem 2. In Theorem 1 all information on the eigenstructure
of \( A \) is collected. This theorem gives rise to relatively simple
computational schemes for the largest eigenvalue \( \lambda_0 \) of \( A \),
the vectors \( p \) and \( q \) previously discussed and the power spectrum
\( \mathcal{S}_n(\theta) \) in (1.6). For the calculation of the power spectrum \( \mathcal{S}_n(\theta) \) this
computational scheme has been worked out in Section IV.
It turns out that $S_j(\theta)$ can be decomposed as (with $M+1=2pN + R$)

$$S_j(\theta) = S_j(\theta) + \sum_{l=1}^{N-1} S_l(\theta) + S_{a_l}(\theta). \quad (1.9)$$

Here $S_j(\theta)$ is the discrete component with spectral lines at $\theta = 1/2p, 2/2p, \cdots, (2p-1)/2p$ (no spectral line at $0$) that are due to the eigenvalues of $A$ of largest modulus (excluding the largest positive eigenvalue). Furthermore, the $S_l(\theta)$, $l = 1, \ldots, N-1$, and $S_{a_l}(\theta)$ constitute the continuous component of $S_j(\theta)$ and are due to the 2p eigenvalues of $A$ of $l$th largest modulus and the $R$-fold eigenvalue 0 of $A$, respectively. These results can be considered as generalizations of these obtained by Chien [2], Justesen [4, Section V, Example 7], and Kerpez [5], in the sense that we consider certain nonintegral $b$ in (1.2). Unfortunately, we do not see how our approach can be extended to the more general matrices in (1.3).

An alternative way to evaluate $S_j(\theta)$ would be by using the general result in [2]. For this one needs the cyclic structure of $A$. Although this structure can be determined, and is even rather simple in this case, the resulting formulas for $S_j(\theta)$ are still less explicit than the ones we obtain.

We conclude this introduction by briefly mentioning a possible application of our results to pilot tracking tones in digital magnetic recording. In digital magnetic recorders part of the low-frequency tone, usually called pilot tone, see [6], is used to store position information. This information is often recorded as a low-frequency tone away from the multiples to avoid interference with the data. This information can then be used to drive back the head to the middle of the track. Since we are dealing with binary data, the obvious technique of adding a sinusoidal waveform to the data cannot be applied. Instead, one can create the effect of a block wave by storing over intervals of large, fixed length, alternately a surplus and a deficit of positive symbols. The frequency of the block wave can be varied by appropriate choice of the interval lengths.

Two relevant questions to be answered are the following. 1) How is storage capacity decreased by including servo position information? 2) How should one choose the frequencies of the pilot tones so as to avoid interference of pilot tones and data? This correspondence addresses these questions under the assumptions that the storage of the surplus/deficit of positive symbols is in accordance with (1.2) and that the data can be stored in accordance with the maxentropic process associated with the transition matrix $B$ in (1.4). Here the effect on the data spectrum of switching from the surplus mode to the deficit mode (and vice versa) has been neglected. It turns out that the maxentropic process gives rise to spectral lines at integer multiples of $1/2p$. Hence, one should choose the pilot tone frequencies away from the multiples to avoid interference of pilot tones and the data.

II. EIGENSTRUCTURE OF $A$

In this section we determine the eigenstructure of the matrix $A$ in (1.1). Throughout we write $M+1=2pN + R$, $m=2p+r$, where $R,r=0,1,\cdots,2p-1$ and $N,l=0,1,\cdots,\cdots$ We define polynomials $P_m, m=0,1,\cdots$, according to the recursion

$$P_{m+1} = \lambda P_m - P_{m-2p}, \quad m=0,1,\cdots \quad (2.1)$$

with the initialization $P_0 = 1$, $P_1 = 0$, $k < 0$, also see [3]. Since

$$\left\{\begin{array}{c}
\lambda (A - \lambda I) \\
\lambda \\
\lambda \\
\lambda \\
\lambda
\end{array}\right. = \left\{\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
0
\end{array}\right. \quad (2.2)$$

it is seen that $x \in \mathbb{C}^{m+1}$ is an eigenvector of $A$ corresponding to $\lambda \in \mathbb{C}$ with nonvanishing 0th component if and only if $x$ is a multiple of $[P_m, \cdots, P_0]$ with $\lambda$ one of the zeros of $P_{m+1}$.

Proposition 1: Define for $m=2p+r$

$$f_{l,r}(z) = \sum_{k=0}^{l} (2p-1)k + l + r \quad (2.3)$$

Then

$$P_m = \lambda^l f_{l,r}(\lambda^{2p}). \quad (2.4)$$

Proof: We have for the $P_m$'s the generating function

$$F(z, \lambda) = (1 - \lambda^r + z^{2p})^{-1} = \sum_{m=0}^{\infty} P_m(z)\lambda^m, \quad (2.5)$$

which follows easily from (2.1). Then (2.4) follows on equating the coefficients of $\lambda^m$ in (2.5).

Proposition 2: All $f_{l,r}$ have $l$ distinct positive zeros $\sigma_{0,l,r}, \sigma_{1,l,r}, \cdots \sigma_{1-l,r}$

Moreover,

$$\sigma_{0,l,r+1} > \sigma_{0,l,r} > \sigma_{1,l,r+1} > \sigma_{1,l,r} > \cdots \quad (2.6)$$

when $r=0,1,\cdots,2p-2$, and

$$\sigma_{0,l,r+1} = \sigma_{0,l,r} > \sigma_{1,l,r+1} > \sigma_{1,l,r} > \cdots \quad (2.7)$$

for $f_{l,r}$ according to (2.4). Note that $f_{l,r}(\infty) > 0$ for all $l, r \geq 0$.

The statements of the proposition and inequalities (2.6) and (2.7) follow from induction with respect to $m = 2p + r$.

Proposition 2 implies that the zeros of $f_{l,r}$ have an interlacing property and that $\sigma_{l,r}$ strictly increases in $m = 2p + r$.

Proposition 3: The nonzero eigenvalues of $A$ are

$$\lambda_{2p+r} = \alpha^{l/2p} e^{2\pi m/2p}, \quad l=0,1,\cdots, N-1, \quad r=0,1,\cdots,2p-1, \quad (2.9)$$

and

$$\lambda_{2p+r} = (P_m(\alpha^{l/2p}) e^{2\pi i m/2p})_{m=0,1,\cdots, M} \quad (2.10)$$

is the eigenvector of $A$ corresponding to $\lambda_{2p+r}$. Here, $\alpha_{l} = \sigma_{l,0,r}$ is the $l$th largest zero of $f_{l,r}$. (Proof: This follows easily from the preceding results.)

Regarding the eigenvalue 0 of $A$ we have the following result.
Proposition 4: Suppose \( R > 0 \) and put
\[
y_r = \left[ P_y(s)'(0), \cdots, P_y(s)'(M) \right]^T, \quad r = 0, 1, \cdots, R - 1.
\]  
(2.11)

Then
\[
A_0 y_0 = 0, \quad A_r y_r = y_{r-1}, \quad r = 1, \cdots, R - 1.
\]
(2.12)

Proof: Let \( r = 0, 1, \cdots, R - 1 \), differentiate (2.2) \( r \) times and set \( \lambda = 0 \). Since \( P_{M+1} \) has an \( R \)-th-order zero at \( 0 \) we obtain (2.12).
\( \square \)

Note: We have
\[
y_r = P_y(s)'(0) = \left[ \begin{array}{cccc} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \end{array} \right]
\]
(2.13)

which shows e.g., that the \( y_r \) of Proposition 4 are orthogonal.

Proposition 5: Denote
\[
x = \left[ x_1, \cdots, x_{2Np-1} \right], \quad y = \left[ y_1, \cdots, y_{R-1} \right]
\]
(2.14)

with \( x_m \) and \( y_r \) given in Propositions 3 and 4, respectively, and let \( \Pi \) be the \((M+1) \times (M+1) \) matrix defined by
\[
z = \left[ z_0, z_1, \cdots, z_M \right]^T \in \mathbb{C}^{M+1}.
\]

Then we have \( X^T \Pi Y = 0 \), and

\[
\Gamma = X^T \Pi X = \left[ \begin{array}{cccc} y_0 & 0 & \cdots & 0 \\ 0 & y_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_{2Np-1} \end{array} \right],
\]
(2.15)

\[
\Delta = Y^T \Pi Y = \left[ \begin{array}{cccc} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_1 & \cdots & 0 \\ \alpha_0 & 0 & \cdots & 0 \end{array} \right],
\]
(2.16)

with
\[
y_m = P_{M+1}(\lambda_m), \quad \alpha_r = (-1)^r (R - 1 - r) R! \left( \begin{array}{c} R + N \\ N \end{array} \right).
\]
(2.17)

Proof: We have \( \Pi \Pi^T = A^T \). Hence, for \( 0 \leq m \leq 2Np-1, 0 \leq r \leq R - 1 \)
\[
x_m \Pi^T = \lambda_m x_{m+1} \Pi y_r.
\]
(2.18)

This implies that \( X^T \Pi Y = 0 \). In what follows \( C_{ij} \) abbreviates "coefficient of \( z^i \) in \( z^j \)". We have for \( \lambda \neq \mu \) by (2.5)
\[
\sum_{i=0}^{M} P_i(\lambda) P_{M-i}(\mu) = C_{ij} \left( \frac{1}{1 - \lambda z + 2\mu z} \right),
\]
(2.19)

\[
\sum_{i=0}^{M} P_i(\lambda) P_{M-i}(\mu) = \frac{P_{M+1}(\lambda) - P_{M+1}(\mu)}{\lambda - \mu}.
\]
(2.20)

When we take \( \lambda = \lambda_m, \mu = \lambda_m \) with \( m \neq n \), the left-hand side of (2.18) becomes \( x_m^T \Pi x_n \), while the right-hand side vanishes since
\[
P_{M+1}(\lambda_m) - P_{M+1}(\lambda_m) = 0.
\]

And when we take \( \lambda = \lambda_m \) and \( \mu = \lambda_m \), we obtain \( x_m^T \Pi x_m = P_{M+1}(\lambda_m) \). This proves the claims about \( X^T \Pi X \). To prove the claims about \( Y^T \Pi Y \) we can use (2.13). Alternatively, we have as

In (2.18)
\[
\sum_{i=0}^{M} P_i(\lambda) P_{M-i}(\mu) = \frac{R!}{(1 - \lambda z + 2\mu z)^{R+M+1}}.
\]
(2.21)

Note: Setting \( \lambda = 0 \) and recalling that \( M = 2Np + R - 1 \) we easily get our result.

The next theorem summarizes the results of this section.

Theorem 1: Let \( S = [X|Y] \) with \( X, Y \) as in (2.14). Then
\[
S^{-1}AS = \left[ \begin{array}{cccc} \Lambda_0 & 0 & \cdots & 0 \\ 0 & \Lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda_{2Np-1} \end{array} \right] = \Lambda,
\]
(2.22)

where
\[

\begin{align*}
\Lambda_0 &= x_0 \left[ \begin{array}{cccc} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{array} \right] \\
\Lambda_1 &= x_1 \left[ \begin{array}{cccc} 0 & \alpha_0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_1 & \cdots & 0 \end{array} \right] \\
\Lambda_{2Np-1} &= x_{2Np-1} \left[ \begin{array}{cccc} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_0 & 0 & \cdots & 0 \end{array} \right]
\end{align*}
\]
(2.23)

with \( \Lambda_m \) given in Proposition 3. Moreover, the inversion of \( S \), which is needed for efficient calculation of \( A^k \), \( k = 0, 1, \cdots \), as required in Section IV, presents no difficulty by formula (2.22).

III. Finite State Machine with Maximum Entropy

In this section we make some comments on the entropy and vector of stationary probabilities of the maxentropic process associated with the skeleton matrix \( A \) in (1.1). The entropy of this process is \( \log_{10} \eta(A) \) where we have explicitly indicated the dependence of the largest eigenvalue of \( A \) on \( M \), and it has been shown in [3] that \( \log_{10} \eta(A) \) increases as a function of \( M \) to the limit
\[
\log_{10} \eta(A) = \frac{2p - 1}{2p} \log_{10} 2p - \frac{1}{2p} \log_{10} \frac{1}{2p}.
\]
(3.1)

We observe that the right-hand side of (3.1) is the entropy of a binary channel generating independent \( \pm 1 \)'s with probabilities \( \frac{p}{2p + (p - 1)/2p} \) and average \( \mu = (p - 1)/p \). We have more

1After completion of this work, the authors became aware of a theorem of M. Biernacki, quoted in P. Schmidt, F. Spitzer, Math. Scand., vol. 8, Sect. 7, pp. 15–38, 1960, that shows that a corresponding limit formula holds for the more general matrices in (1.3).
precisely, according to [3], the third-order approximation
\[
\lambda_0'' = \frac{1 - \pi^2(2p-1)}{2(M+2)^2}
\] (3.2)
to \(\lambda_0''(M)\), and it can also be shown from [3], (3.20), that, stated somewhat imprecisely, the asymptotic form of the eigenvector \(p = [p_0, p_1, \cdots, p_M]^T\) corresponding to \(\lambda_0''(M)\) is given by
\[
\left(2p-1\right)^{\pi/2}/2\sin\left(\frac{\pi(m+1)}{M+2}\right)_{m=0, \cdots, M}, \quad M \to \infty. \quad (3.3)
\]
Observe that for the case \(p=1\) the vector in (3.3) is exactly the eigenvector of \(A\) corresponding to the largest eigenvalue (see [2])
\[
2\cos\frac{\pi}{M+2} = 2\left(1 - \frac{\pi^2}{2(M+2)^2} + \cdots\right) \quad (4.4)
\]
To further appreciate these asymptotic results, we have calculated for \(p=2, M+1=9\) (so that \(N=2, R=1\)) the quantities \(\lambda_0''(M)\), \(\lambda_0''(M)\) and the approximate (3.2). We obtain (since \(f_{2,1}(z) = z^{-2} - 6z + 3\))
\[
\begin{align*}
\lambda_0'(M) &= (3 + 2\sqrt{3})^{1/4} = 1.527878993, \\
\lambda_0''(M) &= \frac{4}{3}^{4/4} = 1.754765351, \\
\lambda_0'''(M) &= \frac{1}{200} = 1.494982753,
\end{align*}
\] (3.4)

We conclude this section by noting that the left eigenvector \(q\) of \(B\) in (1.4) corresponding to eigenvalue 1 (vector of stationary probabilities) is given by (1.5). This is an easy consequence of the facts that \(\Pi A\Pi = A\) and that
\[
B = \lambda_0'^1D^{-1}AD, \quad D = \text{diag}(p_0, p_1, \cdots, p_5). \quad (3.6)
\]
Note that \(q\) is midpoint symmetric, i.e., \(\Pi q = q\), and that the limiting form of \(q\), when \(M \to \infty\), as follows from (3.3), is independent of \(p\), viz. a multiple of
\[
\sin^2\left(\frac{\pi(m+1)}{M+2}\right)_{m=0, \cdots, M}. \quad (3.7)
\]

IV. POWER SPECTRUM OF MAXENTROPIC PROCESS

In this section we present a decomposition of the power spectrum \(S_\theta(\theta)\) of the maxentropic \(t_\theta\) of (1.2) in accordance with the eigenvalue structure of \(A\). The power spectra \(S_\theta(\theta)\) of the actual binary data and \(S_\theta(\theta)\) are related according to (1.8). In view of (1.6) and (1.7) we have
\[
S_\theta(\theta) = \sum_{k=-N}^{N} R(k)e^{-2\pi ik\theta}, \quad R(k) = u^T FB^k u, \quad k = 0, 1, \cdots, (4.1)
\]
where \(F = \text{diag}(q_0, q_1, \cdots, q_M)\) and \(u = p^{-1}[m, m+1, \cdots, m-1, m]^T\) with \(m = [p]\). We have in the present case
\[
M = 2[p], \quad N = \left|\frac{1}{p}\right|, \quad R = 2[p] - 2p + \left|\frac{1}{p}\right| + 1. \quad (4.2)
\]

Lemma 1: Let (see Theorem 1 and (3.6))
\[
S^T \Pi Du = v = \begin{bmatrix} v_p \\ v_m \\ v_{all} \end{bmatrix} \quad (4.3)
\]
with
\[
\begin{align*}
v_p &= \begin{bmatrix} v_{0,0}, v_{0,1}, \cdots, v_{0,2N-1} \end{bmatrix}^T, \\
v_m &= \begin{bmatrix} v_{0,0}, \cdots, v_{m,0}, v_{m+1,0} \end{bmatrix}^T.
\end{align*}
\]

Then
\[
R(k) = R_\alpha(\theta) + R_m(k), \quad k = 0, 1, \cdots, (4.4)
\]
with
\[
\begin{align*}
R_\alpha(\theta) &= \sum_{l=0}^{N-1} p_\theta l \sum_{r=0}^{2p-1} c_{\theta,l} e^{2\pi irk/2p}, \\
R_m(k) &= c_k. \quad (4.5)
\end{align*}
\]
Here
\[
\begin{align*}
\rho_l &= \frac{\lambda_0'}{\lambda_0''}, \\
c_k &= -\lambda_0'^{1/2} \sum_{r=0}^{R-1} c_{\theta,0} \cdots c_{\theta,k} \alpha_{r+k} \quad (4.7)
\end{align*}
\]
and the power spectra \(S_\theta(\theta)\) now follow from the results of Section II.

Lemma 1 shows that \(R_\alpha(\theta)\) contains a periodic component (the term with \(l=0\) and \(N-1\) rapidly decaying components corresponding to the terms with \(l=1, \cdots, N-1\). Due to the special form of \(u\) (the numbers \(c_{\theta,l}\) can be expressed (pretty much as the \(p\)'s in (2.16)) in terms of (derivatives of) the \(P_\theta\)'s at special points. In particular, it can be shown that
\[
\begin{align*}
c_{\theta,l} &= c_{l,2p-l}, \\
c_{l,0}, c_{l,p} \in R; \\
c_{0,0} = 0 \leq c_{0,l}. \quad (4.10)
\end{align*}
\]
As a consequence, certain terms in (4.5) can be combined, and we obtain the following result for the spectrum \(S_\theta(\theta)\).

Theorem 2: We have
\[
S_\theta(\theta) = S_\theta(\theta) + \sum_{l=1}^{N-1} S_\theta(\theta) + S_m(\theta). \quad (4.11)
\]
In this decomposition we have that
\[
S_\theta(\theta) = \sum_{r=1}^{2p-1} c_{l,0} \delta(\theta - \frac{r}{2p}) \quad (4.12)
\]
is the discrete component in \(S_\theta(\theta)\), and that
\[
S_\theta(\theta) = c_{l,0} p(\theta) + c_{l,p} F\left(\theta - \frac{1}{2}\right) + \sum_{l=1}^{p} \sum_{r=1}^{N-1} \text{Re}\left[c_{\theta,l} T_l\left(\theta - \frac{r}{2p}\right)\right] + c_{l,p} T_l\left(\theta + \frac{r}{2p}\right), \quad l = 1, \cdots, N-1. \quad (4.13)
\]

with
\[
T_l(\theta) = 1 - \rho_l^2 + 2 \rho_l \sin 2\pi \theta + \rho_l^2, \quad F_l(\theta) = \text{Re} \ T_l(\theta), \quad (4.14)
\]

and
\[
S_m(\theta) = c_0 + 2 \sum_{k=1}^{R-1} c_k \cos 2\pi k\theta \quad (4.15)
\]
constitute the continuous component of \(S_s(\theta)\). Here \(\rho_\mu, c_\mu, c_k\) are given in Lemma 1.

Proof: This follows from Lemma 1 and (4.10). Furthermore, it is used that when \(0 < \rho < 1\), \(\varphi \in \mathbb{R}\), and

\[
U_\mu(\varphi, \theta) = \sum_{k=-\infty}^{\infty} \rho^{2|k|} e^{i\varphi|k| - 2\pi i k},
\]

then for \(A \in \mathbb{C}\)

\[
AU_\mu(\varphi, \theta) + A^*U_\mu(-\varphi, \theta) = \Re \left( A^* - \rho^2 + 2i j \sin 2\pi (\theta - \varphi) \right)
+ A^* - 2i p \cos 2\pi (\theta + \varphi) + \rho^2 + A - 2i p \cos 2\pi (\theta + \varphi) + \rho^2 \]

(4.17)

And also it is used that

\[
\sum_{k=-\infty}^{\infty} e^{i\varphi|k| - 2\pi i k} = \frac{1}{2} \sum_{k=-\infty}^{\infty} \left\{ \delta(\theta - \varphi - k) + \delta(\theta + \varphi - k) \right\}
\]

(4.18)

with \(\delta\) Dirac's delta function. Here convergence of both series is to be taken in the sense of generalized functions.

Note: The discrete component at \(0\) vanishes since \(c_{0,0} = 0\).

I. INTRODUCTION

Some digital transmission systems require codes so that the frequency spectrum of the encoded message has a desired shape. For example, some digital magnetic recording systems use a code by which the source sequence is encoded into the message having a spectral null at dc.

In this correspondence we think of a directed graph with labeled states as a model of an encoder and a sequence of labels (complex numbers) along a path in the directed graph as an encoded sequence generated by the encoder. We consider the encoded message to be the function of the Markov chain given by a transition probability matrix compatible with the graph whose values are the sequences of labels along the infinite paths of the graph. It has been pointed out that the running digital sum at dc (denoted \(\text{RDS}_0\)) of the encoded sequence plays an important role in constructing and analyzing encoders whose encoded message has a small amount of frequency content at low frequencies. Where for an encoded sequence \(a = a_0 a_1 \cdots a_L\), \(\text{RDS}_0\) of \(a\) is defined by

\[
\text{RDS}_0(a) = \sum_{m=0}^{L} a_m.
\]

Yasuda and Inose [16], [17] proved that for an encoder the following three conditions are equivalent: 1) the encoder satisfies a “finite \(\text{RDS}_0\) condition” (i.e., for every encoded sequence from the encoder, \(\text{RDS}_0\) takes its value in a finite range); 2) the encoder satisfies a “loop-sum-zero condition” (i.e., for every encoded sequence generated by a cycle of the encoder, its \(\text{RDS}_0\) value is 0); 3) the encoded message has a spectral null at \(f\) (i.e., the entire encoded message has zero mean and the power spectral density vanishes at dc). This was rediscovered by [7], [13]. Let \(k\) be a nonnegative integer and \(n\) a positive integer with \(\text{gcd}(k,n) = 1\). Let \(f_k\) be the symbol frequency. Yoshida and Yajima [19] defined the running digital sum at \(f = kf_k/n\) (denoted \(\text{RDS}_f\)) of an encoded sequence \(a = a_0 a_1 \cdots a_L\) to be

\[
\text{RDS}_f(a) = \sum_{m=0}^{L} a_m \exp \left( -j2\pi m/n \right).
\]

where \(j = \sqrt{-1}\). (They call \(\text{RDS}_f(a)\) a weighted digital sum variation of \(a\).) As an extension of the previous result, Yoshida and Yajima [18], and Marcus and Siegel [10] proved that the following three conditions are equivalent: 1) the encoder satisfies a “finite \(\text{RDS}_f\) condition”; 2) for every encoded sequence generated by a cycle of the encoder of length a multiple of \(n\), its \(\text{RDS}_f\) value is 0; 3) the encoded message has a spectral null at \(f\) (i.e., the encoded message has a zero spectral line and the power spectral density vanishes at \(f\)). Moreover, Marcus and Siegel showed that a “coboundary condition at \(f\)” is equivalent to these three conditions, and that the coboundary condition is useful in constructing encoders for a spectral null at \(f\) (canoni-