The vertex-connectivity of a distance-regular graph

Brouwer, A.E.; Koolen, J.H.

Published in:
European Journal of Combinatorics

DOI:
10.1016/j.ejc.2008.07.006

Published: 01/01/2009

Document Version
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the author’s version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

Citation for published version (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal

Take down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Download date: 03. Dec. 2018
The vertex-connectivity of a distance-regular graph

Andries E. Brouwer\textsuperscript{a}, Jack H. Koolen\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, Techn. Univ. Eindhoven, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
\textsuperscript{b} Department of Mathematics, POSTECH, Pohang, South Korea

ARTICLE INFO

Article history:
Available online 28 August 2008

To Eiichi Bannai, on the occasion of his 60th birthday

ABSTRACT

The vertex-connectivity of a distance-regular graph equals its valency.
© 2008 Dr Andries E. Brouwer. Published by Elsevier Ltd. All rights reserved.

1. Introduction

In this paper we prove the following theorem.

**Theorem.** Let $\Gamma$ be a non-complete distance-regular graph of valency $k > 2$. Then the vertex-connectivity $\kappa(\Gamma)$ equals $k$, and the only disconnecting sets of vertices of size not more than $k$ are the point neighbourhoods.

The special case of this theorem where $\Gamma$ has diameter 2 was proved by Brouwer and Mesner \cite{4} more than twenty years ago. The case of diameter 3 was announced by the first author at the conference celebrating Eiichi Bannai’s 60th birthday.

The upper bound $k$ is tight. For example, an icosahedron (with $k = 5$) can be disconnected by removing a hexagon, leaving two triangles, and the line graph of the Petersen graph (with $k = 4$) can be disconnected by removing a 5-coclique, leaving two pentagons.

The edge-connectivity of distance-regular graphs was determined earlier.

**Proposition 1.1** (Brouwer and Haemers \cite{2}). Let $\Gamma$ be a distance-regular graph with more than one vertex. Then its edge-connectivity equals its valency $k$, and the only disconnecting sets of $k$ edges are the sets of edges incident with a single vertex.
2. Tools

Given good information on the eigenvalues, expansion properties follow from the below version of Tanner’s bound.

**Proposition 2.1 (Haemers [5])**. Let \( \Gamma \) be a regular graph of valency \( k \) with second largest eigenvalue \( \theta \) and smallest eigenvalue \( \theta' \). Let \( A \) and \( B \) be two separated sets in \( \Gamma \) of sizes \( a \) and \( b \), respectively. Then

\[
\frac{ab}{(v-a)(v-b)} \leq \left( \frac{\theta - \theta'}{2k - \theta - \theta'} \right)^2.
\]

If the separating set \( S \) has size \( s \), so that \( v - a = b + s \), then an equivalent formulation is

\[
\frac{ab}{vs} \leq \frac{(\theta - \theta')^2}{4(k-a)(k-b)}.
\]

For combinatorial work, the coding-theoretic argument below is useful. We will quote this as the ‘inproduct bound’.

**Lemma 2.2.** Among a set of \( a \) binary vectors of length \( n \) and average weight \( w \) there are two with inner product at least \( w(a w/n - 1)/(a - 1) = w^2/n - w(n-w)/m(a-1) \).

**Proof.** The sum of all pairwise inner products of the vectors is at least \( n \left( \frac{aw/n}{2} \right) \).

**Lemma 2.3.** Let \( \Gamma \) be a distance-regular graph with a separation \( \Gamma \setminus S = A + B \), and let \( \alpha \in A \). If \( \alpha \) has \( s \) neighbours in \( S \), then \( |A| > v(1 - \frac{s}{k})(1 - \frac{k-1}{k}) \).

**Proof.** According to Lemma 4.3(i) (the ‘Shadow Lemma’) and subsequent remark in Brouwer and Haemers [2] one has \( |A| \geq v - \sum_{i=1}^{d} \frac{s_i}{k_i} (k_i + \cdots + k_d) \) where \( s_i = |S \cap \Gamma_i(\alpha)| \), and \( (k_1 + \cdots + k_d)/k_i \geq (k_{i+1} + \cdots + k_d)/k_{i+1} \), so that \( |A| \geq v - \frac{s}{k} (v - 1) - \frac{k-s}{k} (v - 1 - k) > v(1 - \frac{s}{k})(1 - \frac{k-1}{k}) \).

3. Vertex-connectivity of a distance-regular graph

Let us say that a distance-regular graph \( \Gamma \) is OK when its vertex-connectivity equals its valency \( k \), and the only disconnecting sets of size \( k \) are the sets of neighbours of a vertex.

Let \( \Gamma \) be a distance-regular graph of diameter \( d \) at least 3, not a polygon, and suppose \( S \) is a set of vertices of size at most \( k \) such that \( \Gamma \setminus S \) is disconnected, say with separation \( A + B \). Suppose moreover that each of \( A \) and \( B \) contains at least two vertices. We shall obtain a contradiction. Notation is as in BCN [1].

Put \( a = |A| \), \( b = |B| \), \( s = |S| \).

**Lemma 3.1.** In any distance-regular graph of diameter more than 2 one has \( 3\lambda + 4 \leq 2k \).

**Proof.** Suppose \( \alpha \sim \beta \sim \gamma \sim \delta \) is a geodesic. If \( 3\lambda + 3 \geq 2k \), then \( \lambda > 2(b_1 - 1) \), so that not all common neighbours of \( \beta \) and \( \gamma \) are nonadjacent to \( \alpha \) or nonadjacent to \( \delta \). But then \( \alpha \) and \( \delta \) have a common neighbour, contradiction.

**Lemma 3.2.** Neither \( A \) nor \( B \) is a clique.

**Proof.** Suppose \( A \) is a clique of size \( a \). Apply the inproduct bound to the \( a \) characteristic vectors of the sets \( \Gamma(\alpha) \cap S \) (for \( \alpha \in A \)) of length at most \( k \) and weight \( k - (a - 1) \), and find \( \lambda \geq a - 2 + \frac{(k-a+1)(k-a)}{k} \).

The right-hand side is minimal for \( a = (k + 1)/2 \), and hence \( \lambda \geq \frac{3}{4} k - \frac{3}{2} - \frac{1}{4k} \). On the other hand, by Lemma 3.1, \( \lambda \leq \frac{1}{2} k - \frac{4}{3} \) and hence \( k = 3, \lambda = 0 \), contradiction.

**Lemma 3.3.** \( a > 3 \).
Proof. If \( a = 3 \), then \( A \) is a path of length 2, and \( 3 + k \geq a + |S| \geq 3 + 3k - 4 - 2\lambda - (\mu - 1) = 3k - 2\lambda - \mu \), so that \( 2b_1 \leq \mu + 1 \leq b_1 \), impossible. \( \square \)

Lemma 3.4. If \( k = 3 \) then \( \Gamma \) is OK.

Proof. Suppose \( k = 3 \), and pick the separation \( \Gamma \setminus S = A + B \) such that \( S \) has minimal size (at most 3) and \( A \) has minimal size larger than one (so that \(|B| \geq |A| > 3\)), given the size of \( S \). If a point of \( S \) has only one neighbour in \( A \), then \( A \) can be made smaller. If a point of \( S \) has no neighbours in \( A \), then \( S \) can be made smaller. So, we may assume that each point of \( S \) has precisely two neighbours in \( A \) and one in \( B \). Then there is a disconnecting set of at most three edges, not all on a single point, contradicting Proposition 1.1. \( \square \)

Lemma 3.5. If \( \lambda = 0 \) and \( \mu = 1 \) then \( a > 7 \).

Proof. Each point of \( A \) has \( k \) neighbours in \( A \cup S \), and each pair of vertices of \( A \) at distance 2 have a common neighbour. We may assume that \( A \) is connected, and then it has at least \( a - 1 \) edges. We find \( ak - \binom{a}{2} + a - 1 \leq a + |S| \). Now use \( k > 3 \). \( \square \)

Lemma 3.6. The icosahedron is OK.

Proof. This is a special case of the following lemma. \( \square \)

Lemma 3.7. An antipodal 2-cover of a complete graph is OK.

Proof. Let \( \Gamma \) be an antipodal 2-cover of a complete graph \( K_{k+1} \). Since \(|S| \leq k \) there is a pair of antipodal vertices neither of which is in \( S \). If both are in \( A \), then each vertex of \( B \) is adjacent to some vertex of \( A \), impossible. So, we have antipodal \( a_0 \in A \) and \( \beta_0 \in B \). Let \( A' = A \setminus \{a_0\} \) and let \( B' \) be the set of antipodes of \( B \setminus \{\beta_0\} \). The graph \( \Delta := \Gamma(a_0) \) is strongly regular and satisfies \( k_\Delta = 2\mu_\Delta \) (BCN 1.5.3). The sets \( A' \) and \( B' \) are subsets of \( \Delta \) and \(|A'| + |B'| \geq k \) and every vertex of \( A' \) is equal or adjacent to every vertex of \( B' \). Now neither \( A' \) nor \( B' \) is a clique, so if \( a_1, a_2 \) are two nonadjacent vertices in \( A' \) and \( \beta_1, \beta_2 \) two nonadjacent vertices of \( B' \), then \( k_\Delta = 2\mu_\Delta = \mu_\Delta(\alpha_1, \alpha_2) + \mu_\Delta(\beta_1, \beta_2) \geq |B'| + |A'| \geq k = v_\Delta \), impossible. \( \square \)

Lemma 3.8. \( k_2 \geq k \).

Proof. One always has \( \mu \geq b_1 \) and hence \( k_2 \geq k \). If equality holds then by BCN 5.1.1(v) \( \Gamma \) has diameter 3 and is an antipodal 2-cover \( (k_3 = 1) \), so is OK by Lemma 3.7. \( \square \)

Lemma 3.9. \( \max(\lambda + 2, \mu) \geq k(1 + k)/(a + k) \).

Proof. Apply the inproduct bound to the \( a \) characteristic vectors of the sets \( \{\alpha\} \cup \Gamma(\alpha) \) for \( \alpha \in A \), of length at most \( a + k \) and weight \( k + 1 \). \( \square \)

Proposition 3.10. If \( \lambda > 0 \) and \( \mu > 1 \) and \( \lambda + 2 \geq \mu \) then \( \Gamma \) is OK.

Proof. By BCN 4.4.3 we have: either \( \Gamma \) is the icosahedron, or \( \lambda = 0 \), or \( \mu = 1 \), or both \( \theta_1 \leq b_1 - 1 \) and \( -\theta_1 \leq \frac{1}{2}b_1 + 1 \). In the latter case the separation bound gives

\[
\frac{ab}{(v - a)(v - b)} \leq \left( \frac{\frac{3}{2}b_1}{2k + 2 - \frac{1}{2}b_1} \right)^2.
\]

Put \( a = ak, b_1 = \beta(k + 1) \). Since \( \lambda + 2 \geq \frac{k(k + 1)}{\frac{a + k}{a + 3k}} = \frac{1}{1 + \alpha}(k + 1) \), we have \( \beta \leq \frac{\alpha}{1 + \alpha} \). Let \( \gamma \) be the RHS of the separation bound. Then \( \gamma \leq \left( \frac{3\beta}{4\beta} \right)^2 \leq \frac{3\nu}{4 + 3\alpha} \) and \( ab \leq \gamma(v - a)(v - b) \leq \gamma(a + k)(b + k) \). Assuming \( b \geq a \) we may multiply by \( a/b \) and obtain \( a^2 \leq \gamma(a + k)(a + \frac{k}{b}) \leq \gamma(a + k)^2 \). But this is a contradiction. \( \square \)
Now the proof is split into the three cases $\mu = 1$, and $\lambda = 0$, $\mu > 1$, and $2 < \lambda + 2 < \mu$. The first of these will be handled in Lemma 3.13 below. The second in Lemma 3.17.

Call a point in $A \cup B$ a deep point if it has no neighbours in $S$.

**Lemma 3.11.** If $\lambda = 0$ and $k_2 \geq 3k$, then $\Gamma$ is OK.

**Proof.** Let $\sigma, \tau$ be the minimum number of neighbours some point of $A$ resp. $B$ has in $S$. Then $a > \frac{3}{4} v(1 - \frac{\sigma}{k})$ and $b > \frac{3}{4} v(1 - \frac{\tau}{k})$ by Lemma 2.3. Since $a + b < v$ we have $\sigma + \tau > \frac{3}{2} k$. Since $\lambda = 0$ we have $\sigma, \tau \leq \frac{1}{2} k$, so $\sigma, \tau$ are nonzero, that is, neither $A$ nor $B$ has a deep point.

Assume $a \leq b$. Count edges incident with vertices in $S$. One finds $\sigma a + \tau b \leq k^2$, so that $a < 2k$. Since $\sigma, \tau \geq \frac{2}{3} k$ we have $v \leq k + k^2 / \mu$. If $\mu > 1$ then by Lemma 3.9, $\mu > k^2 / (a + k) > \frac{1}{2} k$, so that $v < 4k$, contradiction. If $\mu = 1$, then by the same lemma $a + k > k(k + 1)/2$, but $a < 2k$ and hence $k < 4$. By Lemma 3.5 $a > 7$, contradiction. \(\Box\)

**Lemma 3.12.** If $k_2 \geq 4k$ and $v \geq 6k$, then $\Gamma$ is OK.

**Proof.** Let $\sigma, \tau$ be the minimum number of neighbours some point of $A$ resp. $B$ has in $S$. Then $a > \frac{3}{4} v(1 - \frac{\sigma}{k})$ and $b > \frac{3}{4} v(1 - \frac{\tau}{k})$ by Lemma 2.3. Since $a + b < v$ we have $\sigma + \tau > \frac{3}{2} k$.

If $\sigma > \frac{3}{2} k$ then $\lambda, \mu > \frac{3}{2} k$ and $k_2 < 2k$, contradiction.

So, $\sigma, \tau \leq \frac{3}{2} k$ and $\sigma, \tau$ are nonzero, that is, neither $A$ nor $B$ has a deep point.

Assume $a \leq b$. Count edges incident with vertices in $S$. One finds $\sigma a + \tau b \leq k^2$, so that $a < \frac{3}{2} k$.

On the other hand, $a > \frac{3}{4} v(1 - \frac{\sigma}{k}) \geq \frac{1}{4} v > \frac{3}{2} k$, contradiction. \(\Box\)

**Lemma 3.13.** If $\mu = 1$, then $\Gamma$ is OK.

**Proof.** Since $\mu = 1$ we have (by BCN 1.2.1) lines of size $\lambda + 2$, and $(\lambda + 1)|k$, hence $(\lambda + 1)|b_1$. Since $k_2 = b_1 k$ we have $b_1 < 5$ by Lemma 3.12. This leaves the cases $(k, \lambda) \in \{(3, 0), (4, 0), (5, 0), (4, 1), (6, 1), (6, 2), (8, 3)\}$. The cases with $\lambda = 0$ are settled by Lemmas 3.4 and 3.11. This leaves $(k, \lambda) \in \{(4, 1), (6, 1), (6, 2), (8, 3)\}$.

(i) Suppose $(k, \lambda) = (4, 1)$. Now $\Gamma$ is the line graph of a cubic graph. There are four arrays with $d \geq 3$ (see e.g. Brouwer and Koolen [3]) namely $(4, 2, 1; 1, 1, 4)$ for the line graph of the Petersen graph on 15 vertices, $(4, 2, 2; 1, 1, 2)$ for the flag graph of the Fano plane on 21 vertices, $(4, 2, 2; 2, 1, 1, 2)$ for the flag graph of $GQ(2, 2)$ on 45 vertices, and $(4, 2, 2; 2, 2, 2; 1, 1, 1, 2)$ for the flag graph of $GH(2, 2)$ on 189 vertices.

In these four cases the separation bound yields $a \leq 2$, $a \leq 3$, $a \leq 5$, $a \leq 9$, respectively. Since we have $k_2 = 2k = 8$, the shadow bound (Lemma 2.3) yields $a \geq v/8$. Since also $a > 3$, this settles the case $(k, \lambda) = (4, 1)$.

(ii) Suppose $(k, \lambda) = (6, 1)$ or $(k, \lambda) = (6, 2)$. If $k = 6, \lambda \in \{1, 2\}$, $\mu = 1$, $k_2 \in (24, 18)$, then $\sigma + \tau \geq 3$ (because of $v$), so $\sigma + \tau \geq 4$. Also $\sigma \leq 3$ (because of $\mu$) so $\tau > 0$ and $A, B$ do not have deep points. By the inner product bound (with $w = k = 6$ and $n = a + k$) we have $a \geq 9$. On the other hand, $a \leq k^2 / (\sigma + \tau) \leq 9$. So $a = 9$, and $\sigma + \tau = 4$ and $\sigma a + \tau b \leq k^2$ and $a \leq b$ imply $b = 9$. Now $v \leq a + b + k = 24$ and $v > 1 + k + k_2 \geq 25$, contradiction.

(iii) Suppose $(k, \lambda) = (8, 3)$. Then each point is in 2 cliques of size 5, and $\Gamma$ is the line graph of a graph of valency 5. If $d \geq 4$ then $v > k + 12 + (k_2 + k_4) > 6k$, and $\Gamma$ is OK by Lemma 3.12. So, $d = 3$. Now by BCN 4.2.16, $\Gamma$ is the flag graph of $PG(2, 4)$ on 105 vertices, and we are done again since $v > 6k$. \(\Box\)

**Lemma 3.14.** If $d \geq 4$, or if $d = 3$ and $\Gamma$ is not bipartite, then $\mu \leq \frac{1}{2} k$.

**Proof.** If $d \geq 4$ this is trivial. Suppose $d = 3$ and $\Gamma$ is not bipartite and $\mu > k/2$. If $d(\alpha, \beta) = d(\beta, \gamma) = 2$ and $d(\alpha, \gamma) = 3$, then $\beta$ has $\mu$ common neighbours with each of $\alpha, \gamma$, and none occurs twice, so $\beta$ has more than $k$ neighbours. Contradiction. Hence $p^3_2 = 0$, and the graph $F_2$ is (connected and) distance-regular with distances 0, 1, 2, 3 corresponding to 0, 2, 1, 3 in $\Gamma$. But then $k_2 \leq k$, contradiction. \(\Box\)
Lemma 3.15. If \( d \geq 4 \), or \( d = 3 \) and \( \Gamma' \) is not bipartite, and \( \mu \geq \lambda + 2 \), then \( a > k \).

Proof. If \( a \leq k \) then, by Lemma 3.9, \( \max(\lambda + 2, \mu) > \frac{1}{2}k \). But this contradicts Lemma 3.14. \( \square \)

Lemma 3.16. Suppose \( B \) has a deep point and \( A \) does not. If \( \lambda + 2 \leq \mu \) then there is a separating set smaller than \( S \).

Proof. Let \( B' \) be the set of points in \( B \) with a neighbour in \( S \). Put \( s := |S| \) and \( b' := |B'| \). Each point in \( A \cup B' \) has at least \( \mu \) neighbours in \( S \), so \( \mu(a + b') \leq ks \). Since \( a + k > \frac{k^2}{\mu} \) (by Lemma 3.9) it follows that \( \mu(b' - s) \leq -(k - \mu)(k - s) \leq 0 \) so that \( b' < s \). Since \( B \) has a deep point, \( B' \) is a separating set. \( \square \)

Lemma 3.17. If \( \lambda = 0 \) and \( \mu > 1 \) then \( \Gamma \) is OK.

Proof. Suppose \( \lambda = 0 \) and \( \mu > 1 \).

By Lemma 3.16 either both or neither of \( A \) and \( B \) have a deep point.

If \( A \) and \( B \) have deep points \( \alpha \) and \( \beta \), then \( a, b \geq v(1 - k \mu) \), so that \( 2\mu > k - 1 \). Now \( d \geq d(\alpha, \beta) \geq 4 \) and \( \mu = k/2 \) by Lemma 3.14. We have \( d = 4 \), otherwise \( k/2 = \mu < c_3 \leq b_2 \leq k/2 \) (using BCN 5.4.1) would give a contradiction. Now \( b_2 = k/2 \) and (by BCN 5.8.2) \( c_3 = k - 1 \) so that the graph is an antipodal 2-cover and \( \alpha \) and \( \beta \) are antipodes. Now \( |S| \geq k > k \), as desired.

If neither \( A \) nor \( B \) has a deep point (and \( S \) is minimal) then every point of \( A \) (or \( B \)) has distance 2 to some point of \( B \) (or \( A \)), and therefore has at least \( \mu \) neighbours in \( S \). Counting edges meeting \( S \) we find \( v - k \leq a + b \leq k^2/\mu \).

Now \( v \leq k + \frac{k^2}{\mu} \) and \( k_2 = \frac{k(k - 1)}{\mu} \) gives \( 1 + k \leq v - k - k_2 \leq \frac{k}{\mu} \) so that \( k_3 < \frac{k - 1}{\mu} \) (because \( \mu > 1 \)). On the other hand, \( c_3 \leq k \) and \( b_2 \geq 1 \) imply \( k_3 = \frac{k(k - 1)b_2}{\mu c_3} \geq \frac{k - 1}{\mu} \), contradiction. \( \square \)

Lemma 3.18. If \( d \geq 4 \) then \( \Gamma' \) is OK.

Proof. By Proposition 3.10 and Lemmas 3.17 and 3.13 we may assume \( \lambda > 0 \) and \( \lambda + 2 < \mu \). By BCN Lemma 5.5.5 we have \( a_2 \geq \mu \), and since also \( b_2 \geq \mu \) (since \( d \geq 4 \)) it follows that \( \mu \leq k/3 \) and \( b_1 > 2k/3 \).

By Lemma 3.16 either both or neither of \( A \) and \( B \) have a deep point.

If both \( A \) and \( B \) have a deep point, then \( v > a + b > 2v(1 - \frac{k}{2b_1}) > v \), contradiction.

If neither \( A \) nor \( B \) has a deep point, then \( 1 + k + \frac{k(k - 1)}{\mu} + k + 1 \leq v \leq k + \frac{k^2}{\mu} \), again a contradiction. \( \square \)

Lemma 3.19. If \( \lambda > 0 \) then \( \theta_d \geq -\frac{1}{2}b_1 - 1 \geq -\frac{1}{2}k \).

Proof. By BCN 4.4.3(iii), if \( b_1/(\theta_d + 1) > -2 \), then either \( \lambda = 0 \) or \( \Gamma' \) is the icosahedron, but the icosahedron is OK by Lemma 3.6. \( \square \)

Proposition 3.20. Let \( (u_i) \) be the standard sequence for the second largest eigenvalue \( \theta_1 \). If \( u_{d-1} > 0 \) then \( \theta_1 < a_d \), and for each vertex \( \alpha \) the subgraph \( \Gamma_d(\alpha) \) is connected.

Proof. We have \( c_d u_{d-1} + a_d u_d = \theta_1 u_d \), and \( u_d < 0 \) (since \( (u_i) \) has precisely one sign change), so \( \theta_1 < a_d \). By interlacing \( \Gamma_d(\alpha) \) has eigenvalue \( a_d \) with multiplicity 1, and hence is connected. \( \square \)

Proposition 3.21. If \( \lambda > 0 \) and \( \mu > 1 \) and \( \theta_1 < a_d \), then \( \Gamma' \) is OK.

Proof. By Lemma 3.19 we have \( \theta_d \geq -\frac{1}{2}k \). By Proposition 3.10 we may assume \( \mu \geq \lambda + 2 \).

Put \( a = \alpha k \). Then by Lemma 3.9 \( c_d \geq \mu \geq \frac{k}{1 + \alpha} \), hence \( a_d = k - c_d < \frac{ak}{1 + \alpha} \). Using \( \theta_1 < \frac{ak}{1 + \alpha} \) and \( -\theta_d \leq \frac{1}{2}k \) we find from the separation bound that

\[
\frac{ab}{(v-a)(v-b)} \leq \left( \frac{3\alpha + 1}{3\alpha + 5} \right)^2.
\]
Let $\gamma$ be the RHS of the separation bound. Then $ab \leq \gamma(v-a)(v-b) \leq \gamma(a+k)(b+k)$. Assuming $b \geq a$ we may multiply by $a/b$ and obtain $a^2 \leq \gamma(a+k)(a+b/k) \leq \gamma(a+k)^2$, so that $a \leq k$, contradicting Lemma 3.15. □

Lemma 3.22. Suppose $\lambda > 0$ and $\mu > 1$. If $\theta_1 \leq \frac{1}{2}k$, then $\Gamma$ is OK.

Proof. If $\theta_1 \leq \frac{1}{2}k$, then we can use the bound for separated sets again with $\theta \leq \frac{1}{2}k$ and $\theta' \geq -\frac{1}{2}k$. We find
\[
\frac{ab}{(v-a)(v-b)} \leq \frac{1}{4}
\]
so that $3ab \leq vk$, and if $a \leq b$ then $a \leq b \leq (a+k)k/(3a-k)$, so $(3a+k)(a-k) \leq 0$, that is, $a \leq k$. Now we are done by Lemma 3.15 and Proposition 3.10. □

Lemma 3.23. $\theta_1 \leq b_1 - 1$.

Proof. By BCN 4.4.3(ii) either $\theta_1 \leq b_1 - 1$ or $\mu = 1$ or $\Gamma$ is the icosahedron. But $\mu > 1$ by Lemma 3.13, and $\Gamma$ is not the icosahedron by Lemma 3.6. □

Lemma 3.24. Let $d = 3$. If $\frac{1}{2}k < \theta_1 \leq b_1 - 1$ then $\theta_1 < a_3$.

Proof. Firstly, $\theta_1 > \frac{1}{2}k$ is equivalent to $u_1 > \frac{1}{2}$. Secondly, $\theta_1 \leq b_1 - 1$ is equivalent to $u_0 - 2u_1 + u_2 \geq 0$. Since $u_0 = 1$ this implies that $u_2 \geq 2u_1 - u_0 > 0$. Now $\theta_1 < a_3$ follows by Proposition 3.20. □

Theorem 3.25. $\Gamma$ is OK.

Proof. The cases $\lambda = 0$ and $\mu = 1$ were done in Lemmas 3.17 and 3.13. By Lemma 3.18 we may assume $d = 3$. By Lemmas 3.22–3.24 we have $\theta_1 < a_3$ and now Proposition 3.21 completes the proof. □

Acknowledgment

The second author was supported by the Com2MaC-SRC/ERC program of MOST/KOSEF (grant # R11-1999-054).

References