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The vertex-connectivity of a distance-regular graph

Andries E. Brouwer\textsuperscript{a}, Jack H. Koolen\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, Techn. Univ. Eindhoven, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
\textsuperscript{b} Department of Mathematics, POSTECH, Pohang, South Korea

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To Eiichi Bannai, on the occasion of his 60th birthday

\textbf{A B S T R A C T}

The vertex-connectivity of a distance-regular graph equals its valency. © 2008 Dr Andries E. Brouwer. Published by Elsevier Ltd. All rights reserved.

\section{1. Introduction}

In this paper we prove the following theorem.

\textbf{Theorem.} Let $\Gamma$ be a non-complete distance-regular graph of valency $k > 2$. Then the vertex-connectivity $\kappa(\Gamma)$ equals $k$, and the only disconnecting sets of vertices of size not more than $k$ are the point neighbourhoods.

The special case of this theorem where $\Gamma$ has diameter 2 was proved by Brouwer and Mesner \cite{4} more than twenty years ago. The case of diameter 3 was announced by the first author at the conference celebrating Eiichi Bannai's 60th birthday.

The upper bound $k$ is tight. For example, an icosahedron (with $k = 5$) can be disconnected by removing a hexagon, leaving two triangles, and the line graph of the Petersen graph (with $k = 4$) can be disconnected by removing a 5-coclique, leaving two pentagons.

The edge-connectivity of distance-regular graphs was determined earlier.

\textbf{Proposition 1.1} (Brouwer and Haemers [2]). Let $\Gamma$ be a distance-regular graph with more than one vertex. Then its edge-connectivity equals its valency $k$, and the only disconnecting sets of $k$ edges are the sets of edges incident with a single vertex.

\textit{E-mail addresses:} aeb@cwi.nl, Andries.Brouwer@cwi.nl (A.E. Brouwer), koolen@postech.ac.kr (J.H. Koolen).
2. Tools

Given good information on the eigenvalues, expansion properties follow from the below version of Tanner’s bound.

**Proposition 2.1** (Haemers [5]). Let $\Gamma$ be a regular graph of valency $k$ with second largest eigenvalue $\theta$ and smallest eigenvalue $\theta'$. Let $A$ and $B$ be two separated sets in $\Gamma$ of sizes $a$ and $b$, respectively. Then

$$\frac{ab}{(v-a)(v-b)} \leq \left(\frac{\theta - \theta'}{2k - \theta - \theta'}\right)^2.$$ 

If the separating set $S$ has size $s$, so that $v - a = b + s$, then an equivalent formulation is

$$\frac{ab}{vs} \leq \frac{(\theta - \theta')^2}{4(k-a)(k-b)}.$$ 

For combinatorial work, the coding-theoretic argument below is useful. We will quote this as the ‘inproduct bound’.

**Lemma 2.2.** Among a set of $a$ binary vectors of length $n$ and average weight $w$ there are two with inner product at least $w(aw/n - 1)/(a - 1) = w^2/n - w(n-w)/m(a-1)$.

**Proof.** The sum of all pairwise inner products of the vectors is at least $n\left(\frac{aw}{a-1}\right)$. □

**Lemma 2.3.** Let $\Gamma$ be a distance-regular graph with a separation $\Gamma \setminus S = A + B$, and let $\alpha \in A$. If $\alpha$ has $s$ neighbours in $S$, then $|A| > v(1 - \frac{s}{k})(1 - \frac{k}{k_2})$.

**Proof.** According to Lemma 4.3(i) (the ‘Shadow Lemma’) and subsequent remark in Brouwer and Haemers [2] one has $|A| \geq v - \sum_i \frac{s_i}{k_i}(k_i + \cdots + k_d)$ where $s_i = |S \cap T_i(\alpha)|$, and $(k_i + \cdots + k_d)/k_i \geq (k_{i+1} + \cdots + k_d)/k_{i+1}$, so that $|A| \geq v - \frac{s}{k}(v - 1) - \frac{k-s}{k_2}(v - 1 - k) > v(1 - \frac{s}{k})(1 - \frac{k}{k_2})$. □

3. Vertex-connectivity of a distance-regular graph

Let us say that a distance-regular graph $\Gamma$ is OK when its vertex-connectivity equals its valency $k$, and the only disconnecting sets of size $k$ are the sets of neighbours of a vertex.

Let $\Gamma$ be a distance-regular graph of diameter $d$ at least 3, not a polygon, and suppose $S$ is a set of vertices of size at most $k$ such that $\Gamma \setminus S$ is disconnected, say with separation $A + B$. Suppose moreover that each of $A$ and $B$ contains at least two vertices. We shall obtain a contradiction. Notation is as in BCN [1].

Put $a = |A|$, $b = |B|$, $s = |S|$.

**Lemma 3.1.** In any distance-regular graph of diameter more than 2 one has $3\lambda + 4 \leq 2k$.

**Proof.** Suppose $\alpha \sim \beta \sim \gamma \sim \delta$ is a geodesic. If $3\lambda + 3 \geq 2k$, then $\lambda > 2(b_1 - 1)$, so that not all common neighbours of $\beta$ and $\gamma$ are nonadjacent to $\alpha$ or nonadjacent to $\delta$. But then $\alpha$ and $\delta$ have a common neighbour, contradiction. □

**Lemma 3.2.** Neither $A$ nor $B$ is a clique.

**Proof.** Suppose $A$ is a clique of size $a$. Apply the inproduct bound to the $a$ characteristic vectors of the sets $\Gamma(\alpha) \cap S$ (for $\alpha \in A$) of length at most $k$ and weight $k - (a - 1)$, and find $\lambda \geq a - 2 + (k-a+1)(k-a)/k$. The right-hand side is minimal for $a = (k+1)/2$, and hence $\lambda \geq \frac{3}{4}k - \frac{3}{2} + \frac{1}{4k}$. On the other hand, by Lemma 3.1, $\lambda \leq \frac{1}{2}k - \frac{4}{3}$ and hence $k = 3, \lambda = 0$, contradiction. □

**Lemma 3.3.** $a > 3$. 

Lemma 3.4. If $k = 3$ then $\Gamma$ is OK.

Proof. Suppose $k = 3$, and pick the separation $\Gamma \setminus S = A + B$ such that $S$ has minimal size (at most 3) and $A$ has minimal size larger than one (so that $|B| \geq |A| > 3$), given the size of $S$. If a point of $S$ has only one neighbour in $A$, then $A$ can be made smaller. If a point of $S$ has no neighbours in $A$, then $S$ can be made smaller. So, we may assume that each point of $S$ has precisely two neighbours in $A$ and one in $B$. But then there is a disconnecting set of at most three edges, not all on a single point, contradicting Proposition 1.1. □

Lemma 3.5. If $\lambda = 0$ and $\mu = 1$ then $a > 7$.

Proof. Each point of $A$ has $k$ neighbours in $A \cup S$, and each pair of vertices of $A$ at distance 2 have a common neighbour. We may assume that $A$ is connected, and then it has at least $a - 1$ edges. We find $ak - \binom{a}{2} + a - 1 \leq a + |S|$. Now use $k > 3$. □

Lemma 3.6. The icosahedron is OK.

Proof. This is a special case of the following lemma. □

Lemma 3.7. An antipodal 2-cover of a complete graph is OK.

Proof. Let $\Gamma$ be an antipodal 2-cover of a complete graph $K_{k+1}$. Since $|S| \leq k$ there is a pair of antipodal vertices neither of which is in $S$. If both are in $A$, then each vertex of $B$ is adjacent to some vertex of $A$, impossible. So, we have antipodal $a_0 \in A$ and $\beta_0 \in B$. Let $A' = A \setminus \{a_0\}$ and let $B'$ be the set of antipodes of $B \setminus \{\beta_0\}$. The graph $\Delta := \Gamma(\alpha_0)$ is strongly regular and satisfies $k_\Delta = 2\mu_\Delta \ (BCN \ 1.5.3)$. The sets $A'$ and $B'$ are subsets of $\Delta$ and $|A'| + |B'| \geq k$ and every vertex of $A'$ is equal or adjacent to every vertex of $B'$. Now neither $A'$ nor $B'$ is a clique, so if $a_1, a_2$ are two nonadjacent vertices in $A'$ and $\beta_1, \beta_2$ two nonadjacent vertices of $B'$, then $k_\Delta = 2\mu_\Delta = \mu_\Delta(\alpha_1, \alpha_2) + \mu_\Delta(\beta_1, \beta_2) \geq |B'| + |A'| \geq k = v_\Delta$, impossible. □

Lemma 3.8. $k_2 > k$.

Proof. One always has $\mu \leq b_1$ and hence $k_2 \geq k$. If equality holds then by BCN 5.1.1(v) $\Gamma$ has diameter 3 and is an antipodal 2-cover ($k_3 = 1$), so is OK by Lemma 3.7. □

Lemma 3.9. $\max(\lambda + 2, \mu) \geq (k+1)/(a + k)$.

Proof. Apply the inproduct bound to the $a$ characteristic vectors of the sets $\{\alpha\} \cup \Gamma(\alpha)$ for $\alpha \in A$, of length at most $a + k$ and weight $k + 1$. □

Proposition 3.10. If $\lambda > 0$ and $\mu > 1$ and $\lambda + 2 \geq k$ then $\Gamma$ is OK.

Proof. By BCN 4.4.3 we have: either $\Gamma$ is the icosahedron, or $\lambda = 0$, or $\mu = 1$, or both $\theta_1 \leq b_1 - 1$ and $-\theta_1 \leq \frac{3}{2} b_1 + 1$. In the latter case the separation bound gives

$$\frac{ab}{(v-a)(v-b)} \leq \left(\frac{3}{2} b_1 \right)^2.$$ 

Put $a = ak, b_1 = \beta(k+1)$. Since $\lambda + 2 \geq \frac{k(k+1)}{\alpha + k} = \frac{1}{1+2}(k+1)$, we have $\beta \leq \frac{\alpha}{k+1}$. Let $\gamma$ be the RHS of the separation bound. Then $\gamma \leq \left(\frac{3\beta}{4\sqrt{\beta}}\right)^2 \leq \left(\frac{3\alpha}{4\sqrt{\alpha}}\right)^2$ and $ab \leq \gamma(v-a)(v-b) \leq \gamma(a+k)(b+k)$. Assuming $b \geq a$ we may multiply by $a/b$ and obtain $a^2 \leq \gamma(a+k)(a+\frac{k}{b}) \leq \gamma(a+k)^2$. But this is a contradiction. □
Now the proof is split into the three cases $\mu = 1$, and $\lambda = 0$, $\mu > 1$, and $2 < \lambda + 2 < \mu$. The first of these will be handled in Lemma 3.13 below. The second in Lemma 3.17.

Call a point in $A \cup B$ a deep point if it has no neighbours in $S$.

**Lemma 3.11.** If $\lambda = 0$ and $k_2 \geq 3k$, then $\Gamma$ is OK.

**Proof.** Let $\sigma$, $\tau$ be the minimum number of neighbours some point of $A$ resp. $B$ has in $S$. Then $a > \frac{3}{4}v(1 - \frac{\sigma}{k})$ and $b > \frac{3}{4}v(1 - \frac{\tau}{k})$ by Lemma 2.3. Since $a + b < v$ we have $\sigma + \tau > \frac{k}{2}$. Since $\lambda = 0$ we have $\sigma, \tau \leq \frac{1}{2}k$, so $\sigma, \tau$ are nonzero, that is, neither $A$ nor $B$ has a deep point.

Assume $a \leq b$. Count edges incident with vertices in $S$. One finds $\sigma a + \tau b \leq k^2$, so that $a < 2k$. Since $\sigma, \tau \geq \mu$ we have $v \leq k + k^2/\mu$. If $\mu > 1$ then by Lemma 3.9, $\mu > k^2/(a + k) > \frac{1}{3}k$, so that $v < 4k$, contradiction. If $\mu = 1$, then by the same lemma $a + k > k(k + 1)/2$, but $a < 2k$ and hence $k \leq 4$. By Lemma 3.5 $a > 7$, contradiction. \qed

**Lemma 3.12.** If $k_2 \geq 4k$ and $v \geq 6k$, then $\Gamma$ is OK.

**Proof.** Let $\sigma$, $\tau$ be the minimum number of neighbours some point of $A$ resp. $B$ has in $S$. Then $a > \frac{3}{4}v(1 - \frac{\sigma}{k})$ and $b > \frac{3}{4}v(1 - \frac{\tau}{k})$ by Lemma 2.3. Since $a + b < v$ we have $\sigma + \tau > \frac{k}{3}$. If $\sigma > \frac{6}{7}k$ then $\lambda$, $\mu > \frac{1}{3}k$ and $k_2 < 2k$, contradiction.

So, $\sigma, \tau \leq \frac{1}{3}k$ and $\sigma, \tau$ are nonzero, that is, neither $A$ nor $B$ has a deep point.

Assume $a \leq b$. Count edges incident with vertices in $S$. One finds $\sigma a + \tau b \leq k^2$, so that $a < \frac{3}{2}k$.

On the other hand, $a > \frac{3}{4}v(1 - \frac{\sigma}{k}) \geq \frac{1}{4}v \geq \frac{3}{2}k$, contradiction. \qed

**Lemma 3.13.** If $\mu = 1$, then $\Gamma$ is OK.

**Proof.** Since $\mu = 1$ we have (by BCN 1.2.1) lines of size $\lambda + 2$, and $(\lambda + 1)|k$, hence $(\lambda + 1)|b_1$. Since $k_2 = b_1k$ we have $b_1 < 5$ by Lemma 3.12. This leaves the cases $(k, \lambda) \in \{(3, 0), (4, 0), (5, 0), (4, 1), (6, 1), (6, 2), (8, 3)\}$. The cases with $\lambda = 0$ are settled by Lemmas 3.4 and 3.11. This leaves $(k, \lambda) \in \{(4, 1), (6, 1), (6, 2), (8, 3)\}$.

(i) Suppose $(k, \lambda) = (4, 1)$. Now $\Gamma$ is the line graph of a cubic graph. There are four arrays with $d \geq 3$ (see e.g. Brouwer and Koolen [3]) namely $(4, 2, 1; 1, 1, 4)$ for the line graph of the Petersens graph on 15 vertices, $(4, 2, 2; 1, 1, 2)$ for the flag graph of the Fano plane on 21 vertices, $(4, 2, 2; 1, 1, 1, 2)$ for the flag graph of $GQ(2, 2)$ on 45 vertices, and $(4, 2, 2; 2, 2, 2; 1, 1, 1, 1, 2)$ for the flag graph of $GH(2, 2)$ on 189 vertices.

In these four cases the separation bound yields $a \leq 2$, $a \leq 3$, $a \leq 5$, $a \leq 9$, respectively. Since we have $k_2 = 2k = 8$, the shadow bound (Lemma 2.3) yields $a > v/8$. Since also $a > 3$, this settles the case $(k, \lambda) = (4, 1)$.

(ii) Suppose $(k, \lambda) = (6, 1)$ or $(k, \lambda) = (6, 2)$. If $k = 6$, $\lambda \in \{1, 2\}$, $\mu = 1$, $k_2 \in \{24, 18\}$, then $\sigma + \tau > 3$ (because of $\nu$), so $\sigma + \tau \geq 4$. Also $\sigma \leq 3$ (because of $\mu$) so $\tau > 0$ and $A, B$ do not have deep points. By the inner product bound (with $w = k = 6$ and $n = a + k$) we have $a \geq 9$. On the other hand, $a \leq k^2/(\sigma + \tau) \leq 9$. So $a = 9$, and $\sigma + \tau = 4$ and $\sigma a + \tau b \leq k^2$ and $a \leq b$ imply $b = 9$. Now $v \leq a + b + k = 24$ and $v > 1 + k + k_2 \geq 25$, contradiction.

(iii) Suppose $(k, \lambda) = (8, 3)$. Then each point is in 2 cliques of size 5, and $\Gamma$ is the line graph of a graph of valency 5. If $d \geq 4$ then $v > k + k_2 = (k_3 + k_4) > 6k$, and $\Gamma$ is OK by Lemma 3.12. So, $d = 3$. Now by BCN 4.2.16, $\Gamma$ is the flag graph of $PG(2, 4)$ on 105 vertices, and we are done again since $v > 6k$. \qed

**Lemma 3.14.** If $d \geq 4$, or if $d = 3$ and $\Gamma$ is not bipartite, then $\mu \leq \frac{1}{2}k$.

**Proof.** If $d \geq 4$ this is trivial. Suppose $d = 3$ and $\Gamma$ is not bipartite and $\mu > k/2$. If $d(\alpha, \beta) = d(\beta, \gamma) = 2$ and $d(\alpha, \gamma) = 3$, then $\beta$ has $\mu$ common neighbours with each of $\alpha, \gamma$, and none occurs twice, so $\beta$ has more than $k$ neighbours. Contradiction. Hence $p^3_{22} = 0$, and the graph $F_2$ is (connected and) distance-regular with distances 0, 1, 2, 3 corresponding to 0, 2, 1, 3 in $\Gamma$. But then $k_2 \leq k$, contradiction. \qed
Lemma 3.15. If \( d \geq 4 \), or \( d = 3 \) and \( \Gamma \) is not bipartite, and \( \mu \geq \lambda + 2 \), then \( a > k \).

Proof. If \( a \leq k \) then, by Lemma 3.9, \( \max(\lambda + 2, \mu) > \frac{1}{2}k \). But this contradicts Lemma 3.14. \( \square \)

Lemma 3.16. Suppose \( B \) has a deep point and \( A \) does not. If \( \lambda + 2 \leq \mu \) then there is a separating set smaller than \( S \).

Proof. Let \( B' \) be the set of points in \( B \) with a neighbour in \( S \). Put \( s := |S| \) and \( b' := |B'| \). Each point in \( A \cup B' \) has at least \( \mu \) neighbours in \( S \), so \( \mu(a + b') \leq ks \). Since \( a + k > \frac{k^2}{\mu} \) (by Lemma 3.9) it follows that \( \mu(b' - s) < -(k - \mu)(k - s) \leq 0 \) so that \( b' < s \). Since \( B \) has a deep point, \( B' \) is a separating set.

Lemma 3.17. If \( \lambda = 0 \) and \( \mu > 1 \) then \( \Gamma \) is OK.

Proof. Suppose \( \lambda = 0 \) and \( \mu > 1 \).

By Lemma 3.16 either both or neither of \( A \) and \( B \) have a deep point.

If \( A \) and \( B \) have deep points \( \alpha \) and \( \beta \), then \( a, b > v(1 - \frac{\mu}{\lambda}) \), so that \( 2\mu > k - 1 \). Now \( d \geq d(\alpha, \beta) \geq 4 \) and \( \mu = k/2 \) by Lemma 3.14. We have \( d = 4 \), otherwise \( k/2 = \mu < c_3 \leq b_2 \leq k/2 \) (using BCN 5.4.1) would give a contradiction. Now \( b_2 = k/2 \) and (by BCN 5.8.2) \( \mu = k - 1 \) so that the graph is an antipodal 2-cover and \( \alpha \) and \( \beta \) are antipodes. Now \( |S| \geq k_2 > k \), as desired.

If neither \( A \) nor \( B \) has a deep point (and \( S \) is minimal) then every point of \( A \) (or \( B \)) has distance 2 to some point of \( B \) (or \( A \)), and therefore has at least \( \mu \) neighbours in \( S \). Counting edges meeting \( S \) we find \( v - k \leq a + b \leq k^2/\mu \).

Now \( v \leq k + \frac{k^2}{\mu} \) and \( k_2 = \frac{k(k - 1)}{\mu} \) gives \( 1 + k_3 \leq v - k - k_2 \leq \frac{k}{\mu} \) so that \( k_3 < \frac{k - 1}{\mu} \) (because \( \mu > 1 \)). On the other hand, \( c_3 \leq k \) and \( b_2 \geq 1 \) imply \( k_3 = \frac{k(k - 1)b_2}{\mu c_3} \geq \frac{k - 1}{\mu} \), contradiction. \( \square \)

Lemma 3.18. If \( d = 4 \) then \( \Gamma \) is OK.

Proof. By Proposition 3.10 and Lemmas 3.17 and 3.13 we may assume \( \lambda > 0 \) and \( \lambda + 2 < \mu \). By BCN Lemma 5.5.5 we have \( a_2 \geq \mu \), and since also \( b_2 \geq \mu \) (since \( d = 4 \)) it follows that \( \mu \leq k/3 \) and \( b_1 > 2k/3 \).

By Lemma 3.16 either both or neither of \( A \) and \( B \) have a deep point.

If both \( A \) and \( B \) have a deep point, then \( v > a + b = 2v(1 - \frac{\mu}{\lambda}) > v \), contradiction.

If neither \( A \) nor \( B \) has a deep point, then \( 1 + k + \frac{k(k - 1)}{\mu} + k_1 + 1 \leq v \leq k + \frac{k^2}{\mu} \), again a contradiction. \( \square \)

Lemma 3.19. If \( \lambda > 0 \) then \( \theta_d \geq -\frac{1}{2}b_1 - 1 \geq -\frac{1}{2}k \).

Proof. By BCN 4.4.3(iii), if \( b_1/(\theta_d + 1) > -2 \), then either \( \lambda = 0 \) or \( \Gamma \) is the icosahedron, but the icosahedron is OK by Lemma 3.6. \( \square \)

Proposition 3.20. Let \((u_i)\) be the standard sequence for the second largest eigenvalue \( \theta_1 \). If \( u_{d-1} > 0 \) then \( \theta_1 < a_d \), and for each vertex \( \alpha \) the subgraph \( \Gamma_d(\alpha) \) is connected.

Proof. We have \( c_d u_d - a_d u_d = \theta_1 u_d \), and \( u_d < 0 \) (since \((u_i)\) has precisely one sign change), so \( \theta_1 < a_d \). By interlacing \( \Gamma_d(\alpha) \) has eigenvalue \( a_d \) with multiplicity 1, and hence is connected. \( \square \)

Proposition 3.21. If \( \lambda > 0 \) and \( \mu > 1 \) and \( \theta_1 < a_d \), then \( \Gamma \) is OK.

Proof. By Lemma 3.19 we have \( \theta_d \geq -\frac{1}{2}k \). By Proposition 3.10 we may assume \( \mu \geq \lambda + 2 \).

Put \( a = \alpha k \). Then by Lemma 3.9 \( c_d \geq \mu > \frac{k}{1+\alpha} \), hence \( a_d = k - c_d < \frac{ak}{1+\alpha} \). Using \( \theta_1 < \frac{ak}{1+\alpha} \) and \( -\theta_d \geq \frac{1}{2}k \) we find from the separation bound that

\[
\frac{ab}{(v - a)(v - b)} \leq \left(\frac{3\alpha + 1}{3\alpha + 5}\right)^2.
\]
Let $\gamma$ be the RHS of the separation bound. Then $ab \leq \gamma(v-a)(v-b) \leq \gamma(a+k)(b+k)$. Assuming $b \geq a$ we may multiply by $a/b$ and obtain $a^2 \leq \gamma(a+k)(a+b/k) \leq \gamma(a+k)^2$, so that $a \leq k$, contradicting Lemma 3.15. □

Lemma 3.22. Suppose $\lambda > 0$ and $\mu > 1$. If $\theta_1 \leq 1/k$, then $\Gamma$ is OK.

Proof. If $\theta_1 \leq 1/k$, then we can use the bound for separated sets again with $\theta \leq 1/k$ and $\theta' \geq -1/k$. We find

$$\frac{ab}{(v-a)(v-b)} \leq \frac{1}{4}$$

so that $3ab \leq vk$, and if $a < b$ then $a \leq b \leq (a+k)k/(3a-k)$, so $(3a+k)(a-k) \leq 0$, that is, $a \leq k$. Now we are done by Lemma 3.15 and Proposition 3.10. □

Lemma 3.23. $\theta_1 \leq b_1 - 1$.

Proof. By BCN 4.4.3(ii) either $\theta_1 \leq b_1 - 1$ or $\mu = 1$ or $\Gamma$ is the icosahedron. But $\mu > 1$ by Lemma 3.13, and $\Gamma$ is not the icosahedron by Lemma 3.6. □

Lemma 3.24. Let $d = 3$. If $\frac{1}{2}k < \theta_1 \leq b_1 - 1$ then $\theta_1 < a_3$.

Proof. Firstly, $\theta_1 > 1/k$ is equivalent to $u_1 > 1/2$. Secondly, $\theta_1 \leq b_1 - 1$ is equivalent to $u_0 - 2u_1 + u_2 \geq 0$. Since $u_0 = 1$ this implies that $u_2 \geq 2u_1 - u_0 > 0$. Now $\theta_1 < a_3$ follows by Proposition 3.20. □

Theorem 3.25. $\Gamma$ is OK.

Proof. The cases $\lambda = 0$ and $\mu = 1$ were done in Lemmas 3.17 and 3.13. By Lemma 3.18 we may assume $d = 3$. By Lemmas 3.22–3.24 we have $\theta_1 < a_3$ and now Proposition 3.21 completes the proof. □

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